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Convergence analysis of fixed stress split iterative scheme for small strain anisotropic poroelastoplasticity: A primer

by

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CONVERGENCE ANALYSIS OF FIXED STRESS SPLIT ITERATIVE SCHEME FOR SMALL STRAIN ANISOTROPIC POROELASTOPLASTICITY: A PRIMER

SAUMIK DANA AND MARY F. WHEELER

Abstract. We perform a convergence analysis of the fixed stress split iterative scheme for the Biot system modeling coupled single phase flow and small strain deformation in an anisotropic poroelastoplastic medium. The fixed stress split iterative scheme solves the flow subproblem with stress tensor fixed using a multipoint flux mixed finite element method, followed by the poromechanics subproblem using a conforming Galerkin method in every coupling iteration at each time step. The coupling iterations are repeated until convergence and Backward Euler is employed for time marching. The convergence analysis is based on studying the equations satisfied by the difference of iterates to show that the iterative scheme is contractive.

1. Introduction.

This report serves as a primer to our efforts in arriving at theoretical convergence estimates for the fixed stress split iterative scheme for small strain anisotropic poroelastoplasticity coupled with single phase flow. This work follows up on our previous work [5], where we arrived at a contraction map for the case of anisotropic poroelasticity with tensor Biot parameter.

1.1. Preliminaries.

Given a bounded convex domain $\Omega \subset \mathbb{R}^3$, we use $P_k(\Omega)$ to represent the restriction of the space of polynomials of degree less that or equal to k to Ω and $Q_1(\Omega)$ to denote the space of trilinears on Ω. For the sake of convenience, we discard the differential in the integration of any scalar field χ over Ω as follows

$$\int_\Omega \chi(\mathbf{x}) \equiv \int_\Omega \chi(\mathbf{x})\, dV \qquad (\forall \mathbf{x} \in \Omega)$$

Sobolev spaces are based on the space of square integrable functions on Ω given by

$$L^2(\Omega) \equiv \Big\{\theta : \|\theta\|^2_\Omega := \int_\Omega |\theta|^2 < +\infty\Big\},$$

The inner product of two second order tensors $\mathbf{S}$ and $\mathbf{T}$ is given by (see [8])

$$(i,j=1,2,3) \qquad \mathbf{S} : \mathbf{T} = S_{ij}T_{ij}$$

A fourth order tensor is a linear transformation of a second order tensor to a second order tensor in the following manner (see [8])

$$(i,j,k,l=1,2,3) \qquad \mathbb{P}\mathbf{S} = \mathbf{T} \rightarrow \mathbb{P}_{ijkl}S_{kl} = T_{ij}$$

The dyadic product $\otimes$ of two second order tensors $\mathbf{S}$ and $\mathbf{T}$ is given by (see [8])

$$\mathbb{P} = \mathbf{S} \otimes \mathbf{T} \rightarrow \mathbb{P}_{ijkl} = S_{ij}T_{kl}$$

2. Model equations.

2.1. Flow model.

Let the boundary $\partial\Omega = \Gamma^f_D \cup \Gamma^f_N$ where Γ^f_D is the Dirichlet boundary and Γ^f_N is the Neumann boundary. The fluid mass conservation equation (2.1) in the presence of deformable and anisotropic porous medium with the Darcy

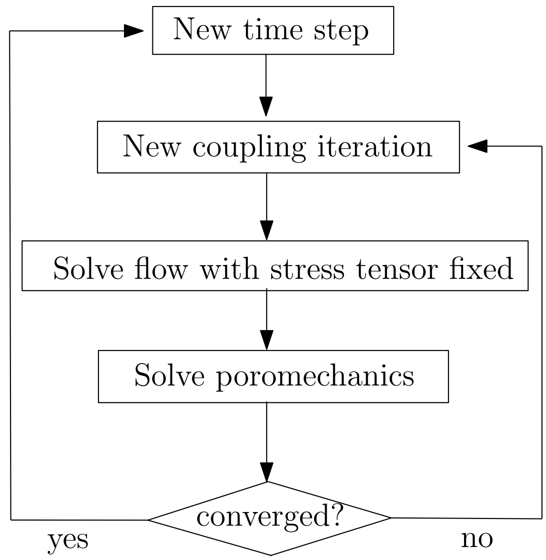

Fig. 1. *Fixed stress split iterative scheme for anisotropic poroelastoplasticity with tensor Biot parameter*

law (2.2) and linear pressure dependence of density (2.3) with boundary conditions (2.4) and initial conditions (2.5) is

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{z} = q \tag{2.1}$$

$$\mathbf{z} = -\frac{\mathbf{K}}{\mu}(\nabla p - \rho_0 \mathbf{g}) = -\boldsymbol{\kappa}(\nabla p - \rho_0 \mathbf{g}) \tag{2.2}$$

$$\rho = \rho_0(1 + c\,(p - p_0)) \tag{2.3}$$

$$p = g \text{ on } \Gamma_D^f \times (0,T],\ \mathbf{z}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N^f \times (0,T] \tag{2.4}$$

$$p(\mathbf{x},0) = p_0(\mathbf{x}),\ \rho(\mathbf{x},0) = \rho_0(\mathbf{x}),\ \phi(\mathbf{x},0) = \phi_0(\mathbf{x})$$

$$(\forall \mathbf{x} \in \Omega) \tag{2.5}$$

where $p : \Omega \times (0,T] \to \mathbb{R}$ is the fluid pressure, $\mathbf{z} : \Omega \times (0,T] \to \mathbb{R}^3$ is the fluid flux, ζ is the increment in fluid content[1], $\mathbf{n}$ is the unit outward normal on Γ_N^f, q is the source or sink term, $\mathbf{K}$ is the uniformly symmetric positive definite absolute permeability tensor, μ is the fluid viscosity, ρ_0 is a reference density, ϕ is the porosity, $\boldsymbol{\kappa} = \frac{\mathbf{K}}{\mu}$ is a measure of the hydraulic conductivity of the pore fluid, c is the fluid compressibility and $T > 0$ is the time interval.

2.2. Poromechanics model. Let the boundary $\partial\Omega = \Gamma_D^p \cup \Gamma_N^p$ where Γ_D^p is the Dirichlet boundary and Γ_N^p is the Neumann boundary. Linear momentum balance for the anisotropic porous solid in the quasi-static limit of interest (2.6) with small strain

[1] [1] defines the increment in fluid content as the measure of the amount of fluid which has flowed in and out of a given element attached to the solid frame

assumption (2.8) with boundary conditions (2.9) and initial condition (2.10) is

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \tag{2.6}$$

$$\mathbf{f} = \rho\phi\mathbf{g} + \rho_r(1-\phi)\mathbf{g} \tag{2.7}$$

$$\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \tag{2.8}$$

$$\mathbf{u}\cdot\mathbf{n}_1 = 0 \text{ on } \Gamma_D^p \times [0,T],\ \boldsymbol{\sigma}^T\mathbf{n}_2 = \mathbf{t} \text{ on } \Gamma_N^p \times [0,T] \tag{2.9}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0} \qquad \forall \mathbf{x} \in \Omega \tag{2.10}$$

where $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^3$ is the solid displacement, ρ_r is the rock density, $\mathbf{f}$ is the body force per unit volume, $\mathbf{n}_1$ is the unit outward normal to Γ_D^p, $\mathbf{n}_2$ is the unit outward normal to Γ_N^p, $\mathbf{t}$ is the traction specified on Γ_N^p, $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{\sigma}$ is the Cauchy stress tensor given by the generalized Hooke's law

$$\boldsymbol{\sigma} = \mathbb{D}\boldsymbol{\epsilon}^e - \boldsymbol{\alpha}p \equiv \mathbb{D}^{ep}\boldsymbol{\epsilon} - \boldsymbol{\alpha}p \tag{2.11}$$

where $\mathbb{D}$ is the fourth order anisotropic elasticity tensor, $\boldsymbol{\alpha}$ is the Biot tensor and $\mathbb{D}^{ep}$ is the elastoplastic tangent operator given in (A.1). The inverse of the generalized Hooke's law (2.11) is given by

$$\boldsymbol{\epsilon} = \mathbb{D}^{{ep}^{-1}}(\boldsymbol{\sigma} + \boldsymbol{\alpha}p) = \mathbb{D}^{{ep}^{-1}}\boldsymbol{\sigma} + \frac{C}{3}\mathbf{B}p \tag{2.12}$$

where $C(> 0)$ is a generalized Hooke's law constant and $\mathbf{B}$ is a generalization of the Skempton pore pressure coefficient B (see [9]) for anisotropic poroelastoplasticity, and is given by

$$\mathbf{B} \equiv \frac{3}{C}\mathbb{D}^{{ep}^{-1}}\boldsymbol{\alpha} \tag{2.13}$$

2.3. Increment in fluid content. The increment in fluid content ζ is given by (see [3])

$$\zeta = \frac{1}{M}p + \boldsymbol{\alpha} : \boldsymbol{\epsilon}^e + \phi^p \equiv Cp + \frac{1}{3}C\mathbf{B} : \boldsymbol{\sigma} + \phi^p \tag{2.14}$$

where $M(> 0)$ is a generalization of the Biot modulus (see [2]) for anisotropic poroelasticity and ϕ^p is a plastic porosity (see [3]).

3. Statement of contraction of the fixed stress split scheme for small strain anisotropic poroelastoplasticity with Biot tensor. We use the notations $(\cdot)^{n+1}$ for any quantity $(\cdot)$ evaluated at time level $n+1$, $(\cdot)^{m,n+1}$ for any quantity $(\cdot)$ evaluated at the m^{th} coupling iteration at time level $n+1$, $\delta_f^{(m)}(\cdot)$ for the change in the quantity $(\cdot)$ during the flow solve over the $(m+1)^{th}$ coupling iteration at any time level and $\delta^{(m)}(\cdot)$ for the change in the quantity $(\cdot)$ over the $(m+1)^{th}$ coupling iteration at any time level. Let $\mathscr{T}_h$ be finite element partition of Ω consisting of distorted hexahedral elements E where $h = \max_{E\in\mathscr{T}_h} diam(E)$. The details of the finite element mapping are given in [4]. The discrete variational statements in terms of coupling iteration differences is : find $\delta^{(m)}p_h \in W_h$, $\delta^{(m)}\mathbf{z}_h \in \mathbf{V}_h$ and $\delta^{(m)}\mathbf{u}_h \in \mathbf{U}_h$

such that

(3.1)
$$C(\delta^{(m)}p_h,\theta_h)_\Omega + \Delta t(\nabla\cdot\delta^{(m)}\mathbf{z}_h,\theta_h)_\Omega + (\delta^{(m)}\phi^p,\theta_h)_\Omega = -\frac{C}{3}(\mathbf{B}:\delta^{(m-1)}\boldsymbol{\sigma},\theta_h)_\Omega$$

(3.2)
$$(\boldsymbol{\kappa}^{-1}\delta^{(m)}\mathbf{z}_h,\mathbf{v}_h)_\Omega = (\delta^{(m)}p_h,\nabla\cdot\mathbf{v}_h)_\Omega$$

(3.3)
$$(\delta^{(m)}\boldsymbol{\sigma}:\boldsymbol{\epsilon}(\mathbf{q}_h))_\Omega = 0$$

where the finite dimensional spaces W_h, $\mathbf{V}_h$ and $\mathbf{U}_h$ are

$$W_h = \{\theta_h : \theta_h| \in P_0(E)\ \forall E \in \mathscr{T}_h\}$$
$$\mathbf{V}_h = \{\mathbf{v}_h : \mathbf{v}_h|_E \leftrightarrow \hat{\mathbf{v}}|_{\hat{E}} \in \hat{\mathbf{V}}(\hat{E})\ \forall E \in \mathscr{T}_h,\ \mathbf{v}_h\cdot\mathbf{n} = 0 \text{ on } \Gamma_N^f\}$$
$$\mathbf{U}_h = \{\mathbf{q}_h = (u,v,w)|_E \in Q_1(E)\ \forall E \in \mathscr{T}_h,\ \mathbf{q}_h = \mathbf{0} \text{ on } \Gamma_D^p\}$$

where P_0 represents the space of constants, Q_1 represents the space of trilinears and the details of $\hat{\mathbf{V}}(\hat{E})$ are given in [4]. The equations (3.1), (3.2) and (3.3) are the discrete variational statements (in terms of coupling iteration differences) of (2.1), (2.2) and (2.6) respectively. The details of (3.1) and (3.2) are given in Appendix B whereas the details of (3.3) are given in Appendix C.

THEOREM 3.1. *The fixed stress split iterative coupling scheme for anisotropic poroelasticity with Biot tensor in which the flow problem is solved first by freezing all components of the stress tensor is a contraction given by*

$$s$$

Proof. • **Step 1: Flow equations**
Testing (3.1) with $\theta_h \equiv \delta^{(m)}p_h$, we get

$$\begin{aligned} &C\|\delta^{(m)}p_h\|^2_\Omega + \Delta t(\nabla\cdot\delta^{(m)}\mathbf{z}_h,\delta^{(m)}p_h)_\Omega + (\delta^{(m)}\phi^p,\delta^{(m)}p_h)_\Omega \\ &= -\frac{C}{3}(\mathbf{B}:\delta^{(m-1)}\boldsymbol{\sigma},\delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.4}$$

Testing (3.2) with $\mathbf{v}_h \equiv \delta^{(m)}\mathbf{z}_h$, we get

$$\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|^2_\Omega = (\delta^{(m)}p_h,\nabla\cdot\delta^{(m)}\mathbf{z}_h)_\Omega \tag{3.5}$$

From (3.4) and (3.5), we get

(3.6)
$$C\|\delta^{(m)}p_h\|^2_\Omega + \Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|^2_\Omega + (\delta^{(m)}\phi^p,\delta^{(m)}p_h)_\Omega = -\frac{C}{3}(\mathbf{B}:\delta^{(m-1)}\boldsymbol{\sigma},\delta^{(m)}p_h)_\Omega$$

• **Step 2: Poromechanics equations**
Testing (3.3) with $\mathbf{q}_h \equiv \delta^{(m)}\mathbf{u}_h$, we get

$$(\delta^{(m)}\boldsymbol{\sigma}:\delta^{(m)}\boldsymbol{\epsilon})_\Omega = 0 \tag{3.7}$$

We now invoke (2.12) to arrive at the expression for change in strain tensor over the $(m+1)^{th}$ coupling iteration as follows

$$\delta^{(m)}\boldsymbol{\epsilon} = \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma} + \frac{C}{3}\mathbf{B}\delta^{(m)}p_h \tag{3.8}$$

Substituting (3.8) in (3.7), we get

$$(\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega + \frac{C}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega = 0 \tag{3.9}$$

- **Step 3: Combining flow and poromechanics equations**

Adding (3.6) and (3.9), we get

$$\begin{aligned} &C\|\delta^{(m)}p_h\|^2_\Omega + \Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|^2_\Omega + (\delta^{(m)}\phi^p, \delta^{(m)}p_h)_\Omega + (\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega \\ &+ \frac{C}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega = -\frac{C}{3}(\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.10}$$

- **Step 4: Variation in fluid content**

In lieu of (2.14), the variation in fluid content in the $(m+1)^{th}$ coupling iteration is

$$\delta^{(m)}\zeta = C\delta^{(m)}p_h + \frac{C}{3}\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma} + \delta^{(m)}\phi^p \tag{3.11}$$

As a result, we can write

$$\begin{aligned} &\frac{1}{2C}\|\delta^{(m)}\zeta\|^2_\Omega - \frac{C}{2}\|\delta^{(m)}p_h\|^2_\Omega - \frac{C}{18}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|^2_\Omega - \frac{1}{2C}\|\delta^{(m)}\phi^p\|^2_\Omega \\ &- (\delta^{(m)}\phi^p, \delta^{(m)}p_h)_\Omega - \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}\phi^p)_\Omega = \frac{C}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.12}$$

From (3.10) and (3.12), we get

$$\begin{aligned} &C\|\delta^{(m)}p_h\|^2_\Omega + \Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|^2_\Omega + (\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega + \frac{1}{2C}\|\delta^{(m)}\zeta\|^2_\Omega \\ &- \frac{C}{2}\|\delta^{(m)}p_h\|^2_\Omega - \frac{C}{18}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|^2_\Omega - \frac{1}{2C}\|\delta^{(m)}\phi^p\|^2_\Omega - \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}\phi^p)_\Omega \\ &= -\frac{C}{3}(\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.13}$$

Adding and subtracting $\frac{C}{6}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|^2_\Omega$ to the LHS of (3.13) results in

$$\begin{aligned} &\frac{C}{6}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|^2_\Omega + \frac{C}{2}\|\delta^{(m)}p_h\|^2_\Omega + \Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|^2_\Omega + (\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega \\ &+ \frac{1}{2C}\|\delta^{(m)}\zeta\|^2_\Omega - \frac{C}{9}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|^2_\Omega - \frac{1}{2C}\|\delta^{(m)}\phi^p\|^2_\Omega - \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta^{(m)}\phi^p)_\Omega \\ &= -\frac{C}{3}(\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.14}$$

In lieu of (2.14) and the fixed stress constraint during the flow solve, the variation in fluid content during the flow solve in the $(m+1)^{th}$ coupling iteration is given by

$$\delta_f^{(m)}\zeta = C\delta_f^{(m)}p_h + \frac{C}{3}\mathbf{B} : \cancelto{\mathbf{0}}{\delta_f^{(m)}\boldsymbol{\sigma}} + \delta_f^{(m)}\phi^p$$

Further, since the pore pressure is frozen during the poromechanical solve, we have $\delta_f^{(m)}p_h = \delta^{(m)}p_h$. As a result, we can write

$$\delta_f^{(m)}\zeta = C\delta^{(m)}p_h + \delta_f^{(m)}\phi^p \tag{3.15}$$

Subtracting (3.15) from (3.11), we can write

$$\delta^{(m)}\zeta - \delta_f^{(m)}\zeta = \frac{C}{3}\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma} + \delta^{(m)}\phi^p - \delta_f^{(m)}\phi^p$$

which implies that

$$\frac{1}{C}\|\delta^{(m)}\zeta - \delta_f^{(m)}\zeta\|_\Omega^2 - \frac{1}{C}\|\delta^{(m)}\phi^p - \delta_f^{(m)}\phi^p\|_\Omega^2 - \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, (\delta^{(m)}\phi^p - \delta_f^{(m)}\phi^p))_\Omega = \frac{C}{9}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|_\Omega^2 \tag{3.16}$$

In lieu of (3.16), we can write (3.14) as

$$\begin{aligned} &\frac{C}{6}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|_\Omega^2 + \overbrace{\frac{C}{2}\|\delta^{(m)}p_h\|_\Omega^2}^{\geq 0} + \overbrace{\Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|_\Omega^2}^{\geq 0} \\ &+ \overbrace{(\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega}^{\geq 0?} + \overbrace{\frac{1}{2C}\|\delta^{(m)}\zeta\|_\Omega^2}^{\geq 0} + \overbrace{\frac{1}{C}\|\delta^{(m)}\phi^p - \delta_f^{(m)}\phi^p\|_\Omega^2}^{\geq 0} \\ &- \Bigg[\overbrace{\frac{1}{C}\|\delta^{(m)}\zeta - \delta_f^{(m)}\zeta\|_\Omega^2 + \frac{1}{2C}\|\delta^{(m)}\phi^p\|_\Omega^2 + \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta_f^{(m)}\phi^p)_\Omega}^{\text{driven to zero by convergence criterion}}\Bigg] \\ &= -\frac{C}{3}(\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \end{aligned} \tag{3.17}$$

• Step 5: Invoking the Young's inequality

Since the sum of the terms on the LHS of (3.17) is nonnegative, the RHS is also nonnegative. We invoke the Young's inequality (see [10]) for the RHS of (3.17) as follows

$$-\frac{C}{3}(\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}, \delta^{(m)}p_h)_\Omega \leq \frac{C}{3}\left(\frac{1}{2}\|\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}\|_\Omega^2 + \frac{1}{2}\|\delta^{(m)}p_h\|_\Omega^2\right) \tag{3.18}$$

In lieu of (3.18), we write (3.17) as

$$\begin{aligned} &\frac{C}{6}\|\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}\|_\Omega^2 + \overbrace{\frac{C}{2}\|\delta^{(m)}p_h\|_\Omega^2}^{>0} + \overbrace{\Delta t\|\boldsymbol{\kappa}^{-1/2}\delta^{(m)}\mathbf{z}_h\|_\Omega^2}^{>0} \\ &+ \overbrace{(\delta^{(m)}\boldsymbol{\sigma} : \mathbb{D}^{ep^{-1}}\delta^{(m)}\boldsymbol{\sigma})_\Omega}^{\geq 0?} + \overbrace{\frac{1}{2C}\|\delta^{(m)}\zeta\|_\Omega^2}^{>0} + \overbrace{\frac{1}{C}\|\delta^{(m)}\phi^p - \delta_f^{(m)}\phi^p\|_\Omega^2}^{\geq 0} \\ &- \Bigg[\overbrace{\frac{1}{C}\|\delta^{(m)}\zeta - \delta_f^{(m)}\zeta\|_\Omega^2 + \frac{1}{2C}\|\delta^{(m)}\phi^p\|_\Omega^2 + \frac{1}{3}(\mathbf{B} : \delta^{(m)}\boldsymbol{\sigma}, \delta_f^{(m)}\phi^p)_\Omega}^{\text{driven to zero by convergence criterion}}\Bigg] \\ &\leq \frac{C}{6}\|\mathbf{B} : \delta^{(m-1)}\boldsymbol{\sigma}\|_\Omega^2 \end{aligned}$$

□

4. On the agenda.

- Use of the Sherman-Morrison formula to arrive at the estimates of the positive definiteness of the elastoplastic compliance tensor
- Design of the convergence criterion

Appendix A. Mathematical theory of small strain elastoplasticity. The important phenomenological aspects of small strain elastoplasticity are

- The existence of an elastic domain, i.e. a range of stresses within which the behaviour of the material can be considered as purely elastic, without evolution of permanent (plastic) strains. The elastic domain is delimited by the so-called yield stress.
- If the material is further loaded at the yield stress, then plastic yielding (or plastic flow), i.e. evolution of plastic strains, takes place.
- Accompanying the evolution of the plastic strain, an evolution of the yield stress itself is also observed. This phenomenon is known as hardening.

The basic components of a general elastoplastic constitutive model are

- the elastoplastic strain decomposition;
- a yield criterion, stated with the use of a yield function;
- a plastic flow rule defining the evolution of the plastic strain;
- a hardening law, characterising the evolution of the yield limit; and
- the elastoplastic tangent operator

A.1. Additive decomposition of the strain tensor. One of the chief hypotheses underlying the small strain theory of plasticity is the decomposition of the total strain, $\boldsymbol{\epsilon}$, into the sum of an elastic (or reversible) component $\boldsymbol{\epsilon}^e$, and a plastic (or permanent) component, $\boldsymbol{\epsilon}^p$,

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p$$

A.2. The yield criterion and the yield surface. A scalar yield function $\Phi(\boldsymbol{\sigma}, \mathbf{A})$ is introduced where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{A}$ is a set of thermodynamical forces. The yield function defines the elastic domain as the set

$$\mathcal{E} = \{\boldsymbol{\sigma} | \Phi(\boldsymbol{\sigma}, \mathbf{A}) < 0\}$$

of stresses for which plastic yielding is not possible. Any stress lying in the elastic domain or on its boundary is said to be plastically admissible. We then define the set of plastically admissible stresses (or plastically admissible domain) as

$$\mathcal{E} = \{\boldsymbol{\sigma} | \Phi(\boldsymbol{\sigma}, \mathbf{A}) \leq 0\}$$

The yield locus, i.e. the set of stresses for which plastic yielding may occur, is the boundary of the elastic domain, where $\Phi(\boldsymbol{\sigma}, \mathbf{A}) = 0$. The yield locus in this case is represented by a hypersurface in the space of stresses. This hypersurface is termed the yield surface and is defined as

$$\mathcal{Y} = \{\boldsymbol{\sigma} | \Phi(\boldsymbol{\sigma}, \mathbf{A}) = 0\}$$

For stress levels within the elastic domain, only elastic straining may occur, whereas on its boundary (at the yield stress), either elastic unloading or plastic yielding (or plastic loading) takes place. This yield criterion can be expressed by

$$\text{If } \Phi < 0 \implies \dot{\epsilon}^p = 0$$

$$\text{If } \Phi = 0 \implies \begin{cases} \dot{\epsilon}^p = 0, \text{ elastic unloading} \\ \dot{\epsilon}^p \neq 0, \text{ plastic loading} \end{cases}$$

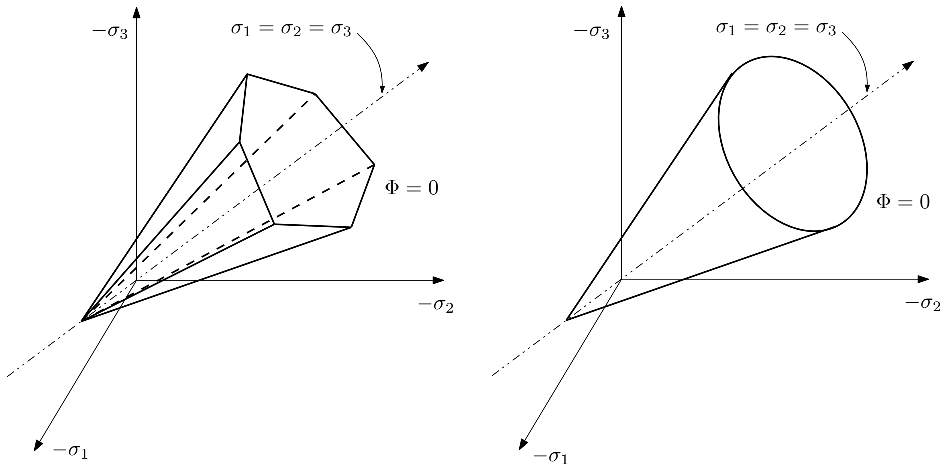

FIG. 2. *M-C yield surface on the left and D-P yield surface to the right; in principal stress space*

A.2.1. The Drucker-Prager yield criterion. The Drucker-Prager yield criterion has been proposed by [7] as a smooth approximation to the Mohr-Coulomb law, which states that plastic yielding is the result of frictional sliding between material particles. The M-C criterion is given as

$$\Phi = (cos\theta - \frac{1}{3}sin\theta\, sin\phi)\sqrt{J_2} + \sigma^{hyd}\, sin\phi - c\, cos\phi$$

where ϕ is the angle of internal friction, c is the cohesion, σ^{hyd} is the hydrostatic stress given by

$$\sigma^{hyd} = \frac{1}{3}tr(\boldsymbol{\sigma})$$

and θ is the Lode angle given by

$$\theta = \frac{1}{3}sin^{-1}\left(\frac{-3\sqrt{3}J_3}{2J_2^{3/2}}\right)$$

where J_2 and J_3 are stress deviator invariants given by

$$J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}$$
$$J_3 = det\, \mathbf{s}$$

and $\mathbf{s}$ is the stress deviator given by

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}(tr\boldsymbol{\sigma})\mathbf{I} = \boldsymbol{\sigma} - \sigma^{hyd}\mathbf{I}$$

The D-P criterion is given as

$$\Phi = \sqrt{J_2} + \eta\sigma^{hyd} - \xi c$$

where the parameters η and ξ are chosen according to the required approximation to the M-C criterion.

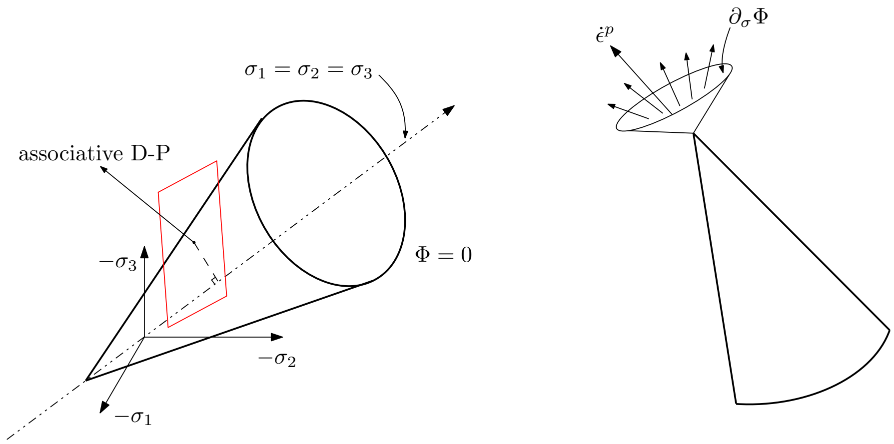

FIG. 3. *Associative D-P flow vector at the cone surface and at the apex*

A.3. Plastic flow rule. The plastic flow rule is given as

$$\dot{\epsilon}^p = \dot{\gamma}\mathbf{N}$$

where $\dot{\gamma}$ is the plastic multiplier and $\mathbf{N}$ is the flow vector.

A.3.1. Flow vector. It is often convenient to define the flow rule in terms of a flow (or plastic) potential. The starting point of such an approach is to postulate the existence of a flow potential with general form $\Psi(\boldsymbol{\sigma}, \mathbf{A})$, from which the flow vector is obtained as

$$\mathbf{N} \equiv \frac{\partial \Psi}{\partial \boldsymbol{\sigma}}$$

A.3.2. The plastic multiplier. The plastic multiplier is non-negative

$$\dot{\gamma} \geq 0$$

and satisfies the complementarity condition,

$$\Phi \dot{\gamma} = 0$$

A.3.3. Associative and non-associative plasticity. A plasticity model is classed as associative if the yield function Φ is taken as the flow potential, i.e.

$$\Psi \equiv \Phi$$

The flow vector is then given by

$$\mathbf{N} \equiv \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}$$

Any other choice of flow potential characterises a non-associative (or non-associated) plasticity model. At points where Φ is non-differentiable in $\boldsymbol{\sigma}$, the isosurfaces of Φ in the space of stresses contain a singularity (corner) where the normal direction is not uniquely defined. A typical situation is schematically illustrated in Figure 4 where two distinct normals, $\mathbf{N}_1$ and $\mathbf{N}_2$, are assumed to exist. In this case, the subdifferential of Φ with respect to $\boldsymbol{\sigma}$, denoted $\partial_{\boldsymbol{\sigma}} \Phi \equiv \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}$, is the set of vectors contained in the cone defined by all linear combinations (with positive coefficients) of $\mathbf{N}_1$ and $\mathbf{N}_2$.

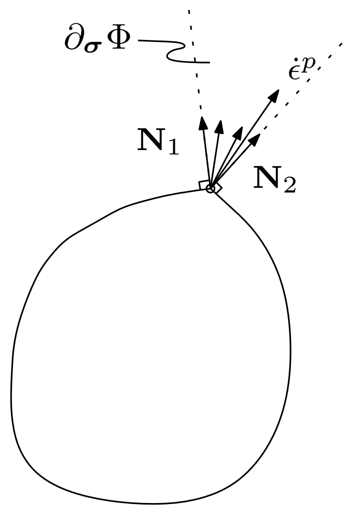

FIG. 4. *Plastic strain increment at corners*

A.4. Hardening law. Hardening is characterised by a dependence of yield stress level upon the history of plastic straining to which the body has been subjected. Hardening is represented by changes in the hardening thermodynamical force, $\mathbf{A}$, during plastic yielding. These changes may, in general, affect the size, shape and orientation of the yield surface, defined by $\Phi(\boldsymbol{\sigma}, \mathbf{A}) = 0$.

A.4.1. Isotropic hardening. A plasticity model is said to be isotropic hardening if the evolution of the yield surface is such that, at any state of hardening, it corresponds to a uniform (isotropic) expansion of the initial yield surface, without translation.

A.5. The elastoplastic tangent operator. The elastoplastic tangent operator in (2.11) for the associated plasticity model is given by

$$\mathbb{D}^{ep} = \mathbb{D} - \frac{1}{H_p + \frac{\partial\Phi}{\partial\boldsymbol{\sigma}} : \mathbb{D}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}}\left(\mathbb{D}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}} \otimes \mathbb{D}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}\right) \tag{A.1}$$

where H_p is the hardening modulus. The reader is refered to [6] for a derivation of the expression for the elastoplastic tangent operator.

Appendix B. Discrete variational statements for the flow subproblem in terms of coupling iteration differences. Before arriving at the discrete variational statement of the flow model, we impose the fixed stress constraint on the strong form of the mass conservation equation (2.1). In lieu of (2.14), we write (2.1) as

$$\frac{\partial}{\partial t}(Cp + \frac{C}{3}\mathbf{B} : \boldsymbol{\sigma} + \phi^p) + \nabla \cdot \mathbf{z} = q$$

$$C\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{z} = q - \frac{C}{3}\mathbf{B} : \frac{\partial \boldsymbol{\sigma}}{\partial t} - \frac{\partial \phi^p}{\partial t} \tag{B.1}$$

Using backward Euler in time, the discrete in time form of (B.1) for the m^{th} coupling iteration in the $(n+1)^{th}$ time step is written as

$$C\frac{1}{\Delta t}(p^{m,n+1} - p^n) + \nabla \cdot \mathbf{z}^{m,n+1}$$
$$= q^{n+1} - \frac{1}{\Delta t}\frac{C}{3}\mathbf{B} : (\boldsymbol{\sigma}^{m,n+1} - \boldsymbol{\sigma}^n) - \frac{1}{\Delta t}(\phi^{p^{m,n+1}} - \phi^{p^n})$$

where Δt is the time step and the source term as well as the terms evaluated at the previous time level n do not depend on the coupling iteration count as they are known quantities. The fixed stress constraint implies that $\boldsymbol{\sigma}^{m,n+1}$ gets replaced by $\boldsymbol{\sigma}^{m-1,n+1}$ i.e. the computation of $p^{m,n+1}$ and $\mathbf{z}^{m,n+1}$ is based on the value of stress updated after the poromechanics solve from the previous coupling iteration $m-1$ at the current time level $n+1$. The modified equation is written as

$$C(p^{m,n+1} - p^n) + \Delta t \nabla \cdot \mathbf{z}^{m,n+1} = \Delta t q^{n+1} - \frac{C}{3}\mathbf{B} : (\boldsymbol{\sigma}^{m,n+1} - \boldsymbol{\sigma}^n) - (\phi^{p^{m,n+1}} - \phi^{p^n})$$

As a result, the discrete weak form of (2.1) is given by

$$C(p_h^{m,n+1} - p_h^n, \theta_h)_\Omega + \Delta t(\nabla \cdot \mathbf{z}_h^{m,n+1}, \theta_h)_\Omega + (\phi^{p^{m,n+1}} - \phi^{p^n}, \theta_h)_\Omega$$
$$= \Delta t(q^{n+1}, \theta_h)_\Omega - \frac{C}{3}(\mathbf{B} : (\boldsymbol{\sigma}^{m-1,n+1} - \boldsymbol{\sigma}^n), \theta_h)_\Omega$$

Replacing m by $m+1$ and subtracting the two equations, we get

$$C(\delta^{(m)} p_h, \theta_h)_\Omega + \Delta t(\nabla \cdot \delta^{(m)} \mathbf{z}_h, \theta_h)_\Omega + (\delta^{(m)} \phi^p, \theta_h)_\Omega = -\frac{C}{3}(\mathbf{B} : \delta^{(m-1)} \boldsymbol{\sigma}, \theta_h)_\Omega$$

The weak form of the Darcy law (2.2) for the m^{th} coupling iteration in the $(n+1)^{th}$ time step is given by

$$(\boldsymbol{\kappa}^{-1}\mathbf{z}^{m,n+1}, \mathbf{v})_\Omega = -(\nabla p^{m,n+1}, \mathbf{v})_\Omega + (\rho_0 \mathbf{g}, \mathbf{v})_\Omega \ \forall \ \mathbf{v} \in \mathbf{V}(\Omega) \tag{B.2}$$

where $\mathbf{V}(\Omega)$ is given by

$$\mathbf{V}(\Omega) \equiv \mathbf{H}(div, \Omega) \cap \{\mathbf{v} : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N^f\}$$

and $\mathbf{H}(div, \Omega)$ is given by

$$\mathbf{H}(div, \Omega) \equiv \{\mathbf{v} : \mathbf{v} \in (L^2(\Omega))^3, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

We use the divergence theorem to evaluate the first term on RHS of (B.2) as follows

$$(\nabla p^{m,n+1}, \mathbf{v})_\Omega = (\nabla, p^{m,n+1}\mathbf{v})_\Omega - (p^{m,n+1}, \nabla \cdot \mathbf{v})_\Omega$$
$$= (p^{m,n+1}, \mathbf{v} \cdot \mathbf{n})_{\partial\Omega} - (p^{m,n+1}, \nabla \cdot \mathbf{v})_\Omega$$
$$= (g, \mathbf{v} \cdot \mathbf{n})_{\Gamma_D^f} - (p^{m,n+1}, \nabla \cdot \mathbf{v})_\Omega \tag{B.3}$$

where we invoke $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_N^f. In lieu of (B.2) and (B.3), we get

$$(\boldsymbol{\kappa}^{-1}\mathbf{z}^{m,n+1}, \mathbf{v})_\Omega = -(g, \mathbf{v} \cdot \mathbf{n})_{\Gamma_D^f} + (p^{m,n+1}, \nabla \cdot \mathbf{v})_\Omega + (\rho_0 \mathbf{g}, \mathbf{v})_\Omega$$

Replacing m by $m+1$ and subtracting the two equations, we get

$$(\boldsymbol{\kappa}^{-1}\delta^{(m)}\mathbf{z}_h, \mathbf{v}_h)_\Omega = (\delta^{(m)} p_h, \nabla \cdot \mathbf{v}_h)_\Omega$$

Appendix C. Discrete variational statement for the poromechanics subproblem in terms of coupling iteration differences. The weak form of the linear momentum balance (2.6) is given by

$$(\nabla \cdot \boldsymbol{\sigma}, \mathbf{q})_\Omega + (\mathbf{f} \cdot \mathbf{q})_\Omega = 0 \qquad (\forall\ \mathbf{q} \in \mathbf{U}(\Omega)) \tag{C.1}$$

where $\mathbf{U}(\Omega)$ is given by

$$\mathbf{U}(\Omega) \equiv \left\{ \mathbf{q} = (u, v, w) : u, v, w \in H^1(\Omega), \mathbf{q} = \mathbf{0} \text{ on } \Gamma_D^p \right\}$$

where $H^m(\Omega)$ is defined, in general, for any integer $m \geq 0$ as

$$H^m(\Omega) \equiv \left\{ w : D^\alpha w \in L^2(\Omega)\ \forall |\alpha| \leq m \right\},$$

where the derivatives are taken in the sense of distributions and given by

$$D^\alpha w = \frac{\partial^{|\alpha|} w}{\partial x_1^{\alpha_1} .. \partial x_n^{\alpha_n}},\ |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

We know from tensor calculus that

$$(\nabla \cdot \boldsymbol{\sigma}, \mathbf{q})_\Omega \equiv (\nabla, \boldsymbol{\sigma}\mathbf{q})_\Omega - (\boldsymbol{\sigma} : \nabla \mathbf{q})_\Omega \tag{C.2}$$

Further, using the divergence theorem and the symmetry of $\boldsymbol{\sigma}$, we arrive at

$$(\nabla, \boldsymbol{\sigma}\mathbf{q})_\Omega \equiv (\mathbf{q}, \boldsymbol{\sigma}\mathbf{n})_{\partial\Omega} \tag{C.3}$$

We decompose $\nabla \mathbf{q}$ into a symmetric part $(\nabla \mathbf{q})_s \equiv \frac{1}{2}\left(\nabla \mathbf{q} + (\nabla \mathbf{q})^T\right) \equiv \boldsymbol{\epsilon}(\mathbf{q})$ and skew-symmetric part $(\nabla \mathbf{q})_{ss}$ and note that the contraction between a symmetric and skew-symmetric tensor is zero to obtain From (C.1), (C.2), (C.3) and (**??**), we get

$$(\boldsymbol{\sigma}\mathbf{n}, \mathbf{q})_{\partial\Omega} - (\boldsymbol{\sigma} : \boldsymbol{\epsilon}(\mathbf{q}))_\Omega + (\mathbf{f}, \mathbf{q})_\Omega = 0$$

which, after invoking the traction boundary condition, results in the discrete weak form for the m^{th} coupling iteration as

$$(\mathbf{t}^{n+1}, \mathbf{q}_h)_{\Gamma_N^p} - (\boldsymbol{\sigma}^{m,n+1} : \boldsymbol{\epsilon}(\mathbf{q}_h))_\Omega + (\mathbf{f}^{n+1}, \mathbf{q}_h)_\Omega = 0$$

Replacing m by $m+1$ and subtracting the two equations, we get

$$(\delta^{(m)}\boldsymbol{\sigma} : \boldsymbol{\epsilon}(\mathbf{q}_h))_\Omega = 0$$

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