Phase-field Modeling of Hydraulic Fracture

Zachary A. Wilson and Chad M. Landis
Department of Aerospace Engineering and Engineering Mechanics
and
The Institute for Computational Engineering and Science
The University of Texas at Austin
210 East 24th Street, C0600
Austin, TX 78712-0235
Phone: (512) 471-4273, Fax: (512) 471-5500
E-mail: landis@utexas.edu

Abstract
In this work a theoretical framework implementing the phase-field approach to fracture is used to couple the physics of flow through porous media and cracks with the mechanics of fracture. The main modeling challenge addressed in this work, which is a challenge for all diffuse crack representations, is on how to allow for the flow of fluid and the action of fluid pressure on the aggregate within the diffuse damage zone of the cracks. The theory is constructed by presenting the general physical balance laws, postulating a kinematic ansatz for an effective porosity, and conducting a consistent thermodynamic analysis to constrain the constitutive relationships. Constitutive equations that reproduce the desired responses at the various limits of the effective porosity are proposed in order to capture Darcy-type flow in the intact porous medium and Stokes-type flow within open cracks. A finite element formulation for the solution of the governing model equations is presented and discussed. Finally, the theoretical and numerical model is shown to compare favorably to several important analytical solutions. More complex and interesting calculations are also presented to illustrate some of the advantageous features of the approach.

Keywords: hydraulic fracture, continuum thermodynamics, finite element methods, nonlinear poroelasticity

1. Introduction
The method of propagating fractures with hydraulically pressurized fluids is common in many engineering applications, particularly in the field of geology. Certainly the most notable of these is the stimulation of oil and gas wells, but other applications include
waste disposal\cite{1}, in situ stress measurement\cite{2}, the stimulation of ground water wells\cite{3,4}, and geothermal reservoir development\cite{5}. The advent of massive hydraulic fracturing and horizontal drilling in recent decades has made the extraction of oil and gas from unconventional reservoirs, particularly shale, economically viable. Additionally, heat extraction from subsurface geothermal systems has the potential to be a significant source of renewable carbon-free energy both world-wide and in the U.S.\cite{6,7}. As such, the optimization of the many applications related to fluid driven fracturing is becoming increasingly important.

Conventional modeling approaches to hydraulically fractured systems have assumed symmetric, planar, bi-wing crack geometries. Analytical 2D models were developed that made use of simplifying geometrical assumptions for fracture height and width in order to arrive at solutions in closed form. One such model, the Perkins-Kern-Nordgren (PKN) model, was formulated for fractures where the crack height is much smaller than crack length. The crack cross-section was assumed to be elliptical and to maintain a constant height along the length of the crack\cite{8,9}. Another popular model, named after the work by Kristianovic, Geertsma, and de Klerk (KGD), was formulated for fractures where the crack height is much larger than crack length. In this case the crack cross section was assumed to have constant width and the pressure within the crack was taken to be constant\cite{10,12}. The coupling of these simplifying geometrical assumptions with fluid mass conservation and the Poiseuille law for the fluid flow in the cracks yielded closed form solutions for the crack growth history. Both models have also been modified to include leak-off of the pressurized fluid into the surrounding porous rock. Later, planar three-dimensional and pseudo-three-dimensional computational models were developed that incorporate anisotropic fracture properties of the rock layers to predict more complex fracture height, length, and width growth profiles. Several commercial production codes are based on these ideas and are in use today\cite{13}. Additional studies, most notably by Detournay and co-workers\cite{14-16}, have revisited planar crack geometries, plane strain and penny shape cracks, and carefully investigated the crack tip behavior and propagation regimes of hydraulic fractures. These asymptotic analyses and analytical solutions provide valuable insights into some of the fundamental behaviors of fluid driven fractures and supply test cases for comparison with numerical methods developed for more complex crack surface evolutions.

Recently research efforts have largely shifted towards the modeling of more complex fracture geometries. Microseismic mappings performed on hydraulically fractured wells have revealed that complex fracture growth is more prevalent than had been initially estimated and that such phenomena occur with increased frequency in unconventional
reservoirs\textsuperscript{[17,18]}. In particular, induced hydraulic fractures interact with pre-existing propped and natural fractures as well as other anisotropic features and inhomogeneities in the material\textsuperscript{[19,20]}. Due to the low permeability of unconventional reservoirs, a proper understanding of fracture evolution is paramount to evaluating and ensuring the desired production of the well. Many of the same issues exist for enhanced geothermal reservoir development along with the additional complications associated with thermal effects and the generation of thermal cracks. Ultimately, the production of energy resources from these types of reservoirs can be improved with the development and utilization of high fidelity numerical tools that model the coupled fluid, thermal, and mechanical behaviors properly while allowing for complex crack growth topologies and interactions.

There exists several works on the numerical modeling of the fluid driven fracture process. Many of these approaches assume that the network of potential crack paths is known, e.g. a rectangular lattice of potential crack surfaces exists a priori, and the fluid loading then acts to open parts of said network as the fluid injection process proceeds\textsuperscript{[21-27]}. The method developed in this work seeks to address cases where the crack path is not known a priori. Other approaches for generic crack path evolution also exist, and each is thought to have certain strengths and weaknesses. Overall, the methods can be grouped into two types of approaches: (i) sharp interface methods where the cracks are represented explicitly and thus require constitutive rules for crack propagation and (ii) diffuse crack models where a significant challenge resides in representing physical behaviors within the cracks like the pressure and flow of fluids.

Of note for sharp interface methods are boundary element methods\textsuperscript{[28]} and discrete crack representations within finite element methods\textsuperscript{[29-35]}. Boundary element methods for determining the evolution of complex crack patterns have the advantage of being highly computationally efficient. Meshes are only needed on the boundaries of the domain and on the crack surfaces. Growth laws for the crack front are postulated relating the front velocity and direction to the local stress intensity factors. The disadvantages of the method are associated with difficulties for the incorporation of nonlinear or time dependent material responses in the bulk elastic material, and also in the handling of topological changes in the crack growth evolution like crack merging or branching. The discrete crack finite element methods usually implement numerical enrichment functions to allow cracks to grow through elements\textsuperscript{[32,33]} or they relocate nodal positions such that cracks grow along element boundaries\textsuperscript{[34]}. Both approaches also implement special element types at the crack front in order to capture the linear elastic near-tip K-fields for elastic problems. Such finite element methods will of course allow for nonlinear and time-dependent behavior in the bulk material, although any nonlinear behavior affecting
the crack tip fields may negate the utility of using the special crack tip elements and any crack growth laws based upon the stress intensities. While the topological changes associated with crack merging and branching can be addressed using level set methods and remeshing, the necessity of additional constitutive rules for crack branching, direction selection, and nucleation remain.

Examples of diffuse crack approaches include peridynamics\cite{36,37}, gradient damage mechanics\cite{38,39}, and the phase-field methods presented herein and by other recent studies\cite{40-52}. Peridynamics is a nonlocal theory in which material points interact with all other material points that reside within a nearby “horizon”. As with the phase-field method, cracks are not tracked explicitly but rather emerge naturally from the model equations. Gradient damage mechanics approaches and phase-field fracture approaches are very similar with minor differences in the details of how localization occurs and the length scale over which damage is able to spread. All of these diffuse crack methods alleviate many of the issues related to the topological changes in crack path evolutions associated with branching, turning, merging, and nucleation. However, there are challenges associated with modeling the fluid flow through cracks and the interaction of pressurized fluids with crack faces.

The phase-field methodology has emerged as a powerful tool for modeling the evolution of microstructures and the interactions of defects in a wide range of materials and physical processes\cite{53-57}. Phase-field methods can be implemented to simulate large-scale evolution of material microstructure and defect motion without the need to explicitly track interfaces or defects. One strength of this modeling approach lies in the fact that there are no additional constitutive rules required within the theory that dictate when a crack should nucleate, grow, change direction, or split into multiple cracks. Cracks and their growth emerge as solutions to the governing partial differential equations of the model. A particularly unique and striking feature of the approach is that all calculations are performed entirely on the initial, undeformed configuration. There is no need to disconnect, eliminate, or move elements or introduce additional discontinuous basis functions, as is commonly done in the discrete crack computational fracture mechanics approaches. This results in a significant simplification of the numerical implementation, and a simple and direct pathway from two-dimensional to three-dimensional applications.

The purpose of this work is to enhance the phase-field/variational approach to fracture by coupling the physics of flow through porous media and cracks with the mechanics of fracture. The main modeling challenge addressed in this work, which is a challenge for
all diffuse crack representations, is on how to allow for the flow of fluid and the action of fluid pressure on the aggregate within the diffuse damage zone of the cracks. There have been a few very recent studies which address the implementation of the phase-field method for fracture in poroelastic media\textsuperscript{[58-60]}. The primary distinction in this work is in the presentation of the governing balance laws, and then the development of constitutive relationships through Coleman and Noll-type analysis procedures and, in particular, the handling of the fluid flow equations, which recover the Stokes equations for the fluid flow within cracks. We begin by developing the general balance laws and conducting a thorough thermodynamic analysis. We then propose constitutive equations that reproduce the desired responses at the various limits of the phase-field variable. In Section 3, we summarize our model equations and outline the numerical methods used to solve the equations. Finally, in Section 4, the approach developed is shown to compare favorably to several important analytical solutions. More complex and interesting calculations are also presented to illustrate some of the advantageous features of the approach.

2. General Theory

The coupled nature of the problem, with different fluid flow regimes occurring through the intact porous solid and through cracks, and the evolution of a phase-field parameter to identify cracks, will be developed and outlined here. Throughout the presentation of the general theory the governing equations will be presented in the reference configuration. The relationships between the current and reference configurations are linked through the deformation gradient. The components of the deformation gradient, $F_{ij}$, are given by,

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad (2.1)$$

where $x_i$ are the Cartesian coordinates of a material point of the aggregate in the current state defined by the spatial domain $V$ bounded by the surface $S$, and $X_j$ are Cartesian coordinates of the same material point in the reference state corresponding to the reference domain $V_0$ bounded by the surface $S_0$. Index notation will be used throughout the presentation with summation assumed over repeated indices, lowercase indices referring to the current configuration, and uppercase indices referring to the reference configuration. The relationship between surface elements in the reference and current configurations is given by Nanson’s formula as $n_i dS = JF_{ij}^{-1}N_j dS_0$, and for volume elements is $dV = JdV_0$, where $n_i$ and $N_j$ are components of the unit normal to
the surface in the current and reference configurations respectively, \( J \) is the determinant of the deformation gradient, and \( F_{\beta}^{-1} \) is the inverse of the deformation gradient such that \( F_{\beta}^{-1} = \partial X_j / \partial x_i \).

### 2.1 Mass Balances

First consider the conservation of mass for a fractured, fluid saturated, porous solid. Following the work of Coussy\cite{61-65} the porous solid and the fluid will together be referred to as the aggregate. In order to develop the theory within the phase-field framework we identify both the reference domain of the aggregate that is not cracked \( V_0 - V_c \), as well as the reference domain of the cracks \( V_c \), which are assumed to be filled completely with fluid. The total reference domain \( V_0 \) contains both cracks and pores and is unchanging while fluid may be convected into and out of this domain. In contrast, the reference crack domain \( V_c \) may evolve over time.

First, the total solid mass of the system remains fixed for all time, and as such, conservation of mass for the solid can be written as,

\[
\frac{D}{Dt} \int_{V_0} (1 - \phi) \rho^s J \, dV_0 = 0, \tag{2.2}
\]

where \( \phi \) is the reference volume fraction of porosity in the aggregate, which is assumed to always be filled by fluid, and \( \rho^s \) is the mass density of the solid. The notation \( D / Dt \) represents the time derivative with respect to an observer attached to a material point of the aggregate. Given that this equation must hold for any arbitrary region of the reference domain, the integrand must be zero pointwise, i.e.

\[
\frac{D}{Dt} \left [(1 - \phi) \rho^s J \right ] \equiv [\mathcal{D}(1 - \phi) \rho^s J] = 0. \tag{2.3}
\]

Equation (2.3) is the mass balance equation for the solid.

The total conservation of mass for the system is given by,

\[
\frac{D}{Dt} \int_{(V_0 - V_c)} (1 - \phi) \rho^s + \phi \rho^f \, dV_0 + \frac{D}{Dt} \int_{V_c} \rho^f J \, dV_c = -\int_{S_0} \rho^f \mathbf{\tilde{w}}_i N_j \, dS_0 + \int_{V_0} \rho^f \mathbf{\tilde{m}} \, dV_0, \tag{2.4}
\]
where $\rho'$ is the mass density of the fluid, $\tilde{w}_I$ is the volume of fluid traversing a unit of reference area of the aggregate per unit of time, and $\tilde{m}$ is the fluid volume injected per unit of reference volume. Note that the nominal fluid flux $\tilde{w}_I$ is related to the true fluid flux $w_i$ as $w_i = \tilde{w}_I F_{ik} / J$. The first term of Eq. (2.4) represents the rate of change of mass contained in regions of the aggregate not containing cracks and the second term is the rate of change of mass contained within cracks in the aggregate. The right-hand side of the equation represents the mass that enters the system through its surface and volume respectively. Using the fact that the solid mass density in the cracks is zero, Equation (2.4) can be re-written as

$$\frac{D}{Dt} \int_{V_0} \left[ (1 - \phi) \rho^* + \phi \rho' \right] J \ dV_0 + \frac{D}{Dt} \int_{V_0} (1 - \phi) \rho' J \ dV_0 = -\int_{S_0} \rho' \tilde{w}_I N_I \ dS_0 + \int_{V_0} \rho' \tilde{m} dV_0 \tag{2.5}$$

Combining (2.2) and (2.5), the conservation of fluid mass of the system is,

$$\frac{D}{Dt} \int_{V_0} \phi \rho' J dV_0 + \frac{D}{Dt} \int_{V_0} (1 - \phi) \rho' J dV_0 = -\int_{S_0} \rho' \tilde{w}_I N_I dS_0 + \int_{V_0} \rho' \tilde{m} dV_0 . \tag{2.6}$$

Equation (2.6) can be manipulated as,

$$\frac{D}{Dt} \int_{V_0} \phi \rho' J dV_0 + \frac{D}{Dt} \int_{V_0} (1 - \phi) \rho' dV_0 + \frac{D}{Dt} \int_{V_0} (1 - \phi) \rho' (J - 1) dV_0 = -\int_{S_0} \rho' \tilde{w}_I N_I dS_0 + \int_{V_0} \rho' \tilde{m} dV_0 . \tag{2.7}$$

This decomposition of the second term is motivated as a separation of the fluid mass changes in closed cracks (which will be argued to vanish) from the fluid mass changes within deformed cracks. Furthermore, the deformation, $(J - 1)$, may be decomposed into two terms, one describing volume contraction and one describing volume expansion,

$$J - 1 = -\langle 1 - J \rangle + \langle J - 1 \rangle , \text{ where } \langle a \rangle = \begin{cases} a , & a \geq 0 \\ 0 , & a < 0 . \end{cases} \tag{2.8}$$

The reference configuration of the cracks $V_c$ is defined such that all cracks are closed in the reference state. If cracks are assumed to be closed in the reference state then the fluid mass in a closed crack must be zero, such that,

$$\int_{V_c} \rho' \ dV_0 = 0 , \tag{2.9}$$
for any arrangement of the reference crack domain $V_c$.

Taking into account the fact that the volume fraction $\phi$ may only take on values between zero and one and noting that $J > 0$, (2.9) implies,

$$
\int_{V_c} (1 - \phi) \rho' dV_0 = 0 \quad \text{and} \quad \int_{V_c} - (1 - \phi) J dV_0 = 0.
$$

Note that the reason that the integral $\int_{V_c} (1 - \phi) J dV_0$ is not necessarily zero is due to the fact that within open cracks the Jacobian of the deformation is singular, and hence this integral cannot be argued to vanish. Thus, the remaining terms in (2.7) are,

$$
\frac{D}{Dt} \int_{V_0} \phi \rho' J dV_0 + \frac{D}{Dt} \int_{V_c} (1 - \phi) \rho' \langle J - 1 \rangle dV_0 = - \int_{S_0} \rho' \tilde{w}_i N_i dS_0 + \int_{V_0} \rho' \tilde{m} dV_0.
$$

Here we have isolated the open cracks. Additional motivation for these steps will be realized when the phase-field approximation for the crack domain is introduced. The phase-field theory for fracture is designed in such a way that the phase-field parameter undergoes a smooth transition from a value signifying damage and fracture to a value indicative of undamaged intact material. Consequently, a length scale is introduced over which this transition takes place. Take $\mu$ to be the phase-field parameter that describes cracks such that $\mu = 0$ indicates a crack and $\mu = 1$ indicates uncracked aggregate. We now introduce a phase-field approximation for the crack domain that allows us to analyze the crack volume integral in an approximate manner. Specifically,

$$
\int_{V_c} a dV_0 \approx \int_{V_0} g_m(\mu) a dV_0,
$$

where $a$ is any arbitrary field. The form of the function $g_m(\mu)$ in (2.12) is chosen such that,

$$
g_m(0) = 1, \quad g_m(1) = 0, \quad \text{and} \quad 0 \leq g_m(\mu) \leq 1.
$$

Note that if there exists any point in $V_0$ such that $\mu < 1$ then for finite fluid mass density,
\[
\int_{V_0} g_m(\mu) \rho' dV_0 > 0. \quad (2.14)
\]

Hence, regardless of the specific form for the field \( \mu \), (2.14) is in contradiction with (2.9). For this reason, the manipulations outlined above are preformed prior to the application of (2.12), as they act to mitigate the inaccuracies introduced by this approximation. A simple \( C^1 \) smooth spline function that satisfies the requirements of (2.13) is,

\[
g_m(\mu) \equiv \begin{cases} 
1 & ; \mu < 0 \\
1 - 3\mu^2 + 2\mu^3 & ; 0 \leq \mu \leq 1 \\
0 & ; \mu > 1
\end{cases} \quad (2.15)
\]

and will be adopted in this work. Of course, this specific form of \( g_m(\mu) \) is only one of any number of functions that would satisfy (2.13) and could be used in any given realization of the theory.

Application of (2.12) to the conservation of fluid mass equation then yields,

\[
\frac{D}{Dt} \int_{V_0} \left[ \phi \rho' J + g_m(\mu) \left( 1 - \phi \right) \rho' \left( J - 1 \right) \right] dV_0 = -\int_{S_0} \rho' \tilde{w}_I N_I dS_0 + \int_{V_0} \rho' \tilde{m} dV_0. \quad (2.16)
\]

Applying the divergence theorem, and taking (2.16) to be valid for any arbitrary volume, the point-wise form of the fluid mass balance becomes,

\[
\left[ \phi \rho' J + g_m(\mu) \left( 1 - \phi \right) \rho' \left( J - 1 \right) \right] = -\left( \rho' \tilde{w}_I \right)_I + \rho' \tilde{m} . \quad (2.17)
\]

It is convenient for the ensuing developments to introduce the effective porosity \( \tilde{\phi} \) as,

\[
\tilde{\phi} \equiv \phi + g_m(\mu) \left( 1 - \phi \right) \left( J - 1 \right) \left( \frac{1}{J} \right), \quad (2.18)
\]

and we may restate (2.17) with a more conventional appearance as,

\[
\left( \tilde{\phi} \rho' J \right) = -\left( \rho' \tilde{w}_I \right)_I + \rho' \tilde{m} . \quad (2.19)
\]
2.2 Momentum Balances

Anticipating how the equations of this theory will eventually be solved numerically, the focus of the momentum balance equations will be placed on the aggregate and then on the fluid. For the aggregate, the balance of linear momentum is written as,

\[
\frac{D}{Dt} \int_{V_0 - V_c} \left( (1 - \phi) \rho^s v_i + \phi \rho^f \left( v_i + \frac{v_i^{fs}}{s} \right) \right) J dV_0 + \frac{D}{Dt} \int_{V_c} \rho^f \left( v_i + \frac{v_i^{fs}}{s} \right) J dV_0
\]

\[
= \int_{S_0} \bar{t}_i dS_0 + \int_{V_0} \bar{b}_i dV_0 - \int_{S_0} \rho^f \bar{w}_i K N_k \left( v_i + \frac{v_i^{fs}}{s} \right) dS_0 + \int_{V_0} \rho^f \bar{m} \left( v_i + \frac{v_i^{fs}}{s} \right) dV_0.
\]  

(2.20)

The components of the traction per unit reference area and body force per reference volume are \( \bar{t}_i \) and \( \bar{b}_i \). Note that the tractions and body forces appearing here are external to the solid/fluid aggregate. Additionally, we adopt the following ansatz that the average fluid velocity relative to the solid aggregate, \( v_i^{fs} \), is related to the true fluid flux by \( w_i = \phi v_i^{fs} \). Notice that in undamaged regions away from cracks the relative velocity is simply \( v_i^{fs} = w_i / \phi \) and in fully cracked regions with large deformations, the relative velocity is \( v_i^{fs} \approx w_i \). The left-hand side of (2.20) represents the rate of change of the total momentum of the solid/fluid aggregate contained within the domain \( V_0 \). The right-hand side of the equation consists of two parts; the sum of the external forces acting on the volume, and the fluid momentum convected and injected into the volume. Note that it is assumed that the fluid injected by external sources is injected into the volume with the same velocity as the existing fluid. Next, the standard hypothesis is adopted such that the first Piola-Kirchhoff stress components, \( P_{Ji} \), are related to the nominal surface traction by,

\[
\bar{t}_i = P_{Ji} N_j \quad \text{on} \quad S_0
\]  

(2.21)

The Cauchy stress is related to the first Piola-Kirchhoff stress in the normal manner as,

\[
\sigma_{ji} = F_{jk} P_{ki} / J.
\]

Recall that the reference crack configuration is designed such that all cracks are closed, and so, the fluid momentum contained within \( V_c \) should be zero, i.e.,
\[ \int V_i \rho^f (v_i + v_i^{f/s}) dV_0 = 0. \] (2.22)

Application of the manipulations described in the previous section along with the phase-field approximation (2.12), the traction-stress relation (2.21), the divergence theorem, the mass balance equations (2.3) & (2.17), and noting that the momentum balance must hold for arbitrary volumes leads to the pointwise form for Newton’s second law for the aggregate,

\[ P_{K_i}^1 + \dot{b}_i = (1 - \phi) \rho^s J \dot{v}_i + \phi \rho^f J (v_i + v_i^{f/s}) + \phi \rho^f J (v_i + v_i^{f/s}) \cdot F_{K_i}^{-1} v_i^{f/s} \] (2.23)

The time derivative, \( \dot{(\cdot)} \) for an observer moving with a fluid particle is given by,

\[ (a) = \dot{a} + a_{\phi} v_i^{f/s} = \dot{a} + a_{\phi} F_{K_i}^{-1} v_i^{f/s}. \] (2.24)

Therefore, (2.23) can be re-written as,

\[ P_{K_i}^1 + \dot{b}_i = (1 - \phi) \rho^s J \dot{v}_i + \phi \rho^f J (v_i + v_i^{f/s}). \] (2.25)

Next, the balance of angular momentum for the aggregate is given by,

\[ \frac{D}{Dt} \int_{V_i} J dV_0 + \int_{S_0} \epsilon_{ijk} x_j \dot{b}_i dS_0 = \int_{S_0} \epsilon_{ijk} x_j \dot{b}_i dS_0 + \int_{V_i} \rho^f \dot{w}^0 N_k \epsilon_{ijk} x_j (v_i + v_i^{f/s}) dV_0 + \int_{V_i} \rho^f \dot{w}^0 N_k \epsilon_{ijk} x_j (v_i + v_i^{f/s}) dV_0 \] (2.26)

Here \( \epsilon_{ijk} \) are the components of the permutation tensor, and \( x_i \) are the components of the position of a particle in the current configuration of the aggregate. Application of the previously established procedures yields the pointwise consequence of the angular momentum balance as,

\[ \epsilon_{ijk} (P_{L_i} F_{i\ell} - J \rho^f w_j v_k - J \rho^f w_k v_j) = 0 \quad \rightarrow \quad P_{K_i}^1 F_{K,jk} = P_{K_i}^1 F_{K,jk}. \] (2.27)

The implication of (2.27) is that the aggregate Cauchy stress is symmetric.

For the fluid, the linear momentum balance is,
\[
\frac{D}{Dt} \int_{V(t)} \phi \rho f \left( v_i + v_i^{f/s} \right) J dV_0 + \frac{D}{Dt} \int_{V(t)} \rho f \left( v_i + v_i^{f/s} \right) J dV_0 = \int_{S_0} \phi \left( \dot{v}_i \right) dS_0 + \int_{V_0} \rho f \left( \dot{v}_i + \ddot{v}_i \right) dV_0 - \int_{S_0} \rho f \left( \dot{w}_i N_k \right) \left( v_i + v_i^{f/s} \right) dS_0 + \int_{V_0} \rho f \left( \ddot{m} \right) \left( v_i + v_i^{f/s} \right) dV_0.
\] 

(2.28)

Here, the components of the net traction on the fluid per unit of aggregate reference area are \( \ddot{v}_i \), the components of the fluid body force per unit of aggregate reference volume supplied by external sources are \( \ddot{b}_i \), and the components of the body force per unit aggregate reference volume that the solid places upon the fluid are \( \ddot{f}_i^{sf} \). Here we adopt the ansatz that the fluid traction \( \ddot{v}_i \) is related to the fluid stress by,

\[
\ddot{v}_i = \bar{\sigma} T_{ki} N_k = \bar{\sigma} \left( \tau_{ji} - p \delta_{ij} \right) J F_{kj}^{-1} N_k,
\] 

(2.29)

where the components of the first Piola-Kirchhoff stress in the fluid are \( T_{ji} \), and the Cauchy fluid stress components have been decomposed into the hydrostatic pressure \( p \) and the deviatoric part \( \tau_{ij} \). Here we have introduced the Kronecker delta components \( \delta_{ij} \). Again, notice that fluid momentum is convected and injected into the volume of the aggregate being considered. Analysis of (2.28) using the previously discussed procedures along with (2.12), (2.24), and (2.29) gives the pointwise form for Newton’s second law for the fluid,

\[
\left( \bar{\sigma} T_{ki} \right)_j + \ddot{b}_i + \ddot{f}_i^{sf} = \bar{\sigma} \rho f \left( v_i + v_i^{f/s} \right)
\] 

(2.30)

The balance of angular momentum for the fluid is,

\[
\frac{D}{Dt} \int_{V(t)} \epsilon_{ijk} \phi \rho f \left( v_i + v_i^{f/s} \right) J dV_0 + \frac{D}{Dt} \int_{V(t)} \epsilon_{ijk} \rho f \left( v_i + v_i^{f/s} \right) J dV_0 = \int_{S_0} \phi \left( \dot{\omega}_k \right) \left( \dot{v}_i \right) dS_0 + \int_{V_0} \rho f \left( \dot{\omega}_k \right) \left( v_k + v_k^{f/s} \right) dV_0 - \int_{S_0} \rho f \left( \dot{\omega}_k \right) \left( \dot{w}_i N_k \right) \left( v_i + v_i^{f/s} \right) dS_0 + \int_{V_0} \rho f \left( \dot{\omega}_k \right) \left( \ddot{m} \right) \left( v_i + v_i^{f/s} \right) dV_0
\] 

(2.31)

The analysis of this integral form provides the pointwise form of angular fluid momentum,

\[
\epsilon_{ijk} \left( \bar{\sigma} T_{kj} F_{li} - J \rho f \left( v_i + v_i^{f/s} \right) v_j \right) = 0 \quad \rightarrow \quad T_{Li} F_{jl} = T_{Lj} F_{li}.
\] 

(2.32)
We note that (2.25) and (2.30), along with the relationship for the fluid force on the solid per unit aggregate volume, $\tilde{f}_i^{bf} = -\tilde{f}_i^{sf}$, can be used to derive Newton’s second law for the solid as,

$$\left(P_{ji} - \tilde{\phi} T_{ji}\right)_j + \tilde{b}_i^j - \tilde{b}_i^j - \tilde{f}_i^{sf} = \left(1 - \tilde{\phi}\right) J \rho^s \dot{v}_i. \quad (2.33)$$

Here we have introduced the definition of the external body force acting on the solid per unit aggregate volume $\tilde{b}_i^s$.

### 2.3 Micro-forces and their Balance

Within this theory the material is allowed to fail or damage according to the phase-field theory of fracture for brittle solids. Using the phase-field damage parameter, following the approach of Fried, Gurtin, and others\[66-68\], we need also to introduce a set of conjugate forces associated with changes in this quantity. To this end, we define $\tilde{\iota}$ as an external surface micro-force such that $\tilde{\iota} \mu$ is the power expended per unit area of aggregate surface by sources external to the volume under consideration, $\tilde{\gamma}$ is an external body micro-force such that $\tilde{\gamma} \mu$ is the power expended per unit aggregate volume by external sources, and $\tilde{\pi}$ is an internal micro-force per unit aggregate volume such that $\tilde{\pi} \mu$ is the power expended internally on the material per unit volume\[69\]. We also assume that on the surface there is a balance between the applied surface micro-force $\tilde{\iota}$ and the material micro-force vector $\tilde{\xi}_i$ such that,

$$\tilde{\iota} = \tilde{\xi}_j N_j \text{ on } S_0. \quad (2.34)$$

It is then also assumed that there exists a net balance of this set of micro-forces such that,

$$\int_{S_0} \tilde{\iota} dS_0 + \int_{V_0} \tilde{\gamma} dV_0 + \int_{V_0} \tilde{\pi} dV_0 = 0. \quad (2.35)$$

Application of the divergence theorem, (2.34), and the argument that the micro-force balance must hold for any arbitrary volume yields the pointwise micro-force balance equation,

$$\tilde{\xi}_{ij,j} + \tilde{\gamma} + \tilde{\pi} = 0. \quad (2.36)$$
2.4 Energy Balances

Next, consider the balance of energy in the solid/fluid aggregate. The integral form for the first law of thermodynamics for a given aggregate volume is written as,

\[ \frac{D}{Dt} \int_{(V_0-V)} \left[ \frac{1}{2} (1-\phi) \rho' v_i v_i + (1-\phi) \rho' e' \right] J dV_0 \]

\[ + \frac{D}{Dt} \int_{(V_0-V)} \left[ \frac{1}{2} \phi \rho' \left( v_i + v_i^{fs} \right) \left( v_i + v_i^{fs} \right) + \frac{1}{2} \phi \rho' A v_i^{fs} v_i^{fs} + \phi \rho' e' \right] J dV_0 \]

\[ + \frac{D}{Dt} \int_{V} \left[ \frac{1}{2} \rho' \left( v_i + v_i^{fs} \right) \left( v_i + v_i^{fs} \right) + \frac{1}{2} \rho' A v_i^{fs} v_i^{fs} + \rho' e' \right] J dV_0 \]

\[ = \int_{S_0} t_i v_i dS_0 + \int_{V_0} \left[ b_i + b_i^{fs} \right] dV_0 + \int_{S_0} \tilde{t}_i v_i^{fs} dS_0 + \int_{V_0} \tilde{m} \rho' e' dV_0 \]

\[ + \int_{S_0} \tilde{t}_i v_i dS_0 + \int_{V_0} \tilde{t}_i v_i^{fs} dV_0 - \int_{S_0} \tilde{q}_i N_j dS_0 + \int_{V_0} \tilde{r} dV_0 \]

\[ - \int_{S_0} \rho' \tilde{w}_K N_k \left[ \frac{1}{2} \left( v_i + v_i^{fs} \right) \left( v_i + v_i^{fs} \right) + \frac{1}{2} A v_i^{fs} v_i^{fs} + e' \right] dS_0 \]

\[ + \int_{V_0} \rho' \tilde{m} \left[ \frac{1}{2} \left( v_i + v_i^{fs} \right) \left( v_i + v_i^{fs} \right) + \frac{1}{2} A v_i^{fs} v_i^{fs} + e' \right] dV_0 \]

(2.37)

Again, notice that the definitions of the body forces are in terms of the forces per unit aggregate reference volume such that the total body force \( \tilde{b}_i \) is simply the sum of the external body force per unit aggregate volume on the fluid \( \tilde{b}_i^{fs} \) and the external body force per unit aggregate volume on the solid \( \tilde{b}_i^{ss} \). Also appearing in the energy balances are the volumetric sources of heat \( \tilde{r} \) supplied per unit reference volume of the aggregate per unit of time. Note that the total heat supply to the aggregate \( \tilde{r} \) includes heat supplied both to the solid and to the fluid, which is partitioned as \( \tilde{r} = \tilde{r}^s + \tilde{r}^{fs} \). Then, \( \tilde{q}_i \) are the components of the net heat flux traversing a unit reference area of the aggregate per unit of time. Likewise, the total heat flux \( \tilde{q}_i \) represents contributions from the solid and the fluid such that it can be partitioned into contributions per unit aggregate reference area as \( \tilde{q}_i = \tilde{q}_i^s + \tilde{q}_i^{fs} \). The additional freshly introduced quantities are the internal energy per unit mass of solid \( e^s \), the internal energy per unit mass of fluid \( e^{fs} \), and the tortuosity \( A \).

The left-hand side of this energy balance equation contains the time rate of change of the kinetic and internal energy contained within a region of the solid/fluid aggregate, and within cracks. Aside from the term containing the tortuosity, each of these terms is
conventional, with the kinetic and internal energies of the solid and fluid partitioned through the porosity. The additional term \( \frac{1}{2} \phi \rho f A v_i^{f/s} v_i^{f/s} \) accounts for the kinetic energy of the fluid per unit aggregate volume that is omitted by the term \( \frac{1}{2} \phi \rho f \left( v_i + v_i^{f/s} \right) \left( v_i + v_i^{f/s} \right) \). Since \( \left( v_i + v_i^{f/s} \right) \) is the average velocity of the fluid, it is valid to claim that the integral of the average velocity is equivalent to the integral of the velocity, however such a relationship does not hold for the square of the velocity and the square of the average velocity. Hence, it is common to introduce the tortuosity to account for this discrepancy. Note that the form for the kinetic energy correction \( \frac{1}{2} \phi \rho f A v_i^{f/s} v_i^{f/s} \) assumes that the fluctuation of the total fluid velocity about its average value is proportional to the average relative velocity of the fluid with respect to the solid, \( v_i^{f/s} \). In general, the tortuosity \( A \) will depend upon the specific structure of the porosity, the porosity itself, and the relative fluid velocity \( v_i^{f/s} \)\[61\].

The first line of the right-hand side of this energy balance equation includes the power expended by the total traction on the aggregate surface, the power expended by the external body forces on the solid and fluid accounted for separately, the power expended by the fluid stress on the relative velocity at the surface (the power expended by the fluid stress on the aggregate velocity \( v_i \) is included in the work done by the total traction), and the power expended by external sources to inject additional fluid volume into the aggregate. The second line of the right-hand side includes the power expended by external micro-forces on the surface and in the volume of the aggregate, and the rate of heat supplied to the aggregate volume across its surface and to the volume directly. Note that since it is an internal force, the internal micro-force \( \tilde{F} \) does not enter the energy balance equation. This is analogous to the fact that the internal force that the solid places upon the fluid, \( \tilde{F}_{i}^{sf} \), does not enter this energy balance equation either.

Finally, the last two lines of the right-hand side represent the energy convected into the volume through its surface, and injected into the volume directly. Note that these terms contain both kinetic and stored energy contributions from the fluid, and it is assumed that the fluid injected directly into the volume arrives with the prevailing kinetic and internal energy of the existing fluid.

Like the mass and momentum, the energy in the closed crack configuration is taken to be zero, or,

\[
\int_{V_0} \left[ \frac{1}{2} \rho f \left( v_i + v_i^{f/s} \right) \left( v_i + v_i^{f/s} \right) + \frac{1}{2} \rho f A v_i^{f/s} v_i^{f/s} + \rho f e_f \right] dV_0 = 0 .
\] (2.38)
Then, the analysis of the energy balance equation requires the use of the mass balances (2.3) and (2.17), the crack volume approximation (2.12), the momentum balances (2.25) and (2.30), and the micro-force balance (2.36), leading to the following pointwise form for the first law for the solid/fluid aggregate,

\[
(1-\phi)J\rho' v' + \bar{\phi} J\rho' e' + \gamma J\rho' (\frac{1}{2} A t^I + t^I) = \left[ P_h - \bar{\phi} T_h + g_n(\mu)(J - \phi) \right](J - 1)^{-1} F_h^{-1} p + \bar{\phi} T_h (v_i + v^{I})
\] 

(2.39)

Here we have introduced the first Piola-Kirchhoff stress in the fluid associated with the deviatoric part of the fluid stress \( \tau_{\mu} \), \( \bar{\tau}_{\mu} = JF_{\mu}^{-1} \tau_{\mu} \). In addition, we have introduced the Heaviside side function,

\[
\langle a \rangle^0 = \begin{cases} 
1, & a \geq 0 \\
0, & a < 0 
\end{cases}
\] 

(2.40)

It is also useful to analyze the energy balances for the solid and fluid separately, noting that the sum of the solid and fluid energies must equate to the solid/fluid aggregate energy. For the solid, the first law energy balance is,

\[
\frac{D}{Dt} \int_{V_0} [\frac{1}{2} (1-\phi) \rho^s v_i v_i + (1-\phi) \rho^s e^s] J dV_0 = \int_{S_0} (\bar{\mu}_s - \bar{\mu}^f) v_i dS_0 + \int_{V_0} \bar{\mu}_s v_i dV_0 \\
+ \int_{S_0} \bar{\mu}_s dS_0 + \int_{V_0} \bar{\mu}_s dV_0 - \int_{S_0} \bar{\gamma}_s N_j dS_0 + \int_{V_0} \bar{\gamma}_s dV_0 + \int_{V_0} \bar{\nu}_s dV_0 + \int_{V_0} \bar{\omega}_s dV_0
\] 

(2.41)

Note, that as for the body forces, the surface traction can be decomposed into its solid and fluid contributions per unit aggregate reference area. In (2.41) the traction associated with the solid is \( \bar{\tau}_s = \bar{\tau}_s - \bar{\tau}^f_i \). The mechanical power density that the fluid transfers to the solid per unit of time is \( \bar{\omega}_s^b \), and the thermal power density that the fluid transfers to the solid per unit of time is \( r_i^b \). The mechanical power per reference volume that the solid transfers to the fluid, \( \bar{\omega}^f \), the force that the solid imparts upon the fluid, \( \bar{\tau}^f \), and the heat that the solid transfers to the fluid, \( \bar{r}^f \), are each the opposites of those from the fluid to the solid. The mechanical power density that the fluid transfers to the solid, \( \bar{\omega}^b \), consists of the dot product of the force density of the fluid on the solid with the solid velocity and the fluid pressure working to open up additional effective porosity. Hence, the power term \( \bar{\omega}^b \), is given by,
\[
\bar{\omega}^h = \tilde{f}_i^h v_i + p J \phi .
\] (2.42)

The pointwise form for the first law for the solid obtained from the analysis of (2.41), (2.3), (2.33), (2.36), and (2.42) is,

\[
(1 - \phi) J \rho^s \epsilon^s = \left[ P_{hi} - \bar{\phi} \bar{T}_{hi} + g_m(\mu)(J - \phi) \left( J - 1 \right)^0 F_{hi}^{-1} p \right] v_{i,j} + \left[ 1 - g_m(\mu) \frac{1}{J} \left( J - 1 \right) \right] p(\phi J) + \tilde{\xi}_i \mu J - \tilde{q}_i^s + \tilde{r}^s - \tilde{r}^g + \left[ g_m'(\mu)(J - 1) p - \bar{\pi} \right] \bar{\mu} .
\] (2.43)

Adopting the following stress partition, the solid stress \( P^s_{hi} \) is defined as,

\[
(1 - \phi) P^s_{hi} = P_{hi} - \bar{\phi} T_{hi} ,
\] (2.44)

and (2.43) can be rewritten as,

\[
(1 - \phi) J \rho^s \epsilon^s = \left[ (1 - \phi) P_{hi}^s - \bar{\phi} J F_{hi}^{-1} p + g_m(\mu)(J - \phi) \left( J - 1 \right)^0 F_{hi}^{-1} p \right] v_{i,j} + \left[ 1 - g_m(\mu) \frac{1}{J} \left( J - 1 \right) \right] p(\phi J) + \left[ g_m'(\mu)(1 - \phi) \langle J - 1 \rangle p - \bar{\pi} \right] \bar{\mu} + \tilde{\xi}_i \bar{\mu}_J - \tilde{q}_i^s + \tilde{r}^s - \tilde{r}^g .
\] (2.45)

Then, the first law energy balance for the fluid is simply the difference between the aggregate energy balance and the solid energy balance.

\[
\bar{\phi} J \rho^f \epsilon^f + \bar{\phi} J \rho^f \left( \frac{1}{2} A v^f v^f v^f \right) = \bar{\phi} J p \left( \frac{\bar{J}^{f}}{\rho^f} \right) + \bar{\phi} J \bar{T} \left( v_i + v^f v^f \right) - \tilde{q}_i^f + \tilde{r}^f + \tilde{r}^g + \left[ p \left( \bar{\phi} J F_{hi}^{-1} \right) - \tilde{r}^g \right] v^f_i .
\] (2.46)

### 2.5 Entropy Inequalities

In this section the entropy inequalities arising from the second law of thermodynamics for the solid, fluid, and solid/fluid aggregate are analyzed. First, consider the continuum form for the second law of thermodynamics for the solid,

\[
\frac{D}{D t} \int_{V_0} (1 - \phi) \rho^s s^s J dV_0 \geq - \int_{S_0} \tilde{q}_i^s \frac{N_i}{\theta^s} dS_0 + \int_{V_0} \tilde{r}^s - \tilde{r}^g dV_0 .
\] (2.47)
Here, the left-hand side of the inequality is the rate of change of the entropy in the solid, and the right-hand side represents the entropy transfer to the solid. Next, we introduce the Helmholtz free energy per unit mass for the solid as,

\[ \psi^s = e^s - s^\theta^s . \]  

(2.48)

Application of (2.3), (2.43), and (2.48) yields the pointwise form for the entropy inequality for the solid as,

\[ (1 - \phi)J\rho^s \dot{\psi}^s \leq \left[ (1 - \phi)P^s - \bar{\phi} J F_{ij}^{-1} p + g_m(\mu) \left( J - \phi \right) \left( J - 1 \right)^{1/2} F_{ij}^{-1} p \right] v_{ij} + \frac{1}{J} \left( J - 1 \right) p (\phi J) + \left[ g_m(\mu) \left( J - 1 \right) p - \bar{\pi} \right] \dot{\mu} + \xi \mu_j - \left( 1 - \phi \right) J \rho^s s^\theta^\prime - \frac{1}{\theta^\prime} \bar{q}^i \theta^i . \]  

(2.49)

We now assume that the Helmholtz free energy of the solid \( \psi^s \), the solid stress \( P^s \), the internal micro-force vector \( \xi_i \), the internal micro force \( \bar{n} \), the solid entropy density \( s^s \), and the heat flux in the solid \( \bar{q}^i \), each can depend upon the temperature in the solid \( \theta^s \), its gradient \( \theta^\prime \), the true porosity \( \phi J \), the phase-field variable \( \mu \), its gradient \( \mu_j \), its rate \( \dot{\mu} \), and the deformation gradient \( F_{ij} \). The velocity gradient is related to the material time derivative of the deformation gradient as, \( \dot{F}_{ik} = F_{ik} v_{ij} \). Equation (2.49) becomes,

\[ 0 \geq \left[ (1 - \phi)J\rho^s \frac{\partial \psi^s}{\partial F_{ij}} - (1 - \phi)P^s - \bar{\phi} J F_{ij}^{-1} p - g_m(\mu) \left( J - \phi \right) \left( J - 1 \right)^{1/2} F_{ij}^{-1} p \right] v_{ij} + \frac{1}{J} \left( J - 1 \right) p (\phi J) + \left[ g_m(\mu) \left( J - 1 \right) p - \bar{\pi} \right] \dot{\mu} \mu_j + \left[ (1 - \phi)J\rho^s \frac{\partial \psi^s}{\partial \mu} - g_m(\mu) \left( J - 1 \right) p + \bar{\pi} \right] \dot{\mu} \mu_j \]  

\[ + \left[ (1 - \phi)J\rho^s \frac{\partial \psi^s}{\partial \mu} \right] \dot{\mu} + \left[ (1 - \phi)J\rho^s \frac{\partial \psi^s}{\partial (\phi J)} - \left( 1 - g_m(\mu) \right) \left( J - 1 \right) p \right] (\phi J) + \left[ \frac{1}{\theta^\prime} \bar{q}^i \theta^i \right] . \]  

(2.50)

Following the procedures of Coleman and Noll[70], Equation (2.50) must hold for all admissible processes associated with arbitrary variations of \( v_{ij} \), \( \mu_j \), \( \dot{\mu} \), \( \bar{n} \), \( (\phi J) \), \( \dot{\theta}^s \), and \( \theta^\prime_j \). The inequality is linear in \( v_{ij} \), \( \mu_j \), \( \dot{\mu} \), \( (\phi J) \), \( \dot{\theta}^s \), and \( \theta^\prime_j \), which implies that the
coefficients contracted with these terms must be zero in order for the inequality to hold for all admissible processes, leading to the following constitutive equations for the solid,

\[
(1 - \phi) P^s = \left(1 - \phi_0\right) J \rho^s \frac{\partial \psi^s}{\partial F_{ij}} + \bar{\phi} J F_{ij}^{-1} p - g_m(\mu)\left(J - \phi\right)\left(J - 1\right) F_{ij}^{-1} p
\]

\[
\tilde{\xi}_i = \left(1 - \phi_0\right) J \rho^s \frac{\partial \psi^s}{\partial \mu_i}
\]

\[
\left(1 - g_m(\mu)\left(\frac{1}{J}\right)\right) p = \left(1 - \phi_0\right) J \rho^s \frac{\partial \psi^s}{\partial (\phi J)}
\]

\[
s^* = -\frac{\partial \psi^s}{\partial \theta^s}
\]

\[
\frac{\partial \psi^s}{\partial \mu} = \frac{\partial \psi^s}{\partial \theta^s} = 0
\]

Notice that for the reference configuration relationships we have used the fact that \((1 - \phi) \rho^s J = (1 - \phi_0) \rho^0 J_0\) from (2.3), where the subscript “0” refers to the values of the quantities in the reference state. The reduced form of the dissipation inequality for the solid is then,

\[
\left[\left(1 - \phi\right) J \rho^s \frac{\partial \psi^s}{\partial \mu} - g_m(\mu)\left(1 - \phi\right)\left(J - 1\right) p + \tilde{\pi}\right] \mu + \frac{1}{\theta^s} \tilde{q}^s_i \theta^s_j \leq 0
\]

This reduced dissipation inequality is satisfied if,

\[
\tilde{\pi} = -\left(1 - \phi_0\right) J \rho^s \frac{\partial \psi^s}{\partial \mu} + g_m(\mu)\left(1 - \phi\right)\left(J - 1\right) p - \beta \mu - \eta \theta^s_j
\]

\[
\tilde{q}^s_i = -\bar{\eta} \mu - \kappa^s_{ij} \theta^s_j
\]

with the positivity condition,

\[
\beta \mu^2 + \mu \eta \theta^s_j + \theta^s_j \bar{\eta} \mu + \theta^s_j \kappa^s_{ij} \theta^s_j \geq 0 \quad \text{for all } \mu, \theta^s_j
\]

Each of the newly introduced material properties, \(\beta, \eta, \bar{\eta}, \), and \(\kappa^s_{ij}\) are allowed to depend upon \(\mu, \mu, \bar{\mu}, (\phi J), \theta^s, \theta^s_j, \) and \(F_{ij}\). Note that \(\kappa^s_{ij}\) is the thermal conductivity tensor for the solid skeleton.
We now consider the fluid. For the fluid, the second law inequality is,

\[
\frac{D}{Dt} \int_{v_{s}} \phi \rho s f J dV_{0} + \frac{D}{Dt} \int_{v_{s}} \rho s f J dV_{0} \geq -\int_{S_{0}} \frac{\tilde{\tau}_{j}^{s f}}{\theta^{f}} dS_{0} + \int_{V_{0}} \rho v_{j} N_{s f} dV_{0} - \int_{S_{0}} \rho v_{j} \bar{w}_{j} N_{s f} dS_{0} + \int_{V_{0}} \rho \tilde{m}_{s f} dV_{0}.
\]  

(2.55)

Here, the left-hand side of the inequality is the rate of change of the entropy in the fluid, and the right-hand side represents the entropy transfer to the fluid including the entropy of the convected and injected fluid. As with (2.38),

\[
\int_{V_{s}} \rho s f J dV_{0} = 0,
\]

(2.56)

and the Helmholtz free energy for the fluid is,

\[
\psi^{f} = e^{f} - s^{f} \theta^{f}.
\]

(2.57)

Application of (2.17), (2.46), (2.56), and (2.57) leads to the pointwise form for the second law inequality for the fluid as,

\[
\tilde{\phi}^{f} \rho^{f} \psi^{f} \leq \tilde{\phi} \rho^{f} - \frac{\tilde{\phi} \rho^{f} s^{f} \theta^{f}}{\rho^{f}} + \tilde{\phi} \bar{\tau}_{j}^{s f} \left( v_{i} + v_{i}^{f f} \right)_{j} - \frac{1}{2} A v_{i}^{f f} + A \left( v_{i}^{f f} \right) v_{i}^{f f} + \left[ p \left( \tilde{\phi} J F_{j}^{-1} \right)_{j} - f_{i}^{f f} \right] v_{i}^{f f} - \frac{1}{\theta^{f}} \tilde{q}_{j}^{f f} \theta^{f}
\]

(2.58)

In contrast to the analysis of the second law for the solid, here we do not invoke the principle of equipresence, i.e. that each dependent quantity depends upon each of the independent quantities. Instead, we take a slightly more restrictive set of assumptions for the analysis of (2.58). We assume that the Helmholtz free energy of the fluid \( \psi^{f} \), the fluid pressure \( p \), and the entropy density of the fluid \( s^{f} \) are dependent only on the density of the fluid \( \rho^{f} \), the fluid velocity gradient \( v^{f} = v_{i j} + v_{i j}^{f f} \), the temperature of the fluid \( \theta^{f} \), and the temperature gradient in the fluid \( \theta^{f} \). The remaining dependent quantities, include the fluid stress \( \bar{\tau}_{j}^{s f} \), the heat flux in the fluid \( \bar{q}_{j}^{f f} \), and the interaction force between the solid and fluid \( \bar{f}_{i j}^{s f} \), each of which can depend upon \( \rho^{f} \), \( \theta^{f} \), \( \theta^{f} \), \( \phi \),
\( \dot{v}_{i,j} \), as well as the relative fluid velocity \( \dot{v}_{i,j}^{fs} \), its rate with respect to the fluid \( \dot{v}_{i,j}^{fs} \), the deformation gradient \( F_{i,j} \), and the phase-field parameter \( \mu \). Equation (2.58) then becomes,

\[
0 \geq \bar{\phi} \frac{\partial \psi}{\partial \rho} \left( \frac{\partial \rho}{\partial \rho} - \frac{p}{\rho} \right) + \bar{\theta} \frac{\partial \psi}{\partial \theta} \left( \frac{\partial \rho}{\partial \theta} + s \right) + \bar{\dot{\theta}} \frac{\partial \psi}{\partial \theta} \left( \dot{\rho} + \phi \frac{\partial \psi}{\partial \theta} \right) + \phi \frac{\partial \psi}{\partial \theta} \left( \dot{v}_{i,j} \right)
\]

\[
+ \bar{\dot{\theta}} \left( \frac{1}{2} A \dot{v}_{i,j}^{fs} + A \left( \dot{v}_{i,j}^{fs} \right) \right) v_{i,j}^{fs} - \left[ p \left( \bar{\phi} JF^{-1} \right) \right] v_{i,j}^{fs} + \frac{1}{\theta} \left( q \dot{\theta} \dot{\rho} - \phi \dot{\theta} \dot{v}_{i,j} \right)
\]

Equation (2.59) must hold for all admissible processes associated with arbitrary variations of \( \rho, \theta, v_{i,j}, \left( \dot{v}_{i,j} \right), \left( \dot{v}_{i,j}^{fs} \right) \). The inequality is linear in \( \rho, \theta, \left( \dot{v}_{i,j} \right), \left( \dot{v}_{i,j}^{fs} \right) \), and the coefficients contracted with these terms must be zero in order for the inequality to hold for all admissible processes, leading to the constitutive equations for the fluid,

\[
p = \left( \rho \right)^{2} \frac{\partial \psi}{\partial \rho}
\]

\[
s^{f} = -\frac{\partial \psi}{\partial \theta},
\]

\[
\frac{\partial \psi}{\partial \theta} = 0, \quad \frac{\partial \psi}{\partial v_{i,j}} = 0
\]

and the reduced dissipation inequality for the fluid,

\[
\left[ \bar{\phi} J \rho \left( \frac{1}{2} A \dot{v}_{i,j}^{fs} + A \left( \dot{v}_{i,j}^{fs} \right) \right) - \left[ p \left( \bar{\phi} JF^{-1} \right) \right] \right] v_{i,j}^{fs} - \left( q \dot{\theta} \dot{\rho} - \phi \dot{\theta} \dot{v}_{i,j} \right) + \frac{1}{\theta} \left( q \dot{\theta} \dot{v}_{i,j} \right) \leq 0.
\]

This reduced dissipation inequality for the fluid is satisfied if,
\[
\hat{\theta}_{ij} = \nu_{ijkl} (w_{k,l} + \bar{\phi} v_{k,l}) + \hat{\tau}_{ji}
\]
\[
\tilde{q}_i^f = -\kappa_i^f \theta_i^f + \tilde{q}_i
\]
\[
\tilde{f}_i^{sf} = -\chi_q v_i^{fs} - \tilde{\phi}_j \hat{\tau}_{ij} + p \Big( \hat{\phi} J \hat{F}^{-1}\Big)_j - \bar{\phi} J \rho \left( \frac{1}{2} A v_i^{fs} - A \left( \hat{v}_i^{fs} \right) + \tilde{f}_i^{sf} \right)
\]

Here we have focused on the direct terms for the deviatoric fluid stress, the fluid heat flux, and the solid/fluid interaction force introducing the positive definite material tensors of the fluid viscosity \( \nu_{ijkl} \), the thermal conductivity of the fluid \( \kappa_i^f \), and the fluid impermeability of the porous solid \( \chi_q \). Each of these material tensors is allowed to depend upon \( \rho^f \), \( \theta^f \), \( \theta_i^f \), \( \bar{\phi} \), \( v_i^{sf} \), \( v_i^{fs} \), and \( \left( v_i^{fs} \right), F_{ij} \), and \( \mu \). The additional functions \( \hat{\tau}_{ji}, \tilde{q}_i, \) and \( \tilde{f}_i^{sf} \) are not written out explicitly for the sake of brevity, but their general construction and positivity constraint is analogous to that spelled out for the solid in (2.53) and (2.54). In practice, each of these functions will usually be taken to vanish, as will the cross terms in (2.53).

At this point there remains one quantity that has not been constrained, the heat transferred from the solid to the fluid, \( \tilde{r}_i^{sf} = -\tilde{r}_i^{fs} \). For this we consider the second law inequality for the solid/fluid aggregate as a whole,

\[
\frac{D}{Dt} \int_{V_s - V_f} \left[ (1 - \phi) \rho^s \theta^s + \phi \rho^f \theta^f \right] J dV_0 + \frac{D}{Dt} \int_{V_f} \rho^f \theta^f J dV_0 
\geq -\int_{S_s} \tilde{q}_i^s \hat{N}_i^s \hat{\theta}^s \theta^s dS_0 + \int_{V_f} \left( \tilde{\phi}^s + \tilde{\phi}^{fs} \right) J dV_0 - \int_{S_s} \rho^f \tilde{\phi}^s \hat{N}_i^s \hat{\theta}^s \theta^s dS_0 + \int_{V_f} \rho^f \tilde{\theta}^s dV_0 \]

Here, the left-hand side of the inequality is the rate of change of the entropy in the aggregate, and the right-hand side represents the entropy transfer to the aggregate including entropy convected and injected into the aggregate by the fluid. Applying the previously derived constitutive equations, the pointwise form is,

\[
0 \leq \frac{1}{\theta^f} \left[ \tilde{\phi} \hat{\tau}_{ij} v_i^{fs} - \tilde{\phi} J \rho \left( \frac{1}{2} A v_i^{fs} - A \left( \hat{v}_i^{fs} \right) + \tilde{f}_i^{fs} \right) \right] v_i^{fs} + \left[ p \left( J \tilde{\phi} F^{-1}\right)_j - \tilde{f}_i^{sf} \right) + \frac{1}{\theta^s} \left[ (1 - \phi) J \rho \frac{\partial \hat{\psi}_s^s}{\partial \mu} - g^s_m (\mu) (1 - \phi) (J - 1) p + \tilde{\pi} \right] \hat{\mu} - \frac{1}{\theta^s} \tilde{q}_i^s \theta^s \right] \]

\[
+ \frac{1}{\theta^s} \left[ \left( 1 - \phi \right) J \rho \hat{v}_s^s - \hat{v}_i^s \mu \right] + \frac{\tilde{f}_i^{sf} \left( \theta^s - \theta^f \right)}{\theta^s \theta^f} \right) \]

\[(2.64)\]
The two terms in brackets have already appeared in the dissipation inequalities for the fluid (2.61) and solid (2.52) and each is already constrained to be non-negative. Hence, (2.64) still does not offer a clear constraint upon the heat transferred between the solid and fluid. However, a reasonable approach is to take the interpretation that each of the components of (2.64) must satisfy the inequality independently of the others. Then, the third term will satisfy the inequality if,

\[ \tilde{r}^{sf} = \tilde{h}^{sf} (\theta^s - \theta^f) . \]  

(2.65)

Here, \( \tilde{h}^{sf} \) is an effective heat transfer coefficient for the porous solid/fluid system, which must be positive and can depend upon each of the independent variables associated with the fluid and the solid.

In the absence of the inclusion of the crack volume in the fluid mass balance, the phase-field micro-force balance laws, and the transition to Stokes flow in the cracks, the equations presented above reduce to the nonlinear poroelastic theories developed by Coussy and co-workers\[61-65\], which have been shown to be in agreement with Biot’s classical theory\[71-73\].

2.6 The Solid Free Energy

At this point the model is quite general and has many material properties that must be specified. In this section a free energy potential is devised that simultaneously models the linear poroelasticity of Biot-type models and the onset and propagation of fluid filled fractures within the medium. To this end a “poro-enthalpy” potential \( \Omega^s \) is proposed that can be derived from the Hemholtz free energy by the following Legendre transformation,

\[ (1 - \phi) J \rho^s \Omega^s = (1 - \phi) J \rho^s \psi^s - \left( J - g_m(\mu) \langle J - 1 \rangle \right) p \phi . \]  

(2.66)

Then, the entropy inequality for the solid (2.49) can be rewritten as,

\[ (1 - \phi) J \rho^s \dot{\Omega}^s \leq \left[ \left( 1 - \phi \right) P^s_{\delta i} - \left( \bar{\psi} - g_m(\mu) \langle J - 1 \rangle^0 \right) p J F_{\delta i}^{-1} \right] v_{\delta i} \]

\[ - \left[ 1 - \left( \frac{1}{J} \right) g_m(\mu) \langle J - 1 \rangle \right] \phi \dot{p} + \left[ g_m^\prime(\mu) \langle J - 1 \rangle p - \bar{\pi} \right] \tilde{\mu} + \tilde{\xi}_v \tilde{\mu} - (1 - \phi) J \rho^s \dot{\theta}^s - \frac{1}{\theta^s} \tilde{g}_i \hat{\theta}^s . \]  

(2.67)
and the constitutive relations outlined by (2.51) and (2.53) can be written in terms of \( \Omega^* \) as,

\[
(1 - \phi) P^* = (1 - \phi_0) J_0 \rho_0^* \frac{\partial \Omega^*}{\partial F_{ij}} + \left( \phi - g_m(\mu) \langle J - 1 \rangle \right) p J F_{ij}^{-1} \\
\xi_j = (1 - \phi_0) J_0 \rho_0^* \frac{\partial \Omega^*}{\partial \mu_j} \\
\left[ 1 - \left( \frac{1}{J} \right) g_m(\mu) \langle J - 1 \rangle \right] \phi J = -(1 - \phi_0) J_0 \rho_0^* \frac{\partial \Omega^*}{\partial p} \\
s^* = -\frac{\partial \Omega^*}{\partial \theta^*} = 0 \\
\frac{\partial \Omega^*}{\partial \mu} = \frac{\partial \Omega^*}{\partial \theta^*} = 0 \\
\tilde{p} = -(1 - \phi_0) J_0 \rho_0^* \frac{\partial \Omega^*}{\partial \mu} + g_m(\mu) \langle J - 1 \rangle p - \beta \mu - \eta \theta^*_j \\
\tilde{q}^*_j = -\eta_j \mu - \kappa_j \theta^*_j
\]  

(2.68)

We consider the following simple poro-enthalpy for reversible, hyperelastic, isothermal behavior,

\[
(1 - \phi) J_0 \rho_0^* \Omega^* = g_\phi(\mu) \Omega^+(F_{ij}) + \Omega^-(F_{ij}) - \left( 1 - g_m(\mu) \langle J - 1 \rangle \right) \left[ \alpha (J - 1) p + \frac{1}{2M} \right] ^2 \\
+ \frac{G}{4 \ell_0^2} \left[ (1 - \mu)^2 + 4 \ell_0^2 \mu_j \mu_{ij} \right]
\]  

(2.69)

Here an additive decomposition of the elastic strain-energy \( \Omega^\epsilon = \Omega^+ + \Omega^- \) has been employed so degradation is driven by the presence of tensile deformations. The material constants \( \alpha, M, \) and \( G \) are a coupling coefficient commonly referred to as the Biot coefficient, the Biot modulus, and Griffith’s critical energy release rate, respectively. The last term in (2.69) is ubiquitous to the second order theory for phase-field fracture theories as originally proposed by Francfort and Marigo\(^{[46]}\) in their pioneering work on the variational approach to fracture, where \( \ell_0 \) is a nonlinear process zone length scale. The degradation function \( g_\phi(\mu) \) is left in general terms for the sake of simplicity. In practice, the “bell-curve” formulation employed by Wilson et al.\(^{[68]}\) will be used,
\[ g_d(\mu) = a \left[ 1 - \left( \frac{a-1}{a} \right)^{\mu^2} \right] \] (2.70)

with \( a > 1 \). The above function necessarily satisfies the properties required for degrading energy in the solid as described by Miehe et al.\cite{48}, namely,

\[ g_d(0) = 0, \quad g_d(1) = 1, \quad g_d'(0) = 0. \] (2.71)

The parameter \( a \) is typically chosen to be close to 1, in which case (2.70) has several advantages when compared to the more conventional \( g_d(\mu) = \mu^2 \). Most notably, from a simple analysis of uniaxial, homogenous behavior, the stress and phase-field \( \mu \) in the solid approach zero more quickly for large values of strain that exceed the critical value for the onset of degradation. This property is particularly useful for materials with very low permeabilities, which, typically, are the materials that are candidates for hydraulic fracture treatments. Similar treatments of the degradation of the strain energy are discussed in Miehe\cite{59}. For more details of the advantages of (2.70) the reader is directed to the aforementioned work by Wilson et al.\cite{68}.

For further clarity, the constitutive relations of (2.68) are given explicitly with the poro-enthalpy potential (2.69) and \( \beta = 0 \) and \( \eta_i = 0 \) as,

\[
\begin{align*}
(1-\phi)P_j^* & = g_d(\mu) \frac{\partial \Omega^+}{\partial F_{ij}} + \frac{\partial \Omega^-}{\partial F_{ij}} + \left[ \phi - \left( \alpha - g_m(\mu) \langle J-1 \rangle^0 (1-\alpha) \right) \right] J F_{ij}^{-1} p \\
\phi J & = \left. \frac{1 - g_m(\mu) \langle J-1 \rangle^0}{1 - \left( \frac{1}{J} \right) g_m(\mu) \langle J-1 \rangle} \left( \alpha \langle J-1 \rangle + \frac{p}{M} \right) \right| J F_{ij}^{-1} p \\
\tilde{\pi} & = -g_d'(\mu) \Omega^+(F_{ij}) + \frac{G}{2\ell_0} (1-\mu) + g_m'(\mu) \left[ 1 - \alpha \langle J-1 \rangle \right] p - \frac{1}{2M} \langle J-1 \rangle^0 \left. \langle J-1 \rangle^0 p^2 \right| J F_{ij}^{-1} p \\
\tilde{\xi}_i & = 2\ell_0 \mu_i
\end{align*}
\] (2.72)

Defining a reference fluid pressure \( p_0 \) such that the porosity at such a pressure in the absence of deformation and degradation from the reference state is \( \phi_0 \), (2.72)-b can be restated as,
\[
\phi J - \phi_0 = \frac{1 - g_m(\mu) (J - 1)^0}{1 - \left(\frac{1}{J}\right) g_m(\mu) (J - 1)} \left(\alpha (J - 1) + \frac{p - p_0}{M}\right).
\] (2.73)

Using (2.44) the total aggregate stress is,

\[
P_{ji} = g_d(\mu) \frac{\partial \Omega^+}{\partial F^i_{kj}} + \frac{\partial \Omega^-}{\partial F^i_{kj}} - \left[\alpha + g_m(\mu) (J - 1)^0 (1 - \alpha)\right] J F^{-1}_{kj} p + \bar{\phi} \tau_{ji}. \tag{2.74}
\]

Additionally, the spectral decomposition of the stored strain energy for an isotopic solid as described in the work of Miehe et al.\[^{[48]}\] is adopted. The strain energy takes the form,

\[
\Omega^+(\varepsilon_{ij}) = \frac{\lambda^s}{2} \varepsilon_{kk}^2 + \mu^s \left(\langle \varepsilon_1 \rangle^2 + \langle \varepsilon_2 \rangle^2 + \langle \varepsilon_3 \rangle^2\right)
\]

\[
\Omega^-(\varepsilon_{ij}) = -\frac{\lambda^s}{2} \langle -\varepsilon_{kk} \rangle^2 - \mu^s \left(\langle -\varepsilon_1 \rangle^2 + \langle -\varepsilon_2 \rangle^2 + \langle -\varepsilon_3 \rangle^2\right)
\]

\[
\varepsilon_{ij} = \frac{1}{2} \left(F_{ik} F_{kj} - \delta_{ij}\right)
\]

where \(\lambda^s\) and \(\mu^s\) are the Lamé material constants, \(\varepsilon_{ij}\) is the Green-Lagrangian finite strain measure, and \(\varepsilon_{(1,2,3)}\) are the principal strains. If we neglect the (typically small) effects of fluid shear stress and adopt the assumptions of infinitesimal deformations such that \((J - 1) \approx \varepsilon_{kk}\) and products of displacement gradients are neglected, the theory recovers linear poroelasticity when the phase-field parameter \(\mu = 1\) (or \(g_m(\mu) = 0\))\[^{[61,73]}\].

### 2.7 The Fluid Momentum Equation

Generally, as a result of small crack openings, hydraulic fractures are characterized by low Reynolds number flows. In this section the fluid momentum equation (2.30) is reduced under the assumptions of isothermal, creeping Newtonian flow. The goal is to simultaneously describe the Darcy-type flow regimes that occur in fully intact regions of the porous aggregate while also properly representing the laminar flow that is present in crack interiors. To this end, the following constitutive relationships are proposed,
\[
\tau_{ij} = g_s(\mu)\nu^f \left[ w_{ij} + w_{ji} - \frac{1}{3} w_{kk} + \bar{\phi} \left( v_{ij} + v_{ji} - \frac{1}{3} v_{kk} \right) \right] \]

\[
\tilde{j}_{fi}^f = p \left( \bar{\phi} J F_{ji}^{-1} \right) - \bar{\phi} \tilde{j}_{fi}^\ast - g_D(\mu) \frac{\nu^f}{\kappa} w_i
\]

Here \( \nu^f \) is the fluid viscosity, \( \kappa \) is the isotropic intrinsic permeability for the aggregate, and \( g_s(\mu) \) and \( g_D(\mu) \) are indicator functions of the phase-field for the Stokes and Darcy flow regimes. The indicator functions are chosen such that,

\[
g_s(\mu) \to 1 \text{ and } g_D(\mu) \to 0 \text{ as } \mu \to 0
\]

\[
g_s(\mu) \to 0 \text{ and } g_D(\mu) \to 1 \text{ as } \mu \to 1
\]

The specific forms chosen are,

\[
\begin{align*}
g_s(\mu) &= \left(1 - \mu \right)^2 \\
g_D(\mu) &= 1 - \left(1 - \mu^4 \right)^{\frac{1}{4}}
\end{align*}
\]

where the Stokes function has been chosen based on its simplicity and ease of implementation and the Darcy function has been chosen such that the impermeability (which is typically quite high) is degraded more rapidly in the presence of cracks. The choices made for the indicator functions in (2.78) will be shown to reproduce the desired behaviors for the numerical experiments conducted, which will be described in the following sections. We note that alternative choices for the indicator functions that comply with (2.77) can be made. However, an exhaustive study of the effects of the form of these functions is beyond the scope of this work. Applying (2.76) and (2.78) within (2.30) and neglecting inertial terms based upon the assumption of low Reynolds number creeping flows, the fluid momentum equation becomes,

\[
\left[ g_s(\mu)\nu^f \left( w_{ij} + w_{ji} - \frac{1}{3} w_{kk} + \bar{\phi} \left( v_{ij} + v_{ji} - \frac{1}{3} v_{kk} \right) \right) J F_{ji}^{-1} \right] - J F_{ji}^{-1} p_j - J g_D(\mu) \frac{\nu^f}{\kappa} w_i = 0
\]

Equation (2.79) is analogous to Brinkman’s Equation\(^{[74]}\) where the material coefficients depend directly on the phase-field parameter \( \mu \). When \( \mu = 1 \) Darcy flow laws are recovered, and when \( \mu = 0 \) the equations for Stokes flow are recovered.
3. Finite Element Method

The numerical experiments presented in the following sections will be for isothermal, isotropic, quasi-static material behavior with incompressible, steady-state fluid flows. To summarize, the following equations will enter into the numerical model:

\[
\begin{align*}
P_{j,i} + \hat{b}_i &= 0 \quad \text{in } V_0 \\
P_{j,i} N_j &= \hat{t}_i \quad \text{on } S_t \\
u_i &= \hat{u}_i \quad \text{on } S_u
\end{align*}
\]

\[
\begin{align*}
\tilde{\xi}_{i,j} + \tilde{\gamma} + \tilde{\pi} &= 0 \quad \text{in } V_0 \\
\tilde{\xi}_{j} N_j &= \tilde{\iota}_i \quad \text{on } S_f \\
\mu &= \hat{\mu} \quad \text{on } S_p
\end{align*}
\]

\[
\begin{align*}
\left(\tau_j J F^{-1} \right)_j - J F^{-1} p_j - J g_\mu(\mu) \frac{\nu_j}{\kappa} w_i &= 0 \quad \text{in } V_0 \\
\left(\tilde{\phi} J + J F^{-1} w_i \right)_j - \tilde{m} &= 0 \quad \text{in } V_0 \\
\tilde{\phi} \left(\tau_j - p \delta_0 \right) J F_j^{-1} N_j &= \tilde{\iota}_i \quad \text{on } S_f \\
w_i &= \tilde{w}_i \quad \text{on } S_w \\
p &= \tilde{p} \quad \text{on } S_p
\end{align*}
\]

Then, the corresponding virtual work statement that will serve as the framework for the finite element analysis is

\[
\int_{V_0} \left[ P_{j,i} \delta F_{j,i} + \tilde{\xi}_{i,j} \delta \mu + \tilde{\gamma} \delta \mu + \left(\tau_j - p J F^{-1}_j\right) \delta w_i + g_D(\mu) \frac{\nu_j}{\kappa} Jw_i \delta w_i - \left(\tilde{\phi} J + J F^{-1}_j w_i \right) \delta p \right] dV_0 . \quad (3.2)
\]

For the numerical method the unknown nodal quantities include the displacements of the aggregate \(u_i\), which is simply the difference between the material point positions in the current and reference configurations, the relative flux of the fluid with respect to the solid \(w_i\), the fluid pressure \(p\), and the phase-field parameter \(\mu\). The porosity \(\phi\) is determined constitutively and the effective porosity \(\tilde{\phi}\) is determined from (2.18). We note that the derivations of the governing equations constructed in the previous sections
were done so in the reference configuration, which is the domain on which the numerical calculations are performed. The importance in distinguishing between the reference configuration and the current configuration lies in a proper representation of the fluid mass and momentum balances, particularly in regions with degradation. However, for the brittle, linearly elastic materials addressed in this work there is no real advantage in distinguishing between reference and current configurations for the aggregate momentum balance (2.25) or micro-force balance (2.36). This is due to the fact that the geologically relevant structures investigated in this work undergo relatively small displacements and deformations. In order to simplify the implementation, we believe it is suitable to adopt small deformation approximations for the aggregate and micro-force balances. To maintain consistency, however, we will not describe those equations here.

In terms of the numerical discretization, standard continuous Galerkin finite element methods are sufficient for the solution of the aggregate equilibrium and micro-force balance equations. However, for the fluid flow equations it is well-known that the elements must satisfy the LBB conditions\cite{75,77}. For the two-dimensional numerical investigations described here we have used Taylor-Hood elements\cite{78} with biquadratic interpolations for the relative flux $w_i$, and bilinear interpolations for the fluid pressure $p$. The evolution equation that appears in the fluid mass balance is solved using an implicit backward Euler approximation. The fully coupled equations are nonlinear and standard Newton-Raphson procedures are implemented to solve for all quantities of interests simultaneously. This is in contrast with many other phase-field approaches where staggered schemes are used to solve for specific quantities while all other fields are held fixed\cite{60}. Quadratic convergence is achieved for the N-R scheme as long as the time steps used are sufficiently small. For matrix inversion, a direct solver from the MUMPS package for the PETSc library is used to solve the problems that are discussed in the next section\cite{79-82}.

4. Results and Discussion

This section presents a series of benchmark solutions that the theory must, at a minimum, recover in order to properly model the hydraulic fracture process. Many of these simple solutions are readily recovered if the crack surfaces are identified explicitly by a discrete crack method, but are not trivial for diffuse crack descriptions like the phase-field approach to fracture. The comparison of the model with these foundational problems for which analytical solutions have been developed serves to establish confidence in the theory for when more complex problems are investigated. The three most fundamental problems that are addressed in this paper are the steady, laminar
flow through parallel crack faces, the fluid-loading of a free surface as in Terzaghi’s consolidation problem, and the uniform pressurization of a plane-strain center crack in an isotropic, impermeable, elastic, infinite solid. After agreement is established for these three fundamental problems, an investigation of the Kristianovic-Geerstma-de Klerk problem (a plane strain center crack in an impermeable, elastic, infinite domain subject to a constant point injection at the crack’s center) is carried out. Finally, a problem that includes multiple cracks interacting and merging is presented to demonstrate the advantageous features of the phase-field method.

4.1 Laminar Flow between Fixed Parallel Plates
A proper representation of the fluid flow field through the system is needed to produce the correct pressure field. This pressure then drives the solution for the deformation of the aggregate and the evolution of the phase-field within the system. We note that there are also shear stresses within the fluid, but that the influence of fluid shear stress on the aggregate is small. Without a correct representation of the fluid flow, particularly within cracks, an accurate description of crack evolution would be hopeless. Sub-surface cracks are characterized by two long length scales, the crack height and crack length, and one small length scale, the width/opening or crack aperture. Thus, the fluid flow inside the crack is well approximated by the flow between two fixed parallel plates such that the crack walls make up the two plates. Because of the relatively small openings of cracks in these sub-surface materials, the flow is laminar. Assuming the fluid is a Newtonian fluid, the solution to the Navier-Stokes equations for this simplified geometry yields a parabolic flow profile where the total fluid flux through a cross-section per unit crack height is

\[ q_y = \frac{(W_n)^3}{12 \nu} p_y, \quad (4.1) \]

The result is proportional to the pressure gradient, \( p_y \), along the length direction of the crack \( y \) (see the inset of Figure 2), and dependent on the cube of the crack opening aperture \( W_n = \Delta u_n \). Thus, it is crucial that the crack opening displacement is properly incorporated into the theory. Due to the fact that the cracks are represented by a diffuse field, \( \Delta u_n \) is not a readily available local quantity for phase-field methods. In order to address this issue, the present theory utilizes a finite deformation framework such that the effects of the crack opening are accounted for in the solution of the fluid flow balances, which are naturally dependent on the deformation within cracks. There is, however, an additional issue that arises for the present theory when numerical
discretizations are implemented. As shown in Figure 1 the analytic solution to the phase-field theory for fracture has a jump discontinuity in displacement. As previously mentioned, standard Galerkin methods are used to discretize the displacement field $u_i$, which forces the resulting field to be $C^0$ continuous. Thus, the opening in the discretized setting occurs over some discretization length scale associated with the element size, as shown in the figure.

**Figure 1**
These plots Illustrate the difference between the analytical and discretized representations of the material stretch normal to a phase-field crack. (a) shows the phase-field profile before deformation (solid line) and after opening (dotted line) in the deformed configuration for the analytical case. (b) shows the phase-field profile before deformation (solid line) and after opening (dotted line) in the deformed configuration for the discretized case. (c) and (d) show the crack opening displacement as a function of location in the reference configuration for the analytical and discretized cases respectively. $h^*$ is a characteristic length scale associated with the discretization (i.e. element size).
For the discretized case the crack opening width can be estimated as,

\[ W_n^h \approx h^e \left(1 + e_n\right), \quad (4.2) \]

where \( e_n \) is the nominal strain of a line element perpendicular to the crack plane and \( h^e \) is the length scale associated with the discretization. It can be shown that the total crack opening displacement is approximated properly by the discretized method (i.e. \( h^e e_n = \Delta u_n^h \approx \Delta u_n \)), but, according to (4.2), the actual crack opening width will be overestimated. In order to account for this discrepancy our approach is to use a scaled viscosity defined as,

\[ \nu_s^f = \nu^f \left(\frac{1 + e_n}{e_n}\right)^3 = \nu^f \left(\frac{\lambda_n}{\lambda_n - 1}\right)^3, \quad (4.3) \]

where \( \lambda_n \) is the stretch ratio of a line element perpendicular to the crack plane and can be determined using the deformation gradient \( F_{ij} \) with gradients of the phase-field as,

\[ \lambda_n = F_{ij} n_i N_j = F_{ij} \frac{\mu \mu_{ij}}{\mu_{ik} \mu_{kj}^{1/2} \mu_{ij}^{1/2}} = \sqrt{\frac{\mu_{ik} \mu_{ij}}{F_{ij}^{-1} F_{kl}^{-1} \mu_{ij}^{1/2} \mu_{kl}^{1/2}}} . \quad (4.4) \]

This is the same measure as that described in Miehe [59]. A potential issue with the scaling (4.3) is that if the stretch is equal to unity or less, the scaled viscosity will approach infinity or become negative. If the phase-field parameter \( \mu \) is not equal to one in these regions, or the Stokes indicator function \( g_s(\mu) > 0 \), the scaling as presently proposed will cause issues in the solution of the fluid mass balance (2.79). For instance, as seen in Figure 1 the stretch is localized to the region of maximum damage, and regions just outside the localization of deformation where the phase-field is still transitioning but the stretch may be unity or less are be subject to the issues described. To address this we cap the scaling at a value as to not overwhelm Darcy’s law. Effectively, the scaling is designed such that a closed crack acts nearly as if it would if the crack did not exist. Thus, the scaling is chosen such that,

\[ \nu_s^f = \nu^f \times \min \left( \frac{\lambda_n}{\left(\lambda_n - 1\right)^3}, 1 + \frac{\ell_0^2}{\kappa} \right), \quad \text{with} \ \epsilon_r \ll 1 . \quad (4.5) \]
These simulations demonstrate the ability of the theory to accurately represent the flow through cracks of different openings. A pressure differential is applied between the top and bottom surfaces. The contour plot above shows the phase-field damage profile, which in the undeformed configuration (not shown) is identical for both the open and closed crack. The solutions for the fluid flow in the vertical direction $w_y$ are shown with blue lines for the case of a closed (a) and open (b) diffuse crack. The color contours and red lines indicate the phase-field parameter in the deformed configuration. When the crack is nearly closed (a) there is only enhanced Darcy-like flow (blue line) in the damaged region, as can be seen in the included insert. However, when the crack opens (b) the proper channel flow (blue line) within the crack is recovered such that the net fluid flux is in agreement with laminar flow through fixed parallel plates. The fluid flows (blue line) are normalized by $w^* = \frac{p_y \ell_0^2}{\mu'}$.

For the calculations in this work $\epsilon_y = 10^{-3}$. Note that it is beneficial to the convergence of the Newton-Raphson scheme if there is a smooth transition between these two limits. One possible construction is detailed in Appendix A. Figure 2 demonstrates how the flow profile evolves for a given pressure gradient with a large versus a small crack.
opening. A crack is imposed by setting $\mu = 0$ for a line of elements through the thickness of a thin strip of material. A pressure drop is imposed in the vertical direction and fluid flows through the strip. The flow evolves according to the opening imposed on the crack, starting as an enhanced Darcy-type flow in the region of damage and transitioning to a full Stokes-type flow.

Total Flux vs. Crack Aperture ($\kappa/\ell_0^2 = 10^{-14}$)

Figure 3

This plot shows the dependence of the fluid flux on crack opening for the flow through a cracked strip of permeable material. The initial length of the strip is $200\ell_0$ and the middlemost element is completely degraded as in Figure 1b. The solid black line corresponds to the analytical solution of flow between two fixed, discrete parallel plates (4.1). The purple, green, and orange lines correspond to cases where the fluid viscosity is not scaled for different element sizes. The red, yellow and blue lines correspond to the scaling (4.5) for various element sizes. The scaling results in excellent agreement with the analytical calculation for crack apertures of practical interest. The damage zone about the crack leads to an enhanced Darcy-type behavior for situations where the crack is essentially closed. This explains the slight discrepancy between the scaled solutions and the Darcy solution (dotted line) where no crack (i.e. no damage) is present. Without the scaling, accurate solutions cannot be expected until the normalized crack aperture is on the order of the mesh size.
Figure 3 illustrates how well the dependence on crack opening is captured with and without the viscosity scaling for various element sizes. The total flux through a strip of length $200\ell_0$ is calculated. The volumetric flow rate is normalized by $q^* = \frac{p_y \ell^3}{\nu_f}$ in the plot. The results clearly demonstrate that if the scaling is not used, accuracy cannot be expected unless the crack opening displacement divided by the mesh size, $\Delta u_n^h/h^c$, is well above unity. The solutions for the scaled viscosity deviate slightly from the result for strictly Darcy flow as a result of the enhanced Darcy zone that can be observed in Figure 2. This is a result of the damage in the neighborhood of the crack and the Darcy degradation function (2.78)b.

A remarkable feature of the viscosity scaling is that there is no explicit dependence on a length scale associated with the discretization. This is in stark contrast with other approaches that utilize the phase-field method to model the hydraulic fracture problem, see\textsuperscript{[59,60]}. The approach adopted in these other models is to use Reynolds lubrication theory directly, i.e. (4.1), by solving the Darcy equation but modifying the permeability based on the crack opening displacement. For instance in Miehe et al. (2015)\textsuperscript{[59]} the approach requires that the stretch normal to the crack plane be multiplied by the square of a length scale associated with the discretization. A similar approach is also used in Mikelic et al. (2015)\textsuperscript{[60]}.

### 4.2 Fluid Loading of a Surface and Consolidation

Perhaps the most classical problem in poromechanics is Terzaghi’s consolidation problem\textsuperscript{[73]}. A rock or soil layer of thickness $L$ rests on a rigid impermeable base and is subject to a constant applied traction $t_y = -p_0$ on the top surface. The more relevant case for pressurized fractures is the case when the applied load is the consequence of the surface being in contact with a fluid at pressure $p_0$. For this class of uniaxial strain problems the governing equations for displacement and fluid pressure according to Biot theory can be decoupled. For constant loading, Biot theory reduces to Terzaghi’s consolidation theory, which gives a homogenous diffusion equation with constant coefficients for the evolution of the fluid pressure field. Thus, the one-dimensional initial-boundary value problem for fluid pressure can be written as,
\[
\frac{\partial p}{\partial t} - c \frac{\partial^2 p}{\partial x^2} = 0 \quad x \in [0, L], \ t \geq 0
\]
\[
p = p_0 \quad x = 0, \ t \geq 0
\]
\[
\frac{\partial p}{\partial x} = 0 \quad x = L, \ t \geq 0
\]
\[
p = \frac{(\nu - \nu_u)}{\alpha(1-2\nu)(1-\nu_u)} p_0 \quad x \in (0,L], \ t = 0^+
\]

(4.6)

Here no distinction is made between the current and reference configurations. The resulting pressure field can be integrated to yield the displacement \( u_x \) (see Detournay\textsuperscript{[73]}). The diffusivity coefficient, \( c \), is dependent on the intrinsic permeability, \( \kappa \), the fluid viscosity, \( \nu_f \), the shear modulus of the aggregate, \( \mu^s \), the Biot coefficient, \( \alpha \), and the Poisson’s ratio of the drained and undrained aggregate, \( \nu \) and \( \nu_u \). We note that the use of the symbol \( \nu \) for Poisson’s ratios and for viscosity is unfortunate here, but are left to simply point this out again to emphasize the distinction. The pore pressure field at the instant of loading corresponds to the homogeneous undrained response (i.e. if the material were loaded but fluid was not allowed to enter or leave the system). The reason that this problem is relevant to the phase-field fracture model is that the diffuse phase-field crack surface should act as a permeable boundary. In other words, the solution to (3.1) subject to the following boundary conditions,

\[
\mu = 0 \quad X \leq 0, \ t \geq 0
\]
\[
p = p_0 \quad X \leq 0, \ t \geq 0
\]
\[
w_z = 0 \quad X = L, \ t \geq 0
\]
\[
u_z = 0 \quad X = L, \ t \geq 0
\]

(4.7)

should approximate the solution to (4.6) for \( 0 \leq X \leq L \) and \( t \leq 0 \). Note that in (4.7) the tractions on the crack surface are not specified. Figure 4 shows the results of calculations when the surface is identified explicitly and when the surface is modeled with the phase-field and compares them with the exact solution provided by Detournay. For the sake of consistency the fluid is taken to be incompressible and, as such, the
undrained Poisson ratio is $\nu_u = 0.5$. The other relevant material parameters are taken as $\alpha = 0.45$, $\nu = 0.22$, and $L = 200\ell_0$. On the left plot is the displacement $u_x$ of the surface at $X = 0$ normalized by the final steady state settlement for $t \to \infty$,

$$u_x^\infty = \frac{p_0 L (1 - \alpha) (1 - 2\nu)}{2\mu' (1 - \nu)},$$

(4.8)

and on the right plot is the fluid pressure $p/p_0$ at the impermeable boundary at $X = L$ as each quantity varies over the dimensionless time $\tau = \left(\frac{ct}{4L^2}\right)$.

Figure 4

These plots show the comparison of the phase-field approach (b) to analytical solutions (a) for the time-dependent poroelastic consolidation process. The pore fluid is incompressible. A normal traction is applied to the left boundary or surface ($X = 0$) at time $t = 0^+$ and the surface displacement (c) and pore pressure at the right boundary ($X = L$) (d) are plotted as functions of time. Note that for the phase-field description the left boundary is not a true boundary, but rather is represented by a damaged phase-field. In other words, this can be thought of as the right half of a symmetric pressurized crack. The plots demonstrate excellent agreement with the discrete and analytical solutions to the problem.
The results demonstrate excellent agreement between Biot theory, mixture theory, and the phase-field method developed herein. The only slight discrepancy that can be noticed is that the settlement of the layer at early times is slightly greater for the phase-field formulation. This is due to the fact that the permeability of the aggregate is increased in regions of degradation, and thus, fluid diffusion happens more quickly near the boundary. This behavior only affects the fields in a boundary layer on the order of the fracture surface length scale \( \ell_0 \). In other words, the pressure response far away from the surface is unchanged.

### 4.3 Pressurized Cracks

The third verification of the model acts to demonstrate that the theory is capable of properly describing the evolution of cracks within the system. As seen in the previous section, pressurized fluid induces tractions on the crack faces and acts to open up the fractures. The cracks should experience the proper deformation and must propagate at the appropriate critical conditions. To demonstrate that the theory does indeed recover the aforementioned phenomena, a plane-strain center crack in an infinite, isotropic, impermeable, elastic medium is analyzed. If the crack is subject to a uniform pressure then this problem lends itself to a simple linear elastic fracture mechanics solution with which comparisons can be made. The propagation of the crack for fixed pressure conditions is unstable, so the modeling approach is to set a crack at a desired length by imposing that the phase-field be zero along the desired length of elements. Then, the entire system is subject to a uniform pressure that is increased until the crack begins to propagate. The infinite domain is represented with high fidelity by using a Dirichlet-to-Neumann map\(^{[83,84]}\) on the outermost boundary of the circular finite element mesh. By conducting the study for various imposed crack lengths, the theory can be verified against the linear elastic fracture mechanics solution. According to linear elastic fracture mechanics, the relationship between the critical pressure and the crack length is,

\[
\frac{p_c}{\sigma_0} = \sqrt{\frac{\ell_0}{\pi a}} \tag{4.9}
\]

where \( a \) is the crack half-length and \( \sigma_0 = \sqrt{E'G_c/\ell_0} \). Figure 5 shows the results of these calculations. As expected, the phase-field theory reproduces the LEFM result with remarkable accuracy with the exception of when the crack length is short with respect to the length scale \( \ell_0 \). This is to be expected given that the assumption of a small-scale process zone is no longer valid. The length scale \( \ell_0 \) can be shown to be directly analogous to the process zone size from cohesive zone descriptions of cracks like the
Dugdale-Barenblatt model. As such, discrepancies arise for short cracks where the process zone length scale becomes significant in relation to the crack length \( a \) and this represents a deviation from a regime where linear elastic fracture mechanics is applicable.

![Phase-field method](image)

**Figure 5**
This plot is a comparison of the phase-field method to the linear elastic fracture mechanics solution for the critical pressure values for propagation of a plane-strain center crack imbedded in an infinite medium. For the phase-field method a crack of given length is designated with Dirichlet boundary conditions and the pressure is increased for the entire mesh until the crack begins to grow unstably. The infinite medium is modeled with a Dirichlet-to-Neumann mapping on the boundary elements of a circular mesh. The phase-field method proposed in this work is in excellent agreement with classical linear elastic fracture mechanics predictions.

### 4.4 The Kristianovic-Geerstma-de Klerk Problem
Now we discuss the apparently simple problem of a plane strain hydraulic fracture in an impermeable, elastic, infinite domain subject to the constant rate injection of an incompressible, Newtonian fluid. This geometry is typically termed the KGD fracture geometry after Kristianovic, Geerstma, and de Klerk\[^{10,11}\]. In the last two decades, a group of researchers lead by E. Detournay has revisited this problem, as well as a few other simple geometries like the penny-shaped crack, and conducted state-of-the-art analytical and numerical studies\[^{14-16,85}\]. A concise summary of their results for impermeable rocks can be found in Detournay\[^{14}\]. A particularly notable result in these works is the introduction of a dimensionless parameter comparing the material fracture toughness to the fluid viscosity,
\[ K = 4K_{ic}\left(3\pi^2E^\nu Q\nu^f\right)^{-1/4}. \] (4.10)

This dimensionless toughness can be used to delineate the dominating regime of fracture propagation for the KGD crack, toughness versus viscosity-dominated propagation. Based on the assumption that the crack aperture is very small in the systems of interest and inertial terms may be neglected, the hydraulic fracture problem involves two mechanisms of dissipation that drive the fracture propagation process. In the toughness-dominated regime, where \( K \to \infty \), the energy dissipated by the flow of viscous fluid is small compared to the energy that is dissipated at the crack tip through the creation of new fracture surfaces. In the viscosity-dominated regime, where \( K \to 0 \), the energy dissipated by the flow of viscous fluid is far greater than the dissipation due to the creation of new fracture surfaces. In fact, a primary conclusion of Detournay (2004)\[14] is that when \( K < 1 \) the solution can be accurately approximated by a limit solution constructed on the assumption that the rock has zero toughness, and when \( K > 4 \) the solution may be approximated by a limit solution constructed on the assumption that the injected fluid has zero viscosity. Since no assumptions related to the viscosity or toughness have been made in the construction of this phase-field theory, the model should be able to capture both limiting regimes, and additionally, the transition between them. Figure 6 shows the results of a plane-strain crack subjected to a constant injection rate for \( K = 9.989 \) (toughness dominated) and \( K = 0.919 \) (viscosity dominated). A small initial crack of half-length \( a = 5\ell_0 \) is imbedded in an infinite domain (again using a Dirichlet-to-Neumann map on the outer boundary to model the infinite domain) and the crack is subjected to a constant injection rate \( \dot{m} \) at its center. For slow enough rates, the crack behaves as a toughness dominated crack and for high rates of injection, the crack propagates in the viscosity-dominated regime. The calculations are in very good agreement with the limit solutions developed and outlined by Detournay\[14]. The slight discrepancies may be explained by the fact that the permeability \( \kappa/\nu^f \) in the phase-field calculations is not zero, but very small. Additionally, the location of the crack tip is “blurred” by a Dugdale-Barrenblatt type of process zone at the tip region. The results are also expected to improve with mesh refinement. The toughness-dominated regime is characterized by uniform pressure in the crack, and thus, the results in this regime correlate to the results where a uniform pressure is imposed for the whole domain. On the other hand, the viscosity-dominated regime is characterized by a non-uniform pressure distribution along the length of the crack. Dotted lines on the viscosity-dominated regime plot illustrate how the crack would behave for the prescribed injection rate if the assumption of uniform pressure
were adopted. The difference highlights the necessity of properly modeling the flow within the crack and, in particular, its dependence on crack opening.

![KGD crack geometry](image)

**Figure 6**
These plots show the phase-field simulation comparison to the solutions from Ref. [14] for the 2D KGD crack configuration. (a) Shows results for a case when fracture toughness is the dominant dissipative mechanism \( (K = 9.989) \). (b) Shows results for the case when fluid viscosity is the dominant dissipative mechanism \( (\kappa = 0.919) \). The phase-field method captures both the toughness and viscosity-dominated regimes of propagation accurately. Note that models assuming inviscid fluid behavior cannot capture the viscosity dominated regime and would predict behaviors associated with the dashed line in (b).
A simulation that illustrates the capabilities of the model. There is a line injection of fluid (out of plane direction) into the middle crack at a constant rate. Solutions for the fluid pressure and phase-field fracture parameter are shown for four different time steps, (a) $V/\ell_0^3 = 0.15$, (b) $V/\ell_0^3 = 0.55$, (c) $V/\ell_0^3 = 0.6$, and (d) $V/\ell_0^3 = 1.75$. 

Figure 7
4.5 Multiple Crack Interaction

Here, for illustrative purposes, we show results of a calculation where fluid is injected into a small center crack (half-length \( a = 10\ell_0 \)) that then propagates and merges with two “natural” outer cracks. The surface normal displacements of the outer boundaries are held fixed but fluid exchange is allowed at these surfaces. After merging, fluid fills the natural cracks and they begin to propagate from each end in a deflected direction. Contour plots describing the fluid pressure and crack geometry are shown for four different time steps in Figure 7. The fluid is again assumed to be incompressible, the Biot coefficient is \( \alpha = 0.4 \), and the initial porosity is \( \phi_0 = 0.1 \). As in the consolidation solution, an increase in pore pressure is observed in the undamaged material surrounding the crack. Figure 8 shows the injection pressure as a function of fluid volume injected into the middle crack. The merging of the three cracks corresponds with a drop in the injection pressure as it takes time for the natural cracks to completely open and fill with fluid. Although it is not necessarily geologically relevant in its simplicity, this simulation provides an example of the capabilities of the modeling approach to capture complex behaviors such as crack merging and non-planar propagation.

![Figure 8](image)

**Figure 8**

A plot of the injection pressure versus the volume of fluid injected into the middle crack. The dip in pressure corresponds to the moment that the middle crack merges with the outer cracks. Labels are included that correspond to the snap shots in Figure 8.
5. Summary
This work has outlined the development of a phase-field model for describing the fracturing of a fluid-saturated porous continuum. The governing equations were derived in a general, large deformation framework through the means of fundamental balance laws and Coleman-Noll type procedures. In addition, specific constitutive equations were provided to recover Biot theory in the bulk and incompressible Stokes flow inside cracks. The numerical implementation of these equations was briefly described and several simple solutions were analyzed to verify the expected behavior of the model. With the help of a finite-deformation framework for the governing equations and a mesh-independent viscosity scaling approach, the theory was shown to recover the proper dependence of fluid flow on crack opening displacement. Results were also verified against simple problems in consolidation theory and linear elastic fracture mechanics. Next, the significantly more complex problem of a plane-strain crack subject to uniform fluid injection was investigated. The model was shown to compare favorably to the asymptotic solutions developed by Detournay et al.\cite{detournay2009} for both situations where fracture toughness is the dominant dissipation mechanism and where fluid viscous drag is the dominant dissipation mechanism. Successfully representing both regimes without introducing any regime-dependent constitutive relationships demonstrates the robustness of the modeling approach. Lastly, a model problem that demonstrates the true advantages of a phase-field approach to fracture was described. Crack interactions that resulted in crack path deflection and merging were observed. These phenomena were shown to have a significant impact on fluid injection pressure response. With the confidence established by the favorable comparison to simple analytical solutions, we believe that the phase-field model for fracture can be a powerful tool in understanding many of the more complex behaviors that occur in the hydraulic fracture process.

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**Appendix A**

It is advantageous to numerical computations based on Newton-Raphson type schemes to have smooth tangent stiffness components. Thus, convergence results for the coupled N-R scheme can be improved by adopting a smooth rendition of the viscosity scaling outlined by (4.5),

\[
\nu^f_s = \nu^f \times \min\left(\left(\frac{\lambda_n}{\lambda_n - 1}\right)^3, 1 + \epsilon_v \frac{\ell_0^2}{\kappa}\right).
\]  

(A.1)

A critical stretch ratio \( \Lambda_n > 1 \) can be determined such that,

\[
\left(\frac{\Lambda_n}{\Lambda_n - 1}\right)^3 = 1 + \epsilon_v \frac{\ell_0^2}{\kappa} \Rightarrow \Lambda_n = \left[1 + \epsilon_v \frac{\ell_0^2}{\kappa}\right]^{1/3} \left[1 + \epsilon_v \frac{\ell_0^2}{\kappa}\right]^{1/3} - 1.
\]  

(A.2)

Then, a smooth piece-wise scaling function based on \( \Lambda_n \) can be defined as,

\[
F_v(\lambda_n) \equiv \begin{cases} 
\left(\frac{\Lambda_n}{\Lambda_n - 1}\right)^3 & ; \quad \lambda_n \leq 1 \\
\frac{\lambda_n(\lambda_n - 2) + (1 - 2\Lambda_n)^2}{5 + \lambda_n(\lambda_n - 2) + 4\Lambda_n(\Lambda_n - 2)} \left[\left(\frac{\Lambda_n}{\Lambda_n - 1}\right)^3\right] & ; \quad 1 \leq \lambda_n \leq 2\Lambda_n - 1 \\
\left(\frac{\lambda_n}{\lambda_n - 1}\right)^3 & ; \quad \lambda_n \geq 2\Lambda_n - 1
\end{cases}
\]  

(A.3)

such that a smooth rendition of (4.5) is \( \nu^f_s = \nu^f F_v(\lambda_n) \).
References


