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Mixed formulation of a linearized lubrication fracture model in a poro-elastic medium

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Abstract

We analyze and discretize a mixed formulation for a linearized lubrication fracture model in a poro-elastic medium. The displacement of the medium is expressed in primary variables while the flows in the medium and fracture are written in mixed form, with an additional unknown for the pressure in the fracture. The fracture is treated as a non-planar surface or curve according to the dimension, and the lubrication equation for the flow in the fracture is linearized. The resulting equations are discretized by finite elements adapted to primal variables for the displacement and mixed variables for the flow. Stability and a priori error estimates are derived. A fixed-stress algorithm is proposed for decoupling the computation of the displacement and flow and a numerical experiment is included.

Keywords. poro-elasticity; Biot; lubrication; mixed formulation; finite-elements; fixed stress split algorithm.

1 Introduction

The injection of large volumes of fluids in the subsurface such as during carbon sequestration or during hydraulic fracturing operations can cause geomechanical deformation of the rock mass in the vicinity of the injection well. In addition, recovery predictions from a fractured reservoir are essential for long term production in shale oil and gas fields. Understanding interactions between in-situ stresses, injection fluid pressure and fracture is a difficult and challenging issue because of the complexity of rock properties and physical aspects of rock failure and fracture. In this work, we consider a simplified model for the coupled reservoir-fracture flow which accounts for varying reservoir geometries and complexities including non-planar fractures. Here we utilize different flow models such as Darcy flow and Reynolds' lubrication equation for fractures and reservoir respectively to closely capture the physics. Furthermore, the geomechanics effects have been included by considering Biot's model. An accurate modeling of solid deformations necessitates a better estimation of fluid pressure inside fractures. We model the fractures and reservoirs explicitly, which allows us to capture the flow details and impact of fractures more accurately. The approach presented here is in contrast with existing averaging approaches such as dual and discrete dual porosity models where the effects of fractures are averaged out. The coupled reservoir-fracture flow problem is discretized by a mixed finite element method, because this method is locally mass conservative and the flux values are continuous, see Ingram *et al* [18] and Wheeler *et al* [26]. The pressure degrees of freedom are defined at the grid cell centers, similar to the finite difference scheme widely used in petroleum reservoir simulations. Moreover, our motivation in applying a mixed formulation is that in realistic engineering settings, it is necessary to transport proppant; thus local conservation is essential. The coupled flow and geomechanics model developed, for fractured porous medium, has the following advantages:

1. The fracture flow problem is resolved explicitly resulting in an accurate fracture pressure used as a traction boundary condition for reservoir geomechanics.
2. A physically accurate formulation of fractures and reservoir flow problems is achieved by using different constitutive equations and capillary pressure curves for each of the two domains in the case of multiphase flow.
3. Non-planar fractures can be captured using a coarser mesh (lower computational cost) due non-planar faces of the general hexahedral elements inherent to the discretization.

In this work, we prove existence and uniqueness of the solution of a coupled linearized system with one fracture under fairly weak assumptions on the data. To this date, the analysis of the coupled non-linear system is still an open problem. Our results here represent an extension of a previous article, see [17], in which a continuous Galerkin method for flow was analyzed. In the present situation, switching from a continuous Galerkin scheme to a mixed scheme is not completely straightforward, because the discontinuous approximation of the pressure in the fracture (such as piecewise constants in each element) requires a special analysis in coupling the flow in the fracture with that in the reservoir. This coupling requires the derivation of an inf-sup condition in a norm that is weaker than that used in the mixed form of the exact problem, compare (4.20) and (3.1). Such discrepancy in the norms, that arises from the discontinuity of the pressure, complicates the numerical analysis. The resulting system is then solved by a fixed stress splitting algorithm introduced and analyzed by Mikelić & Wheeler in [22] for a Biot system without fracture, and in Girault *et al* [13] with a fracture. Of course, the numerical experiment reported in this work is applied to the fully non-linear system, where the permeability in the fracture is related to its width.
The Biot system without fracture has been analyzed by a number of authors who established existence, uniqueness, and regularity, see Showalter [25] and references therein, Phillips & Wheeler [23], Girault *et al* [14]. Several articles by Mikelić *et al* (see for instance [4], [9]) treat homogenization of flows through fractured porous media. Another approach consists in treating a fracture as a thin domain in the framework of domain decomposition. We refer the reader to the extensive work of Jaffré, Roberts and co-authors on Darcy flow, see [1,21].
After this introduction, the paper is organized as follows. The modeling equations are described in Section 2. In Section 2.4, the equations are linearized and set into variational formulations. Existence and uniqueness of solutions of the linearized formulation are established in Section 3.3. In Section 4, we propose and analyze a fully discrete scheme: backward Euler in time, continuous Galerkin for elasticity and mixed finite elements for flow, more precisely RT_k on simplices and enhanced BDM on quadrilaterals or hexahedra. The enhanced BDM elements are necessary to guarantee sufficient accuracy in the case of quadrilaterals or hexahedra, which cannot be achieved by RT_k elements. In Section 5 we

present a fixed-stress method as a decoupling computational algorithm. Numerical results are presented in Section 6.1.

1.1 Notation

Let Ω be a bounded domain (open and connected) of $I\!R^d$, where the dimension $d = 2$ or 3, with a Lipschitz continuous boundary $\partial\Omega$, and let Γ be an open subset of $\partial\Omega$ with positive measure. When $d = 3$, we assume that the boundary of Γ is also Lipschitz continuous. Let $\mathfrak{D}(\Omega)$ be the space of all functions that are infinitely differentiable and with compact support in Ω and let $\mathfrak{D}'(\Omega)$ be its dual space, i.e., the space of distributions in Ω. As usual, for $1 \le p < \infty$, we define the Banach space $W^{1,p}(\Omega)$ by

$$W^{1,p}(\Omega) = \{v \in L^p(\Omega)\,;\, \nabla v \in L^p(\Omega)^d\},$$

normed by

$$|v|_{W^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)} \quad , \quad \|v\|_{W^{1,p}(\Omega)} = \big(\|v\|^p_{L^p(\Omega)} + |v|^p_{W^{1,p}(\Omega)}\big)^{\frac{1}{p}},$$

with the usual modification when $p = \infty$. When $p = 2$, $W^{1,2}(\Omega)$ is the classical Hilbert Sobolev space $H^1(\Omega)$. The space of traces of functions of $H^1(\Omega)$ on Γ (or on any Lipschitz curve in $\overline{\Omega}$) is $H^{\frac{1}{2}}(\Gamma)$, which is a proper subspace of $L^2(\Gamma)$. Its dual space is denoted by $H^{-\frac{1}{2}}(\Gamma)$. Several equivalent norms can be used on this space. Here, it is convenient to use the semi-norm and norm, see for example [20]:

$$|v|_{H^{\frac{1}{2}}(\Gamma)} = \Big(\int_\Gamma\int_\Gamma \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^d} d\boldsymbol{x}\, d\boldsymbol{y}\Big)^{\frac{1}{2}} \ , \ \|v\|_{H^{\frac{1}{2}}(\Gamma)} = \big(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{\frac{1}{2}}(\Gamma)}\big)^{\frac{1}{2}}. \tag{1.1}$$

Then we define

$$H^1_0(\Omega) = \{v \in H^1(\Omega)\,;\, v|_{\partial\Omega} = 0\},$$

and more generally

$$H^1_{0,\Gamma}(\Omega) = \{v \in H^1(\Omega)\,;\, v|_\Gamma = 0\}.$$

For a vector $\boldsymbol{v}$ in $I\!R^d$, recall the strain (or symmetric gradient) tensor $\boldsymbol{\varepsilon}(\boldsymbol{v})$:

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big). \tag{1.2}$$

In the sequel we shall use Poincaré's, Korn's, and some trace inequalities. Poincaré's inequality in $H^1_{0,\Gamma}(\Omega)$ reads: There exists a constant $\mathcal{P}_\Gamma$ depending only on Ω and Γ such that

$$\forall v \in H^1_{0,\Gamma}(\Omega)\,,\ \|v\|_{L^2(\Omega)} \le \mathcal{P}_\Gamma |v|_{H^1(\Omega)}. \tag{1.3}$$

Next, recall Korn's first inequality in $H^1_{0,\Gamma}(\Omega)^d$: There exists a constant C_κ depending only on Ω and Γ such that

$$\forall \boldsymbol{v} \in H^1_{0,\Gamma}(\Omega)^d\,,\ |\boldsymbol{v}|_{H^1(\Omega)} \le C_\kappa \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)}. \tag{1.4}$$

We shall use the following trace inequality in $H^1(\Omega)$: There exists a constant C_τ depending only on Ω and Γ such that

$$\forall \varepsilon > 0\,,\ \forall v \in H^1(\Omega)\,,\ \|v\|_{L^2(\Gamma)} \leq \varepsilon \|\nabla\, v\|_{L^2(\Omega)} + \big(\frac{C_\tau}{\varepsilon} + \varepsilon\big)\|v\|_{L^2(\Omega)}. \tag{1.5}$$

This inequality follows for instance from the interpolation inequality (see Brenner & Scott [5])

$$\forall v \in H^1(\Omega)\,,\ \|v\|_{L^2(\Gamma)} \leq C\|v\|^{\frac{1}{2}}_{L^2(\Omega)}\|v\|^{\frac{1}{2}}_{H^1(\Omega)},$$

and Young's inequality. Besides (1.5), by combining (1.3) and (1.4), we immediately derive the alternate trace inequality, with a constant C_D depending only on Ω and Γ:

$$\forall \boldsymbol{v} \in H^1_{0,\Gamma}(\Omega)^d\,,\ \|\boldsymbol{v}\|_{L^2(\Gamma)} \leq C_D\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)}. \tag{1.6}$$

As far as the divergence operator is concerned, we shall use the space

$$H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in L^2(\Omega)^d\,;\ \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla\cdot\boldsymbol{v}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a,b[$ with values in a functional space, say X (cf. [20]). More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number r, $1 \leq r \leq \infty$, we define

$$L^r(a,b;X) = \{f \text{ measurable in }]a,b[\,;\ \int_a^b \|f(t)\|^r_X dt < \infty\},$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|^r_X dt\right)^{\frac{1}{r}},$$

with the usual modification if $r = \infty$. This space is a Banach space if X is a Banach space, and for $r = 2$, it is a Hilbert space if X is a Hilbert space. To simplify, we sometimes denote derivatives with respect to time with a prime and we define for any r, $1 \leq r \leq \infty$,

$$W^{1,r}(a,b;X) = \{f \in L^r(a,b;X)\,;\ f' \in L^r(a,b;X)\}.$$

For any $r \geq 1$, as the functions of $W^{1,r}(a,b;X)$ are continuous with respect to time, we define

$$W^{1,r}_0(a,b;X) = \{f \in W^{1,r}(a,b;X)\,;\ f(a) = f(b) = 0\},$$

and we denote by $W^{-1,r}(a,b;X)$ the dual space of $W^{1,r'}_0(a,b;X)$, where r' is the dual exponent of r, $\frac{1}{r'} + \frac{1}{r} = 1$.

2 Domain and model formulations

Let the reservoir Ω be a bounded domain of $\mathbb{R}^d$ $d = 2$ or 3, with a piecewise smooth Lipschitz boundary $\partial\Omega$ and exterior normal $\boldsymbol{n}$. Let the fracture $\mathcal{C} \Subset \Omega$ be a simple closed piecewise smooth curve with endpoints $\boldsymbol{a}$ and $\boldsymbol{b}$ when $d = 2$ or a simple closed piecewise smooth surface with piecewise smooth Lipschitz boundary $\partial\mathcal{C}$ when $d = 3$. The reservoir contains both the matrix and the fractures; thus the reservoir matrix is $\Omega \setminus \mathcal{C}$.

2.1 Equations in $\Omega \setminus \mathcal{C}$

The displacement of the solid is modeled in $\Omega \setminus \mathcal{C}$ by the quasi-static Biot equations for a linear elastic, homogeneous, isotropic, porous solid saturated with a slightly compressible viscous fluid (see [3]). The constitutive equation for the Cauchy stress tensor $\boldsymbol{\sigma}^{\rm por}$ is

$$\boldsymbol{\sigma}^{\rm por}(\boldsymbol{u}, p) = \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha\, p\, \boldsymbol{I}, \tag{2.1}$$

where $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{u}$ is the solid's displacement, p is the fluid pressure, $\boldsymbol{\sigma}$ is the effective linear elastic stress tensor:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + 2\, G\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{2.2}$$

see (1.2) for the definition of $\boldsymbol{\varepsilon}(\boldsymbol{u})$. Here $\lambda > 0$ and $G > 0$ are the Lamé constants and $\alpha > 0$ is the dimensionless Biot coefficient. Then the balance of linear momentum in the solid reads

$$-\mathrm{div}\, \boldsymbol{\sigma}^{\rm por}(\boldsymbol{u}, p) = \boldsymbol{f} \quad \text{in}\, \Omega \setminus \mathcal{C}, \tag{2.3}$$

where $\boldsymbol{f}$ is a body force, i.e., a gravity loading term. For the fluid, we use a linearized slightly compressible single-phase model. Let p_r be a reference pressure, $\rho_f > 0$ the fluid phase density, $\rho_{f,r} > 0$ a constant reference density relative to p_r, and c_f the fluid compressibility. We consider the simplified case when ρ_f is a linear function of pressure:

$$\rho_f = \rho_{f,r}\big(1 + c_f(p - p_r)\big). \tag{2.4}$$

Next, let φ^* denote the fluid content (or reservoir fluid fraction) of the medium defined by

$$\varphi^* = \varphi(1 + \nabla \cdot \boldsymbol{u}),$$

where φ is the porosity of the medium. For a poroelastic material with small deformation, φ^* can be approximated by

$$\varphi^* = \varphi_0 + \alpha\nabla \cdot \boldsymbol{u} + \frac{1}{M}p, \tag{2.5}$$

where φ_0 is the initial porosity and M a Biot constant. The velocity of the fluid $\boldsymbol{v}^D$ in $\Omega \setminus \mathcal{C}$ obeys Darcy's Law:

$$\boldsymbol{v}^D = -\frac{1}{\mu_f}\boldsymbol{K}\big(\nabla p - \rho_f g \nabla \eta\big), \tag{2.6}$$

where $\boldsymbol{K}$ is the absolute permeability tensor, assumed to be symmetric, bounded, uniformly positive definite in space and constant in time, $\mu_f > 0$ is the constant fluid viscosity, g is the gravitation constant, and η is a signed distance in the vertical direction, variable in space, but constant in time. The fluid mass balance in $\Omega \setminus \mathcal{C}$ reads

$$\frac{\partial}{\partial t}\big(\rho_f \varphi^*\big) + \nabla \cdot (\rho_f \boldsymbol{v}^D) = q, \tag{2.7}$$

where q is a mass source or sink term taking into account injection into or extraction from the reservoir. Let us neglect small quantities by means of the following approximations:

$$\frac{1}{M}(1 + c_f(p - p_r)) \approx \frac{1}{M}\ ,\ c_f\big(\varphi_0 + \alpha \nabla \cdot \boldsymbol{u} + \frac{1}{M} p\big) \approx c_f \varphi_0\ ,\ \rho_{f,r}(1 + c_f(p - p_r))\alpha \approx \rho_{f,r}\alpha,$$
$$\rho_{f,r}(1 + c_f(p - p_r))\boldsymbol{v}^D \approx \rho_{f,r}\boldsymbol{v}^D\ ,\ \rho_{f,r}(1 + c_f(p - p_r))g\nabla\eta \approx \rho_{f,r} g \nabla \eta.$$

Then by substituting (2.4), (2.5), and (2.6) into (2.7), and setting $\tilde{q} = \frac{q}{\rho_{f,r}}$, we obtain

$$\frac{\partial}{\partial t}\Big(\big(\frac{1}{M} + c_f \varphi_0\big)p + \alpha \nabla \cdot \boldsymbol{u}\Big) - \nabla \cdot \Big(\frac{1}{\mu_f}\boldsymbol{K}\big(\nabla p - \rho_{f,r} g \nabla \eta\big)\Big) = \tilde{q}. \tag{2.8}$$

Thus the poro-elastic system we are considering for modeling the displacement $\boldsymbol{u}$ and pressure p in $\Omega \setminus \mathcal{C}$ is governed by (2.1), (2.3) and (2.8).

2.2 Equation in $\mathcal{C}$

For the moment, we assume that the fluid pressure p belongs at least to $H^1(\Omega)$; therefore it has a well defined trace on $\mathcal{C}$. We denote by $\overline{\nabla}$ the surface gradient operator on $\mathcal{C}$. It is the tangential trace of the gradient, that is well defined for functions in $H^1(\Omega)$, cf. for example [15]. The width of the fracture is represented by a non-negative function w defined on $\mathcal{C}$; it is the jump of the displacement $\boldsymbol{u}$ in the normal direction. Since the medium is elastic and the energy is finite, w must be bounded and must vanish on the boundary of the fracture. Then the volumetric flow rate $\mathcal{Q}$ on $\mathcal{C}$ satisfies

$$\mathcal{Q} = -\frac{w^3}{12\mu_f}\big(\overline{\nabla} p_c - \rho_f g \overline{\nabla} \eta\big),$$

and the conservation of mass in the fracture reads

$$\frac{\partial}{\partial t}(\rho_f w) = -\overline{\nabla} \cdot (\rho_f \mathcal{Q}) + q_W - q_L,$$

where q_W is a known injection term into the fracture and q_L is an unknown leakage term from the fracture into the reservoir matrix that guarantees the conservation of mass in the system. Then neglecting again small quantities and setting $\tilde{q}_W = \frac{q_W}{\rho_{f,r}}$, $\tilde{q}_L = \frac{q_L}{\rho_{f,r}}$, we derive the lubrication equation in $\mathcal{C}$:

$$\frac{\partial}{\partial t} w - \overline{\nabla} \cdot \big(\frac{w^3}{12\mu_f}(\overline{\nabla} p_c - \rho_{f,r} g \overline{\nabla} \eta)\big) = \tilde{q}_W - \tilde{q}_L. \tag{2.9}$$

In order to specify the relation between the displacement $\boldsymbol{u}$ of the medium and the width w of the fracture, let us distinguish the two sides (or faces) of $\mathcal{C}$ by the superscripts + and −; a specific choice must be selected but is arbitrary. To simplify the discussion, we use a superscript $\star$ to denote either $+$ or $-$. Let $\Omega^\star$ denote the part of Ω adjacent to $\mathcal{C}^\star$ and let $\boldsymbol{n}^\star$ denote the unit normal vector to $\mathcal{C}$ exterior to $\Omega^\star$, $\star = +, -$. As the fracture is represented by two geometrically coincident surfaces, the normal vectors are related by $\boldsymbol{n}^- = -\boldsymbol{n}^+$. For any function f defined in $\Omega \setminus \mathcal{C}$ that has a trace, let $f^\star$ denote the trace of f on $\mathcal{C}^\star$, $\star = +, -$. Then we define the jump of f on $\mathcal{C}$ in the direction of $\boldsymbol{n}^+$ by

$$[f]_{\mathcal{C}} = f^+ - f^-.$$

The width w is the jump of $\boldsymbol{u} \cdot \boldsymbol{n}^-$ on $\mathcal{C}$:

$$w = -[\boldsymbol{u}]_{\mathcal{C}} \cdot \boldsymbol{n}^+. \tag{2.10}$$

Therefore the only unknown in (2.9) is the leakage term $\tilde{q}_L$.
Summarizing, the equations in $\Omega \setminus \mathcal{C}$ are (2.3) and (2.8), and the equation in $\mathcal{C}$ is (2.9); the corresponding unknowns are $\boldsymbol{u}$, p and $\tilde{q}_L$. These equations are complemented in the next section by interface, boundary and initial conditions.

2.3 Interface, boundary, and initial conditions

Let $\boldsymbol{\tau}_j^\star$, $1 \le j \le d-1$, be a set of orthonormal tangent vectors on $\mathcal{C}^\star$, $\star = +, -$. The balance of the normal traction vector and the conservation of mass yield the interface conditions on each side (or face) of $\mathcal{C}$:

$$(\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p))^\star \boldsymbol{n}^\star = -p_c \boldsymbol{n}^\star \ , \ \star = +, -. \tag{2.11}$$

Then the continuity of p through $\mathcal{C}$ yields

$$[\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p)]_{\mathcal{C}} \boldsymbol{n}^\star = \boldsymbol{0}.$$

Formula (2.11) also implies

$$\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p)\boldsymbol{n}^\star \cdot \boldsymbol{n}^\star = -p_c \ , \ \boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p)\boldsymbol{n}^\star \cdot \boldsymbol{\tau}^\star = 0. \tag{2.12}$$

With the above approximations, the conservation of mass at the interface is expressed as

$$\frac{1}{\mu_f}[\boldsymbol{K}(\nabla p - \rho_{f,r} g \nabla \eta)]_{\mathcal{C}} \cdot \boldsymbol{n}^+ = \tilde{q}_L. \tag{2.13}$$

General conditions on the exterior boundary $\partial\Omega$ of Ω can be prescribed for the poro-elastic system, but to simplify our analysis, we assume that the displacement $\boldsymbol{u}$ vanishes as well as the flux $\boldsymbol{K}(\nabla p - \rho_{f,r} g \nabla \eta) \cdot \boldsymbol{n}$. According to the above hypotheses on the energy and medium, we assume that w is bounded in $\mathcal{C}$ and vanishes on $\partial\mathcal{C}$. Finally, considering that the time derivative in (2.8) acts on $(\frac{1}{M} + c_f \varphi_0)p + \alpha \nabla \cdot \boldsymbol{u}$, we prescribe at initial time (see [25]):

$$\Big((\frac{1}{M} + c_f \varphi_0)p + \alpha \nabla \cdot \boldsymbol{u}\Big)(0) = (\frac{1}{M} + c_f \varphi_0)p_0 + \alpha \nabla \cdot \boldsymbol{u}_0, \tag{2.14}$$

where p_0 is measured and all other initial data are deduced from it: $\boldsymbol{u}_0$ is the displacement associated with p_0 by (2.3) at initial time, the trace of p_0 on $\mathcal{C}$ is denoted by p_c^0, and the initial value of w is deduced from the normal jump of $\boldsymbol{u}_0$ on $\mathcal{C}$. Strictly speaking, the

pressure does not have sufficient regularity in time to define its initial value; therefore p_0 in (2.14) cannot be related to $p(0)$. However, for practical purposes, we shall assume that p is sufficiently smooth, so that $p(0)$ is indeed p_0. Therefore the complete problem statement, called Problem (Q), is:
Find $\boldsymbol{u}$, p, and $\tilde{q}_L$ satisfying (2.1), (2.3), (2.8) in $\Omega \setminus \mathcal{C}$ and (2.9) in $\mathcal{C}$, for all time $t \in]0, T[$, with the interface conditions (2.11) and (2.13) on $\mathcal{C}$ and initial condition (2.14):

$$
(Q) \quad
\begin{gathered}
-\operatorname{div} \boldsymbol{\sigma}^{\mathrm{por}}(\boldsymbol{u}, p) = \boldsymbol{f} \ \text{ in } \Omega \setminus \mathcal{C}, \\
\boldsymbol{\sigma}^{\mathrm{por}}(\boldsymbol{u}, p) = \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha\, p\, \boldsymbol{I} \ \text{ in } \Omega \setminus \mathcal{C}, \\
\frac{\partial}{\partial t}\Big(\big(\tfrac{1}{M} + c_f \varphi_0\big) p + \alpha \nabla \cdot \boldsymbol{u} \Big) - \nabla \cdot \Big(\tfrac{1}{\mu_f} \boldsymbol{K} \nabla \big(p - \rho_{f,r} g \eta\big) \Big) = \tilde{q} \ \text{ in } \Omega \setminus \mathcal{C}, \\
\frac{\partial}{\partial t} w - \overline{\nabla} \cdot \big(\tfrac{w^3}{12 \mu_f} \overline{\nabla} (p - \rho_{f,r} g \eta) \big) = \tilde{q}_W - \tilde{q}_L \ \text{ in } \mathcal{C}, \\
(\boldsymbol{\sigma}^{\mathrm{por}}(\boldsymbol{u}, p))^{\star} \boldsymbol{n}^{\star} = -p|_{\mathcal{C}} \boldsymbol{n}^{\star} \ , \ \star = +, - \ \text{ on } \mathcal{C}, \\
\tfrac{1}{\mu_f} [\boldsymbol{K} \nabla (p - \rho_{f,r} g \eta)]_{\mathcal{C}} \cdot \boldsymbol{n}^{+} = \tilde{q}_L \ \text{ on } \mathcal{C}, \\
\text{where} \quad w = -[\boldsymbol{u}]_{\mathcal{C}} \cdot \boldsymbol{n}^{+}, \\
\text{with the boundary conditions } \boldsymbol{u} = \boldsymbol{0} \ , \ \boldsymbol{K} \nabla (p - \rho_{f,r} g \eta) \cdot \boldsymbol{n} = 0 \ \text{ on } \partial\Omega, \\
\text{and the initial condition at time } t = 0, \\
\Big(\big(\tfrac{1}{M} + c_f \varphi_0\big) p + \alpha \nabla \cdot \boldsymbol{u} \Big)(0) = \big(\tfrac{1}{M} + c_f \varphi_0\big) p_0 + \alpha \nabla \cdot \boldsymbol{u}_0.
\end{gathered}
$$

2.4 Variational formulation

Here, we use a mixed formulation for the flow because it leads to locally conservative schemes.

2.4.1 Spaces

We shall see below that the width function w acts as a weight on the flow velocity in the fracture. For the practical applications we have in mind, w has the following properties when $d = 3$; the statement easily extends to $d = 2$:

Hypothesis 2.1. *The non-negative function w is H^1 in time and is smooth in space away from the fracture's front, i.e., the boundary $\partial\mathcal{C}$. It vanishes on $\partial\mathcal{C}$ and in a neighborhood of any point of $\partial\mathcal{C}$, w is asymptotically of the form:*

$$
w(x, y) \simeq x^{\frac{1}{2}+\varepsilon} f(y), \ \textit{with small } \varepsilon > 0, \tag{2.15}
$$

where y is locally parallel to the fracture's front, x is the distance to $\partial\mathcal{C}$, and f is smooth.

The spaces for our unknowns are described below. To simplify the notation, the spaces related to $\mathcal{C}$ are written $L^2(\mathcal{C})$, $H^{\frac{1}{2}}(\mathcal{C})$, etc, although they are are defined in the interior of $\mathcal{C}$. Regarding $\boldsymbol{x}$, it is convenient (but not fundamental) to introduce an auxiliary partition of Ω into two non-overlapping subdomains Ω^+ and Ω^- with Lipschitz interface Γ containing $\mathcal{C}$, $\Omega^{\star}$ being adjacent to $\mathcal{C}^{\star}$, $\star = +, -$. The precise shape of Γ is not important as long as

Ω^+ and Ω^- are both Lipschitz. Let $\Gamma^\star = \partial\Omega^\star \setminus \Gamma$. For any function f defined in Ω, we extend the star notation to $\Omega^\star$ and set $f^\star = f_{|\Omega^\star}$, $\star = +,-$. Let $W = H^1(\Omega^+ \cup \Omega^-)$ with norm

$$\|v\|_W = \big(\|v^+\|^2_{H^1(\Omega^+)} + \|v^-\|^2_{H^1(\Omega^-)}\big)^{\frac{1}{2}}.$$

The space for the displacement is $L^\infty(0,T;\boldsymbol{V})$, where $\boldsymbol{V}$ a closed subspace of $H^1(\Omega \setminus \mathcal{C})^d$:

$$\boldsymbol{V} = \{\boldsymbol{v} \in W^d\,;\, [\boldsymbol{v}]_{\Gamma\setminus\mathcal{C}} = \boldsymbol{0}, \boldsymbol{v}^\star_{|\Gamma^\star} = \boldsymbol{0}, \star = +,-\}, \tag{2.16}$$

with the norm of W^d:

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} = \big(\sum_{i=1}^d \|v_i\|^2_W\big)^{\frac{1}{2}}. \tag{2.17}$$

As stated previously, the pressure p is essentially in $H^1(\Omega)$ (see more precisely (2.26)), but to set the problem in mixed form, we reduce the regularity of p and take p in $L^\infty(0,T;L^2(\Omega))$. As the functions of $L^2(\Omega)$ have no trace, we introduce an additional variable p_c in the space $L^2(0,T;H^{\frac{1}{2}}(\mathcal{C}))$ that is treated as an unknown variable and is intended to represent the pressure's trace on $\mathcal{C}$.
We associate with the pressure in $\Omega \setminus \mathcal{C}$ an auxiliary velocity $\boldsymbol{z}$ defined by

$$\boldsymbol{z} = -\boldsymbol{K}\nabla(p - \rho_{f,r} g\eta), \tag{2.18}$$

and we associate with the pressure in $\mathcal{C}$, a surface velocity $\boldsymbol{\zeta}$ defined by

$$\boldsymbol{\zeta} = -w^{\frac{3}{2}}\overline{\nabla}(p_c - \rho_{f,r} g\eta). \tag{2.19}$$

The space for the reservoir matrix velocity is $L^2(0,T;\boldsymbol{Z})$, where

$$\boldsymbol{Z} = \{\boldsymbol{q} \in H(\mathrm{div};\Omega^+ \cup \Omega^-)\,;\, [\boldsymbol{q}]\cdot\boldsymbol{n}^+ = 0 \text{ on } \Gamma\setminus\mathcal{C} \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \tag{2.20}$$

normed by

$$\|\boldsymbol{q}\|_{\boldsymbol{Z}} = \big(\|\boldsymbol{q}\|^2_{H(\mathrm{div};\Omega^+)} + \|\boldsymbol{q}\|^2_{H(\mathrm{div};\Omega^-)}\big)^{\frac{1}{2}}. \tag{2.21}$$

Strictly speaking, in (2.20) we should write $[\boldsymbol{q}\cdot\boldsymbol{n}^+] = 0$ on $\Gamma\setminus\mathcal{C}$. However, since $\boldsymbol{n}^+$ does not jump, we abuse the notation and write it as $[\boldsymbol{q}]\cdot\boldsymbol{n}^+ = 0$. Let $\boldsymbol{n}_{\Omega^*}$ be the unit normal to $\partial\Omega^*$, exterior to Ω^*. The trace properties of $H(\mathrm{div};\Omega^*)$ imply that $\boldsymbol{q}\cdot\boldsymbol{n}_{\Omega^*}$ belongs to $H^{-\frac{1}{2}}(\partial\Omega^*)$ which is defined globally (see for example [15]). However, as $\boldsymbol{q}\cdot\boldsymbol{n}$ vanishes on $\partial\Omega$, following the work of Galvis and Sarkis in [10], we can prove first that the jump $[\boldsymbol{q}]\cdot\boldsymbol{n}^+$ belongs to $H^{-\frac{1}{2}}(\Gamma)$, and since it vanishes on $\Gamma\setminus\boldsymbol{C}$ then it is well-defined in $H^{-\frac{1}{2}}(\mathcal{C})$ with continuous dependence on $\|\boldsymbol{q}\|_{\boldsymbol{Z}}$. Therefore $\boldsymbol{Z}$ is a closed subspace of $H(\mathrm{div};\Omega^+\cup\Omega^-)$ and of $H(\mathrm{div};\Omega\setminus\mathcal{C})$.
The space for the velocity in the fracture is $L^2(0,T;\boldsymbol{Z}_{\mathcal{C}})$, where

$$\boldsymbol{Z}_{\mathcal{C}} = \{\boldsymbol{q}_c \in L^2(\mathcal{C})^{d-1}\,;\, \overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{q}_c) \in H^{-\frac{1}{2}}(\mathcal{C})\}, \tag{2.22}$$

equipped with the graph norm

$$\|\boldsymbol{q}_c\|_{\boldsymbol{Z}_{\mathcal{C}}} = \big(\|\boldsymbol{q}_c\|^2_{L^2(\mathcal{C})} + \|\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{q}_c)\|^2_{H^{-\frac{1}{2}}(\mathcal{C})}\big)^{\frac{1}{2}}. \tag{2.23}$$

This space is closely related to the pressure's space in the fracture introduced in [17]):

$$H^1_w(\mathcal{C}) = \{z \in H^{\frac{1}{2}}(\mathcal{C})\,;\, w^{\frac{3}{2}}\overline{\nabla}\, z \in L^2(\mathcal{C})^{d-1}\}, \tag{2.24}$$

equipped with the norm

$$\|z\|_{H^1_w(\mathcal{C})} = \big(\|z\|^2_{H^{\frac{1}{2}}(\mathcal{C})} + \|w^{\frac{3}{2}}\overline{\nabla}\, z\|^2_{L^2(\mathcal{C})}\big)^{\frac{1}{2}}, \tag{2.25}$$

so that p belongs to

$$Q = \{q \in H^1(\Omega)\,;\, q|_{\mathcal{C}} \in H^1_w(\mathcal{C})\}\,. \tag{2.26}$$

Remark 2.1. Strictly speaking, the space $\boldsymbol{Z}_{\mathcal{C}}$ does not correspond to a standard mixed space for the velocity in the fracture since the divergence of its functions is in $H^{-\frac{1}{2}}(\mathcal{C})$ instead of $L^2(\mathcal{C})$. We cannot prescribe this last regularity because the leakage term $\tilde{q}_L$ is not a data: it is the jump in the normal fluxes, which in general cannot be expected to be in $L^2(\mathcal{C})$. Thus the pressure p_c in the fracture must be taken in $H^{\frac{1}{2}}(\mathcal{C})$. This extra regularity will be relaxed in the numerical applications because the discrete jump in the normal fluxes is always in $L^2(\mathcal{C})$. □

Recall the properties of $H^1_w(\mathcal{C})$ established in [17]:

Theorem 2.1. *Under Hypothesis 2.1, $H^1_w(\mathcal{C})$ is a separable Hilbert space, $W^{1,\infty}(\mathcal{C})$ is dense in $H^1_w(\mathcal{C})$, and the following Green formula holds for all θ in $H^1_w(\mathcal{C})$ such that $\overline{\nabla}\cdot(w^3\overline{\nabla}\,\theta)$ belongs to $H^{-\frac{1}{2}}(\mathcal{C})$:*

$$\forall \lambda \in H^1_w(\mathcal{C})\,,\; -\langle \overline{\nabla}\cdot(w^3\overline{\nabla}\,\theta), \lambda\rangle_{\mathcal{C}} = \int_{\mathcal{C}} w^{\frac{3}{2}}\overline{\nabla}\,\theta\cdot w^{\frac{3}{2}}\overline{\nabla}\,\lambda. \tag{2.27}$$

The notation $\langle\cdot,\cdot\rangle_{\mathcal{C}}$ stands for a duality pairing on $\mathcal{C}$. With this result, we can prove that $\boldsymbol{Z}_{\mathcal{C}}$ has the following properties.

Proposition 2.1. *Let w belong to $L^\infty(\mathcal{C})$. Then $\boldsymbol{Z}_{\mathcal{C}}$ is a separable Hilbert space. Moreover, if w satisfies Hypothesis 2.1, the following Green formula holds for all θ_c in $H^1_w(\mathcal{C})$:*

$$\forall \boldsymbol{q}_c \in \boldsymbol{Z}_{\mathcal{C}}\,,\; -\int_{\mathcal{C}} w^{\frac{3}{2}}\overline{\nabla}\,\theta_c\cdot\boldsymbol{q}_c = \langle \theta_c, \overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{q}_c)\rangle_{\mathcal{C}}. \tag{2.28}$$

Proof. To show that $\boldsymbol{Z}_{\mathcal{C}}$ is a Hilbert space, it suffices to prove that it is complete. Let $(\boldsymbol{z}_n)_{n\geq 1}$ be a Cauchy sequence of functions in $\boldsymbol{Z}_{\mathcal{C}}$. Then there exists a function $\boldsymbol{z}$ in $L^2(\mathcal{C})^{d-1}$ and a function $v \in H^{-\frac{1}{2}}(\mathcal{C})$ such that

$$\lim_{n\to\infty}\boldsymbol{z}_n = \boldsymbol{z} \;\text{ in } L^2(\mathcal{C})^{d-1} \quad , \quad \lim_{n\to\infty}\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{z}_n) = v \;\text{ in } H^{-\frac{1}{2}}(\mathcal{C}).$$

In order to prove that $v = \overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{z})$, we take any φ in $H^1_0(\mathcal{C})$. Then

$$\langle\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{z}_n), \varphi\rangle_{\mathcal{C}} = -\int_{\mathcal{C}} w^{\frac{3}{2}}\boldsymbol{z}_n\cdot\overline{\nabla}\,\varphi = -\int_{\mathcal{C}}\boldsymbol{z}_n\cdot(w^{\frac{3}{2}}\overline{\nabla}\,\varphi).$$

Passing to the limit in the first and last term, we obtain

$$\langle v, \varphi \rangle_{\mathcal{C}} = -\int_{\mathcal{C}} \boldsymbol{z} \cdot (w^{\frac{3}{2}} \overline{\nabla} \varphi) = \langle \overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{z}), \varphi \rangle_{\mathcal{C}},$$

whence $v = \overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{z})$.

Proving the separability of $\boldsymbol{Z}_{\mathcal{C}}$ is fairly classical. Let $E = L^2(\mathcal{C})^{d-1} \times H^{-\frac{1}{2}}(\mathcal{C})$ normed by

$$\|v = (\boldsymbol{v}_1, v_2)\|_E = \big(\|\boldsymbol{v}_1\|^2_{L^2(\mathcal{C})} + \|v_2\|_{H^{-\frac{1}{2}}(\mathcal{C})}\big)^{\frac{1}{2}},$$

and let Φ be the mapping: $\boldsymbol{Z}_{\mathcal{C}} \mapsto E$ defined by

$$\forall \boldsymbol{z} \in \boldsymbol{Z}_{\mathcal{C}}\,,\ \Phi(\boldsymbol{z}) = \big(\boldsymbol{z}, \overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{z})\big).$$

Then Φ is an isometry and the argument of the completeness proof above yields that the range of Φ, $\mathcal{R}(\Phi)$, is closed in E. Since E is separable, so is $\mathcal{R}(\Phi)$. Then the separability of $\boldsymbol{Z}_{\mathcal{C}}$ follows from the fact that it is isometrically isomorphic to $\mathcal{R}(\Phi)$.

To prove Green's formula (2.28), we use the density of $W^{1,\infty}(\mathcal{C})$ into $H^1_w(\mathcal{C})$ stated in Theorem 2.1: Let $(p_n)_{n\geq 1}$ be a sequence of functions of $W^{1,\infty}(\mathcal{C})$ that tend to θ_c in $H^1_w(\mathcal{C})$. Then

$$\forall \boldsymbol{q}_c \in \boldsymbol{Z}_{\mathcal{C}}\,,\ -\int_{\mathcal{C}} w^{\frac{3}{2}} \overline{\nabla}\, p_n \cdot \boldsymbol{q}_c = -\int_{\mathcal{C}} \overline{\nabla}(w^{\frac{3}{2}} p_n) \cdot \boldsymbol{q}_c + \int_{\mathcal{C}} p_n \overline{\nabla}(w^{\frac{3}{2}}) \cdot \boldsymbol{q}_c.$$

Since $w^{\frac{3}{2}} p_n$ belongs to $H^1_0(\mathcal{C})$, we can write

$$-\int_{\mathcal{C}} \overline{\nabla}(w^{\frac{3}{2}} p_n) \cdot \boldsymbol{q}_c = \langle w^{\frac{3}{2}} p_n, \overline{\nabla} \cdot \boldsymbol{q}_c \rangle_{\mathcal{C}} = \langle p_n, w^{\frac{3}{2}} \overline{\nabla} \cdot \boldsymbol{q}_c \rangle_{\mathcal{C}}.$$

Hence

$$-\int_{\mathcal{C}} w^{\frac{3}{2}} \overline{\nabla}\, p_n \cdot \boldsymbol{q}_c = \langle p_n, w^{\frac{3}{2}} \overline{\nabla} \cdot \boldsymbol{q}_c + \overline{\nabla}(w^{\frac{3}{2}}) \cdot \boldsymbol{q}_c \rangle_{\mathcal{C}} = \langle p_n, \overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{q}_c) \rangle_{\mathcal{C}},$$

and (2.28) follows by letting n tend to infinity in the first and last term above. □

Finally, in view of the jump and boundary conditions of Problem (Q), we see that $\boldsymbol{z}$ must satisfy the essential jump condition:

$$\frac{1}{\mu_f} [\boldsymbol{z}]_{\mathcal{C}} \cdot \boldsymbol{n}^+ = -\tilde{q}_L \text{ on } \mathcal{C}. \tag{2.29}$$

Consequently, the leakage term is in $L^2(0, T; H^{-\frac{1}{2}}(\mathcal{C}))$ and can be eliminated by substituting (2.29) into the lubrication equation (2.9).

2.4.2 Mixed formulation

To simplify, we denote the scalar products in space by parentheses; if the domain of integration is not indicated, then it is understood that the integrals are taken over $\Omega^+ \cup \Omega^-$. The mixed variational formulation of Problem (Q) reads: For given $\boldsymbol{f} \in L^2((\Omega \setminus \mathcal{C}) \times]0, T[)^d$,

$\tilde{q} \in L^2(\Omega\times]0,T[)$, and $\tilde{q}_W \in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, find $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V})$, $p \in L^\infty(0,T;L^2(\Omega))$, $p_c \in L^2(0,T;H^{\frac{1}{2}}(\mathcal{C}))$, $\boldsymbol{z} \in L^2(0,T;\boldsymbol{Z})$, and $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{Z}_\mathcal{C})$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{V}\,,\; 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})\big) + \lambda\big(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v}\big) - \alpha\big(p,\nabla\cdot\boldsymbol{v}\big) + \big(p_c,[\boldsymbol{v}]_\mathcal{C}\cdot\boldsymbol{n}^+\big)_\mathcal{C} = \big(\boldsymbol{f},\boldsymbol{v}\big)\,, \quad (2.30)$$

$$\forall \theta \in L^2(\Omega)\,,\; \Big(\frac{\partial}{\partial t}\big((\frac{1}{M}+c_f\varphi_0)p + \alpha\nabla\cdot\boldsymbol{u}\big),\theta\Big) + \frac{1}{\mu_f}\big(\nabla\cdot\boldsymbol{z},\theta\big) = \big(\tilde{q},\theta\big), \quad (2.31)$$

$$\begin{aligned}\forall \theta_c \in H^{\frac{1}{2}}(\mathcal{C})\,,\; -\langle\frac{\partial}{\partial t}[\boldsymbol{u}]_\mathcal{C}\cdot\boldsymbol{n}^+,\theta_c\rangle_\mathcal{C} + \frac{1}{12\mu_f}\langle\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{\zeta}),\theta_c\rangle_\mathcal{C} - \frac{1}{\mu_f}\langle[\boldsymbol{z}]_\mathcal{C}\cdot\boldsymbol{n}^+,\theta_c\rangle_\mathcal{C} \\ = \langle\tilde{q}_W,\theta_c\rangle_\mathcal{C},\end{aligned} \quad (2.32)$$

$$\forall \boldsymbol{q} \in \boldsymbol{Z}\,,\; \big(\boldsymbol{K}^{-1}\boldsymbol{z},\boldsymbol{q}\big) = (p,\nabla\cdot\boldsymbol{q}) - \langle p_c,[\boldsymbol{q}]_\mathcal{C}\cdot\boldsymbol{n}^+\rangle_\mathcal{C} + \big(\nabla(\rho_{f,r}g\eta),\boldsymbol{q}\big), \quad (2.33)$$

$$\forall \boldsymbol{q}_c \in \boldsymbol{Z}_\mathcal{C}\,,\; \big(\boldsymbol{\zeta},\boldsymbol{q}_c\big)_\mathcal{C} = \langle p_c,\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{q}_c)\rangle_\mathcal{C} + \big(w^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r}g\eta),\boldsymbol{q}_c\big)_\mathcal{C}, \quad (2.34)$$

subject to the initial condition (2.14):

$$\Big((\frac{1}{M}+c_f\varphi_0)p + \alpha\nabla\cdot\boldsymbol{u}\Big)(0) = \big(\frac{1}{M}+c_f\varphi_0\big)p_0 + \alpha\nabla\cdot\boldsymbol{u}_0.$$

From the assumptions on the data and the choice of spaces for the solution we infer that

$$\frac{\partial}{\partial t}\big((\frac{1}{M}+c_f\varphi_0)p+\alpha\nabla\cdot\boldsymbol{u}\big) \in L^2((\Omega\setminus\mathcal{C})\times]0,T[)\,,\; \frac{\partial}{\partial t}\big([\boldsymbol{u}]_\mathcal{C}\cdot\boldsymbol{n}^+\big) \in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C})). \quad (2.35)$$

Thus, the first part of (2.35) implies that the initial condition (2.14) is meaningful.
We have the following equivalence result.

Theorem 2.2. *Let* $\boldsymbol{f} \in L^2((\Omega\setminus\mathcal{C})\times]0,T[)^d$, $\tilde{q} \in L^2(\Omega\times]0,T[)$, $\tilde{q}_W \in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, *and assume Hypothesis 2.1 holds. Suppose that Problem (Q) has a solution with the following regularity:* $p \in L^\infty(0,T;Q)$, $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V})$, $\tilde{q}_L \in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, *and such that* (2.35) *holds. Then by defining* $\boldsymbol{z}$ *and* $\boldsymbol{\zeta}$ *through* (2.18) *and* (2.19) *respectively, and by setting* p_c *the trace of* p *on* $\mathcal{C}$, *this solution also satisfies* (2.30)–(2.34) *and* (2.14). *Conversely, any solution of the mixed formulation* (2.30)–(2.34) *and* (2.14) *with* w *and* $\tilde{q}_L$ *defined respectively by* (2.10) *and* (2.29), *also solves Problem (Q).*

Proof. Consider first the flow equations in (Q). We set $p_c = p|_\mathcal{C}$ and we have in particular $p_c \in L^2(0,T;H^1_w(\mathcal{C}))$. From the assumptions on the time derivative of p and $\boldsymbol{u}$ and the regularity of $\tilde{q}$, we infer from the definition (2.18) of $\boldsymbol{z}$ and the third line of (Q) that $\boldsymbol{z}$ belongs to $H(\text{div};\Omega\setminus\mathcal{C})$, and thus $\boldsymbol{z} \in L^2(0,T;\boldsymbol{Z})$, owing to the boundary conditions in the next to last line of (Q). Then the third line of (Q) gives (2.31). Similarly, the definition of w and the assumption (2.35) on its time derivative, the assumption on $\tilde{q}_W$ and $\tilde{q}_L$ and its formula (2.29), and the fourth and seventh lines of (Q) imply that $\overline{\nabla}\cdot(w^3\overline{\nabla}\,p_c)$ belongs to $L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$. Then we infer from the trace space $H^1_w(\mathcal{C})$ for p on $\mathcal{C}$ and the definition (2.19) of $\boldsymbol{\zeta}$ that $\boldsymbol{\zeta}$ belongs to $L^2(0,T;\boldsymbol{Z}_\mathcal{C})$ and the fourth, sixth, and seventh lines of (Q) yield (2.32). Next, we turn to the elasticity equation. The assumptions on $\boldsymbol{f}$, $\boldsymbol{u}$ and p imply that each row of $\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u},p)$ belongs to $L^2(0,T;H(\text{div};\Omega^\star))$, thus implying that the normal trace of $\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u},p)$ on $\partial\Omega^\star$ is well defined, $\star = +,-$. Thus we can take the scalar product of the first line of (Q) with $\boldsymbol{v}$ in $\boldsymbol{V}$, apply Green's formula in $\Omega^\star$, see for instance [15], and it remains to recover the boundary term in (2.30). We know that the normal trace of $\boldsymbol{\sigma}^{\rm por}(\boldsymbol{u},p)$ is continuous through $\Gamma \setminus \mathcal{C}$. More precisely, each row of $\boldsymbol{\sigma}^{\rm por}(\boldsymbol{u},p)$ belongs to $L^2(0,T;V_{\rm div})$, where

$$V_{\rm div} = \{\boldsymbol{v} \in L^2(\Omega)^d\,;\, \nabla\cdot\boldsymbol{v} \in L^2(\Omega^\star), \star = +,-, [\boldsymbol{v}]_{\Gamma\setminus\mathcal{C}}\cdot\boldsymbol{n}^+ = 0\}.$$

Hence the fifth line of (Q) is meaningful. Furthermore, since p_c belongs to $L^2(0,T;H^{\frac{1}{2}}(\mathcal{C}))$, the product $p_c\boldsymbol{n}$ is in $L^2(0,T;L^r(\mathcal{C})^d)$ for any real $r \geq 1$ when $d=2$ and $r \in [1,4]$ when $d=3$. Therefore $\boldsymbol{\sigma}^{\rm por}(\boldsymbol{u},p)\boldsymbol{n}|_{\mathcal{C}}$ belongs to $L^2(0,T;L^r(\mathcal{C})^d)$, and consequently the boundary term in (2.30) reduces to an integral on $\mathcal{C}$, as $\boldsymbol{v}$ vanishes on $\Gamma^\star$. This justifies the derivation of (2.30). Finally, we consider the velocity equations. An application of the usual Green formula in (2.18) yields (2.33), and (2.34) follows from (2.19) and (2.28).
Conversely, consider a solution of (2.30)–(2.34) starting from (2.14). By choosing $\boldsymbol{q} \in \mathfrak{D}(\Omega^*)^d$ in (2.33), $* = +,-$, we obtain

$$\boldsymbol{z} = -\boldsymbol{K}\nabla(p - \rho_{f,r}g\eta) \quad \text{in } \Omega^*, * = +,-, \tag{2.36}$$

which implies that p belongs to $H^1(\Omega^*)$. Next, by taking the scalar product of (2.36) with $\boldsymbol{q} \in H^1_0(\Omega)^d$ and applying the usual Green formula, we derive

$$\forall \boldsymbol{q} \in H^1_0(\Omega)^d\,,\; (\boldsymbol{K}^{-1}\boldsymbol{z},\boldsymbol{q}) = (p,\nabla\cdot\boldsymbol{q}) - \int_\Gamma [p]_\Gamma \boldsymbol{q}\cdot\boldsymbol{n}^+ + (\nabla(\rho_{f,r}g\eta),\boldsymbol{q}). \tag{2.37}$$

Then comparing with (2.33), we infer that

$$\forall \boldsymbol{q} \in H^1_0(\Omega)^d\,,\; \int_\Gamma [p]_\Gamma \boldsymbol{q}\cdot\boldsymbol{n}^+ = 0,$$

whence $[p]_\Gamma = 0$, therefore $p \in H^1(\Omega)$ and (2.37) reduces to

$$\forall \boldsymbol{q} \in H^1_0(\Omega)^d\,,\; (\boldsymbol{K}^{-1}\boldsymbol{z},\boldsymbol{q}) = -(\nabla(p - \rho_{f,r}g\eta),\boldsymbol{q}),$$

thus implying (2.18). Then, by substituting (2.18) into (2.31), we recover the third line of (Q). Next, by taking the scalar product of (2.36) with $\boldsymbol{q} \in H^1(\Omega^*)^d$, $* = +,-$, $\boldsymbol{q} = \boldsymbol{0}$ on $\partial\Omega$, applying Green's formula and comparing with (2.33), we obtain:

$$\int_{\mathcal{C}} p[\boldsymbol{q}]_{\mathcal{C}}\cdot\boldsymbol{n}^+ = \int_{\mathcal{C}} p_c[\boldsymbol{q}]_{\mathcal{C}}\cdot\boldsymbol{n}^+.$$

Hence $p_c = p|_{\mathcal{C}}$. Now, from (2.30), we derive as above the first two lines of (Q) in Ω^*. The essential homogeneous Dirichlet boundary condition is included in the space $\boldsymbol{V}$. To recover the interface condition, take the scalar product of this first line with $\boldsymbol{v} \in V$ such that $\boldsymbol{v}^- = \boldsymbol{0}$, $\boldsymbol{v}$ sufficiently smooth in Ω^+, not zero on $\mathcal{C}$, and apply Green's formula. Comparing with (2.30), this gives

$$\langle \boldsymbol{\sigma}^{\rm por}(\boldsymbol{u}^+,p)\boldsymbol{n}^+,\boldsymbol{v}^+\rangle_{\mathcal{C}} = -\int_{\mathcal{C}} p_c\boldsymbol{n}^+\cdot\boldsymbol{v}^+.$$

With the same argument in Ω^-, we recover on one hand the fifth line of (Q) and on the other hand the first two lines in $\Omega\setminus\mathcal{C}$. Finally, we define $\tilde{q}_L$ by (2.13) and w by (2.10). Then (2.34) yields (2.19) in the sense of distributions on $\mathcal{C}$ and (2.32) implies the fourth line of (Q). □

In the sequel, we suppose that the assumptions of Theorem 2.2 hold.

3 Stability and existence of the mixed formulation's solution

From now on we restrict our study to a linearized version of the mixed problem where the factor $w^{\frac{3}{2}}$ in (2.32) and (2.34) is assumed to be known. This would be the case in a time-stepping algorithm, where w is taken at the previous time step.
The geometrical setting is the one described in Section 2. The estimates derived in this section are a priori in the sense that they are obtained under the assumption that the mixed problem has a solution.

3.1 A first set of a priori estimates

We derive here a set of a priori estimates under basic regularity assumptions on the solution. Albeit basic, these estimates cannot be derived without an inf-sup condition on the pressure p_c in the fracture.

3.1.1 An inf-sup condition for p_c

Lemma 3.1. *There exists a constant $\beta > 0$ such that*

$$\forall p_c \in H^{\frac{1}{2}}(\mathcal{C}) \,,\; \sup_{\boldsymbol{q}\in \boldsymbol{Z}} \frac{\langle p_c, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+\rangle_{\mathcal{C}}}{\|\boldsymbol{q}\|_{\boldsymbol{Z}}} \geq \beta \|p_c\|_{H^{\frac{1}{2}}(\mathcal{C})}. \tag{3.1}$$

Proof. By duality we write:

$$\|p_c\|_{H^{\frac{1}{2}}(\mathcal{C})} = \sup_{g\in H^{-\frac{1}{2}}(\mathcal{C})} \frac{\langle p_c, g\rangle_{\mathcal{C}}}{\|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}},$$

and the proof relies on relating g to a suitable function $\boldsymbol{q}$ in $\boldsymbol{Z}$. We proceed in two steps.
1) We propose to extend g by zero to $\partial\Omega^+$. For this, let $E(g)$ be defined by

$$\forall \varphi \in H^{\frac{1}{2}}(\partial\Omega^+) \,,\; \langle E(g), \varphi\rangle_{\partial\Omega^+} = \langle g, \varphi\rangle_{\mathcal{C}}.$$

Then

$$|\langle E(g), \varphi\rangle_{\partial\Omega^+}| \leq \|g\|_{H^{-\frac{1}{2}}(\mathcal{C})} \|\varphi\|_{H^{\frac{1}{2}}(\mathcal{C})} \leq \|g\|_{H^{-\frac{1}{2}}(\mathcal{C})} \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega^+)}.$$

Thus $E(g) \in H^{-\frac{1}{2}}(\partial\Omega^+)$ and

$$\|E(g)\|_{H^{-\frac{1}{2}}(\partial\Omega^+)} \leq \|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}.$$

Moreover, for all $\varphi \in H^{\frac{1}{2}}(\partial\Omega^+)$ that vanish on $\mathcal{C}$ (i.e. $\varphi \in H^{\frac{1}{2}}_{00}(\partial\Omega^+ \setminus \mathcal{C})$, we have $\langle E(g), \varphi\rangle_{\partial\Omega^+} = 0$. This means that $E(g) = 0$ on $\partial\Omega^+ \setminus \mathcal{C}$. Finally, for all $\varphi \in H^{\frac{1}{2}}_{00}(\mathcal{C})$, we have

$$\langle E(g), \varphi\rangle_{\mathcal{C}} = \langle E(g), \varphi\rangle_{\partial\Omega^+} = \langle g, \varphi\rangle_{\mathcal{C}}.$$

Hence $E(g)$ is the desired extension.
2) As $E(g)$ belongs to $H^{-\frac{1}{2}}(\partial\Omega^+)$, there exists $\boldsymbol{q}^+$ in $H(\mathrm{div};\Omega^+)$ such that (see for instance [15])

$$\boldsymbol{q}^+ \cdot \boldsymbol{n}^+ = E(g) \quad \text{on } \partial\Omega^+,$$

and, with a constant C that depends only on Ω^+ and $\mathcal{C}$

$$\|\boldsymbol{q}^+\|_{H(\mathrm{div};\Omega^+)} \le C\|E(g)\|_{H^{-\frac{1}{2}}(\partial\Omega^+)} \le C\|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}.$$

Furthermore

$$\boldsymbol{q}^+ \cdot \boldsymbol{n}^+ = 0 \quad \text{on } \partial\Omega^+ \setminus \mathcal{C}.$$

Now, we choose $\boldsymbol{q}^- = \boldsymbol{0}$ in Ω^-. Then $\boldsymbol{q}$ is in $\boldsymbol{Z}$,

$$[\boldsymbol{q}] \cdot \boldsymbol{n}^+ = g \quad \text{on } \mathcal{C},$$

and

$$\|\boldsymbol{q}\|_{\boldsymbol{Z}} \le C\|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}. \tag{3.2}$$

Thus

$$\|p_c\|_{H^{\frac{1}{2}}(\mathcal{C})} = \sup_{g\in H^{-\frac{1}{2}}(\mathcal{C})} \frac{\langle p_c, E(g)\rangle_{\mathcal{C}}}{\|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}} = \sup_{g\in H^{-\frac{1}{2}}(\mathcal{C})} \frac{\langle p_c, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+\rangle_{\mathcal{C}}}{\|g\|_{H^{-\frac{1}{2}}(\mathcal{C})}} \le C \sup_{\boldsymbol{q}\in\boldsymbol{Z}} \frac{\langle p_c, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+\rangle_{\mathcal{C}}}{\|\boldsymbol{q}\|_{\boldsymbol{Z}}},$$

and this yields (3.1), with $\beta = \frac{1}{C}$. □

Note that the bilinear form $\langle p_c, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+\rangle_{\mathcal{C}}$ is continuous on the product space $H^{\frac{1}{2}}(\mathcal{C}) \times \boldsymbol{Z}$. We associate with this form the operator B and its dual operator B' defined by

$$\forall \boldsymbol{q} \in \boldsymbol{Z}, \forall p_c \in H^{\frac{1}{2}}(\mathcal{C})\,,\ \langle B\,\boldsymbol{q}, p_c\rangle = \langle p_c, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+\rangle_{\mathcal{C}} = \langle \boldsymbol{q}, B' p_c\rangle.$$

The kernel of B in $\boldsymbol{Z}$ is the space

$$H_0(\mathrm{div};\Omega) = \{\boldsymbol{q} \in H(\mathrm{div};\Omega)\,;\ \boldsymbol{q}\cdot\boldsymbol{n} = 0 \ \text{ on } \partial\Omega\}.$$

Indeed, $B\,\boldsymbol{q} = 0$ is equivalent to $[\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+ = 0$, which means that $\boldsymbol{q}\cdot\boldsymbol{n}^+$ does not jump on $\mathcal{C}$. Then the inf-sup condition (3.1) has the following consequence.

Corollary 3.1. *Let $\boldsymbol{z} \in \boldsymbol{Z}$ and $p \in L^2(\Omega)$ be such that*

$$\forall \boldsymbol{q} \in H_0(\mathrm{div};\Omega)\,,\ (p, \nabla\cdot\boldsymbol{q}) - \big(\boldsymbol{K}^{-1}\boldsymbol{z}, \boldsymbol{q}\big) + \big(\nabla(\rho_{f,r} g\eta), \boldsymbol{q}\big) = 0. \tag{3.3}$$

Then there exists a unique p_c in $H^{\frac{1}{2}}(\mathcal{C})$ such that p_c, p, and $\boldsymbol{z}$ satisfy (2.33) *and*

$$\|p_c\|_{H^{\frac{1}{2}}(\mathcal{C})} \le \frac{1}{\beta}\Big(\big(\|p\|^2_{L^2(\Omega)} + |\rho_{f,r} g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}} + \|\boldsymbol{K}^{-1}\boldsymbol{z}\|_{L^2(\Omega\setminus\mathcal{C})}\Big), \tag{3.4}$$

where β is the constant of (3.1).

Proof. It stems from (3.1) and the Babuška-Brezzi's theory (see for instance [2], [6] or [15]) that the mapping B' is an isomorphism from $H^{\frac{1}{2}}(\mathcal{C})$ onto the subspace of $\boldsymbol{Z}'$:

$$\{\boldsymbol{\ell} \in \boldsymbol{Z}'\,;\ \forall \boldsymbol{w} \in H_0(\mathrm{div};\Omega)\,,\ \langle \boldsymbol{\ell}, \boldsymbol{w}\rangle = 0\}.$$

Now, for given $\boldsymbol{z} \in \boldsymbol{Z}$ and $p \in L^2(\Omega)$ satisfying (3.3), let $\boldsymbol{\ell}$ denote the mapping

$$\forall \boldsymbol{q} \in \boldsymbol{Z}\,,\ \langle \boldsymbol{\ell}, \boldsymbol{q}\rangle = (p, \nabla\cdot\boldsymbol{q}) - \big(\boldsymbol{K}^{-1}\boldsymbol{z}, \boldsymbol{q}\big) + \big(\nabla(\rho_{f,r} g \eta), \boldsymbol{q}\big).$$

Clearly, $\boldsymbol{\ell}$ belongs to $\boldsymbol{Z}'$ and by assumption $\boldsymbol{\ell}$ vanishes on $H_0(\mathrm{div};\Omega)$. Therefore there exists a unique p_c in $H^{\frac{1}{2}}(\mathcal{C})$ such that

$$\forall \boldsymbol{q} \in \boldsymbol{Z}\,,\ \langle B' p_c, \boldsymbol{q}\rangle = \langle \boldsymbol{\ell}, \boldsymbol{q}\rangle,$$

i.e.

$$\langle p_c, [\boldsymbol{q}]_{\mathcal{C}}\cdot\boldsymbol{n}^+\rangle_{\mathcal{C}} = (p, \nabla\cdot\boldsymbol{q}) - \big(\boldsymbol{K}^{-1}\boldsymbol{z}, \boldsymbol{q}\big) + \big(\nabla(\rho_{f,r} g \eta), \boldsymbol{q}\big),$$

and the estimate (3.4) follows easily from this and (3.1). □

3.1.2 Stability estimates

As specified at the beginning of this section, we assume that the mixed problem has a sufficiently smooth solution p, $\boldsymbol{u}$, p_c, $\boldsymbol{z}$, and $\boldsymbol{\zeta}$; its precise regularity will be stated further on. A preliminary stability equality is derived by testing (2.31) with $\theta = p$, (2.30) with $\boldsymbol{v} = \boldsymbol{u}'$, (2.32) with $\theta_c = p_c$, (2.33) with $\boldsymbol{q} = \boldsymbol{z}$ and (2.34) with $\boldsymbol{q}_c = \boldsymbol{\zeta}$. Recall that scalar products in space are denoted by parentheses and if the domain of integration is not indicated, then it is understood that the integrals are taken over $\Omega^+ \cup \Omega^-$. Thus we obtain the five equalities:

$$\frac{1}{2}\Big(\frac{1}{M} + c_f\varphi_0\Big)\frac{d}{dt}\|p(t)\|^2_{L^2(\Omega)} + \alpha(\nabla\cdot\boldsymbol{u}'(t), p(t)) + \frac{1}{\mu_f}(\nabla\cdot\boldsymbol{z}(t), p(t)) = (\tilde{q}(t), p(t)), \quad (3.5)$$

$$\begin{aligned} G\frac{d}{dt}\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{\lambda}{2}\frac{d}{dt}\|\nabla\cdot\boldsymbol{u}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} - \alpha(p(t), \nabla\cdot\boldsymbol{u}'(t)) + \langle p_c(t), [\boldsymbol{u}'(t)]_{\mathcal{C}}\cdot\boldsymbol{n}^+\rangle_{\mathcal{C}} \\ = (\boldsymbol{f}(t), \boldsymbol{u}'(t)), \\ -\langle [\boldsymbol{u}'(t)]_{\mathcal{C}}\cdot\boldsymbol{n}^+, p_c(t)\rangle_{\mathcal{C}} + \frac{1}{12\mu_f}\langle\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)), p_c(t)\rangle_{\mathcal{C}} - \frac{1}{\mu_f}\langle[\boldsymbol{z}]_{\mathcal{C}}\cdot\boldsymbol{n}^+, p_c(t)\rangle_{\mathcal{C}} \\ = \langle\tilde{q}_W(t), p_c(t)\rangle_{\mathcal{C}}, \\ \frac{1}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} = \frac{1}{\mu_f}(p(t), \nabla\cdot\boldsymbol{z}(t)) - \frac{1}{\mu_f}\langle[\boldsymbol{z}(t)]_{\mathcal{C}}\cdot\boldsymbol{n}^+, p_c(t)\rangle_{\mathcal{C}} + \frac{1}{\mu_f}(\nabla(\rho_{f,r}g\eta), \boldsymbol{z}(t)), \\ \frac{1}{12\mu_f}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})} = \frac{1}{12\mu_f}\langle\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)), p_c(t)\rangle_{\mathcal{C}} + \frac{1}{12\mu_f}\big(w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta), \boldsymbol{\zeta}(t)\big)_{\mathcal{C}}. \end{aligned} \quad (3.6)$$

The second equation is problematic because we have no information on the time derivative of $\boldsymbol{u}$ that appears in its right-hand side. Therefore, supposing $\boldsymbol{f}$ is differentiable in time, we write

$$(\boldsymbol{f}(t), \boldsymbol{u}'(t)) = \frac{d}{dt}(\boldsymbol{f}(t), \boldsymbol{u}(t)) - (\boldsymbol{f}'(t), \boldsymbol{u}(t)),$$

and use instead

$$\begin{aligned} G\frac{d}{dt}\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{\lambda}{2}\frac{d}{dt}\|\nabla\cdot\boldsymbol{u}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} - \alpha(p(t), \nabla\cdot\boldsymbol{u}'(t)) + \langle p_c(t), [\boldsymbol{u}'(t)]_{\mathcal{C}}\cdot\boldsymbol{n}^+\rangle_{\mathcal{C}} \\ = \frac{d}{dt}(\boldsymbol{f}(t), \boldsymbol{u}(t)) - (\boldsymbol{f}'(t), \boldsymbol{u}(t)). \end{aligned} \quad (3.7)$$

In addition, for the sake of convenience, we rewrite the last two equations as

$$\begin{aligned}\frac{1}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} - \frac{1}{\mu_f}(p(t),\nabla\cdot\boldsymbol{z}(t)) + \frac{1}{\mu_f}\langle[\boldsymbol{z}(t)]_{\mathcal{C}}\cdot\boldsymbol{n}^+, p_c(t)\rangle_{\mathcal{C}}\\ = \frac{1}{\mu_f}(\nabla(\rho_{f,r}g\eta),\boldsymbol{z}(t)),\end{aligned} \tag{3.8}$$

$$\frac{1}{12\mu_f}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})} - \frac{1}{12\mu_f}\langle\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)), p_c(t)\rangle_{\mathcal{C}} = \frac{1}{12\mu_f}\big(w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta),\boldsymbol{\zeta}(t)\big)_{\mathcal{C}}. \tag{3.9}$$

When adding the five equations (3.5) through (3.9), we find the following stability equality

$$\begin{aligned}&\frac{1}{2}\big(\frac{1}{M}+c_f\varphi_0\big)\frac{d}{dt}\|p(t)\|^2_{L^2(\Omega)} + G\frac{d}{dt}\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{\lambda}{2}\frac{d}{dt}\|\nabla\cdot\boldsymbol{u}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\\ &+\frac{1}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{1}{12\mu_f}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})} = (\tilde{q}(t),p(t)) + \frac{d}{dt}(\boldsymbol{f}(t),\boldsymbol{u}(t)) - (\boldsymbol{f}'(t),\boldsymbol{u}(t))\\ &+\langle\tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}} + \frac{1}{\mu_f}(\nabla(\rho_{f,r}g\eta),\boldsymbol{z}(t)) + \frac{1}{12\mu_f}\big(w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta),\boldsymbol{\zeta}(t)\big)_{\mathcal{C}}.\end{aligned} \tag{3.10}$$

Corollary 3.1 will be used to control the pressure p_c in the fracture; it appears in the fourth term of the right-hand side of (3.10), but cannot be absorbed by any term of the left-hand side.

Theorem 3.1. *Let the data satisfy* $\boldsymbol{f}\in H^1(0,T;L^2(\Omega\setminus\mathcal{C})^d)$, $\tilde{q}\in L^2(\Omega\times]0,T[)$, $\tilde{q}_W\in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, *and suppose that* w *verifies Hypothesis 2.1 and* $\rho_{f,r}g\eta$ *is independent of time and belongs both to* $H^1(\Omega\setminus\mathcal{C})$ *and* $H^1(\mathcal{C})$. *If* $p\in H^1(0,T;L^2(\Omega))$, $\boldsymbol{u}\in H^1(0,T;H^1(\Omega\setminus\mathcal{C})^d)$, $p_c\in L^2(0,T;H^{\frac{1}{2}}(\mathcal{C}))$, $\boldsymbol{z}\in L^2(0,T;\boldsymbol{Z})$ *and* $\boldsymbol{\zeta}\in L^2(0,T;\boldsymbol{Z}_{\mathcal{C}})$ *is a solution of the mixed problem* (2.30)–(2.34) *and* (2.14), *then it satisfies the following a priori bound almost everywhere in* $]0,T[$:

$$\begin{aligned}&\big(\frac{1}{M}+c_f\varphi_0\big)\|p(t)\|^2_{L^2(\Omega)} + G\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|^2_{L^2(\Omega\setminus\mathcal{C})} + \lambda\|\nabla\cdot\boldsymbol{u}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{1}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)}\\ &+\frac{1}{12\mu_f}\|\boldsymbol{\zeta}\|^2_{L^2(\mathcal{C}\times]0,t[)} \le C\Big[\big(\frac{1}{M}+c_f\varphi_0\big)\|p(0)\|^2_{L^2(\Omega)} + G\|\boldsymbol{\varepsilon}(\boldsymbol{u}(0))\|^2_{L^2(\Omega\setminus\mathcal{C})} + \lambda\|\nabla\cdot\boldsymbol{u}(0)\|^2_{L^2(\Omega\setminus\mathcal{C})}\\ &+\|\boldsymbol{u}(0)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{f}(0)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\tilde{q}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)} + \|\boldsymbol{f}\|^2_{H^1(0,t;L^2(\Omega\setminus\mathcal{C})^d)} + \|\tilde{q}_W\|^2_{L^2(0,t;H^{-\frac{1}{2}}(\mathcal{C}))}\\ &+t|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})} + \frac{1}{12\mu_f}\|w^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r}g\eta)\|^2_{L^2(\mathcal{C}\times]0,t[)}\Big]\exp(t),\end{aligned} \tag{3.11}$$

$$\|p_c\|_{L^2(0,t;H^{\frac{1}{2}}(\mathcal{C}))} \le \frac{\sqrt{2}}{\beta}\Big(\|p\|^2_{L^2(\Omega\times]0,t[)} + \|\boldsymbol{K}^{-1}\boldsymbol{z}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)} + t|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}\Big)^{\frac{1}{2}}, \tag{3.12}$$

with the constant β *of* (3.1) *and a constant* C *that depends on* α, $\|\boldsymbol{K}^{-\frac{1}{2}}\|_{L^\infty(\Omega\setminus\mathcal{C})}$, $\|\boldsymbol{K}^{\frac{1}{2}}\|_{L^\infty(\Omega\setminus\mathcal{C})}$, $1/(\frac{1}{M}+c_f\varphi_0)$, $1/\mu_f$, *but is independent of* t.

Proof. Deriving a bound from the stability equality (3.10) and the pressure bound (3.4) is straightforward. Of course (3.12) is an immediate consequence of (3.4), and it suffices to derive (3.11). We integrate (3.10) over $]0,t[$, $t>0$ and bound the terms in the right-hand side with positive constants δ_i that will be adjusted at the end. The first term is bounded by

$$\Big|\int_0^t (\tilde{q}(t),p(t))\Big| \le \frac{1}{2}\Big(\delta_1\big(\frac{1}{M}+c_f\varphi_0\big)\|p\|^2_{L^2(\Omega\times]0,t[)} + \frac{1}{\delta_1}\frac{1}{\frac{1}{M}+c_f\varphi_0}\|\tilde{q}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)}\Big).$$

For the second term, we use Poincaré's and Korn's inequalities (1.3) and (1.4):

$$\Big|\int_0^t \frac{d}{dt}(\boldsymbol{f}(t),\boldsymbol{u}(t))\Big| \le \frac{1}{2}\Big(\delta_2 G\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|^2_{L^2(\Omega\setminus\mathcal{C})} + C^2\frac{1}{\delta_2}\frac{1}{G}\|\boldsymbol{f}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{u}(0)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{f}(0)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big),$$

where C is the product of the constants in (1.3) and (1.4). Similarly, the third term has the bound

$$\Big|\int_0^t (\boldsymbol{f}'(t),\boldsymbol{u}(t))\Big| \le \frac{1}{2}\Big(\delta_3 G\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)} + C^2\frac{1}{\delta_3}\frac{1}{G}\|\boldsymbol{f}'\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)}\Big).$$

For the fourth term, applying (3.4), we write

$$\begin{aligned}
\Big|\langle \tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}}\Big| &\le \frac{1}{\beta}\|\tilde{q}_W(t)\|_{H^{-\frac{1}{2}}(\mathcal{C})}\Big(\big(\|p(t)\|^2_{L^2(\Omega)}+|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}} + \|\boldsymbol{K}^{-\frac{1}{2}}\|_{L^\infty(\Omega\setminus\mathcal{C})}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|_{L^2(\Omega\setminus\mathcal{C})}\Big)\\
&\le \frac{\delta_4}{2}\big(\frac{1}{M}+c_f\varphi_0\big)\big(\|p(t)\|^2_{L^2(\Omega)}+|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}\big) + \frac{\delta_5}{2\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\\
&\quad + \frac{1}{2\beta^2}\Big(\frac{1}{\delta_4}\frac{1}{\frac{1}{M}+c_f\varphi_0} + \frac{\mu_f}{\delta_5}\|\boldsymbol{K}^{-\frac{1}{2}}\|^2_{L^\infty(\Omega\setminus\mathcal{C})}\Big)\|\tilde{q}_W(t)\|^2_{H^{-\frac{1}{2}}(\mathcal{C})}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Big|\int_0^t \langle \tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}}\Big| &\le \frac{\delta_4}{2}\big(\frac{1}{M}+c_f\varphi_0\big)\big(\|p\|^2_{L^2(\Omega\times]0,t[)} + t|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}\big)\\
&+\frac{\delta_5}{2\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)} + \frac{1}{2\beta^2}\Big(\frac{1}{\delta_4}\frac{1}{\frac{1}{M}+c_f\varphi_0} + \frac{\mu_f}{\delta_5}\|\boldsymbol{K}^{-\frac{1}{2}}\|^2_{L^\infty(\Omega\setminus\mathcal{C})}\Big)\|\tilde{q}_W\|^2_{L^2(0,t;H^{-\frac{1}{2}}(\mathcal{C}))}.
\end{aligned}$$

The fifth term has the bound

$$\Big|\int_0^t \frac{1}{\mu_f}(\nabla(\rho_{f,r}g\eta),\boldsymbol{z}(t))\Big| \le \frac{1}{2}\frac{\delta_6}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)} + \frac{1}{2}\frac{t}{\mu_f}\frac{1}{\delta_6}\|\boldsymbol{K}^{\frac{1}{2}}\|^2_{L^\infty(\Omega\setminus\mathcal{C})}|\rho_{f,r}g\eta|^2_{H^1(\Omega\setminus\mathcal{C})}.$$

Finally, we have for the last term

$$\Big|\int_0^t \frac{1}{12\mu_f}\int_{\mathcal{C}} w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta)\boldsymbol{\zeta}(t)\Big| \le \frac{\delta_7}{24\mu_f}\|\boldsymbol{\zeta}\|^2_{L^2(\mathcal{C}\times]0,t[)} + \frac{1}{\delta_7}\frac{1}{24\mu_f}\|w^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r}g\eta)\|^2_{L^2(\mathcal{C}\times]0,t[)}.$$

With a suitable choice of positive parameters δ_i, $1\le i\le 7$, all terms in the above right-hand sides that involve the solution can be absorbed by the corresponding terms in the left-hand side of (3.10). This yields (3.11). In view of Corollary 3.1, (3.12) is immediate. □

3.2 Additional a priori estimates

Theorem 3.1 gives no information on the divergence of $\boldsymbol{z}$ or on the surface divergence of $w^{\frac{3}{2}}\boldsymbol{\zeta}$ on $\mathcal{C}$. As is usual, a bound for these quantities requires an estimate on the time derivative of p and $\boldsymbol{u}$, and this will also yield a bound for the leakage term $\tilde{q}_L$.
In order to estimate these time derivatives, we test (2.31) with p' and (2.32) with p_c', then we differentiate (2.30), (2.33), and (2.34) in time, and test them respectively with $\boldsymbol{u}'$, $\boldsymbol{z}$, and $\boldsymbol{\zeta}$. By summing these five equations we obtain

$$\begin{aligned}&\Big(\frac{1}{M}+c_f\varphi_0\Big)\|p'(t)\|^2_{L^2(\Omega)}+2G\|\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\|^2_{L^2(\Omega\setminus\mathcal{C})}+\lambda\|\nabla\cdot(\boldsymbol{u}'(t))\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{1}{2\mu_f}\frac{d}{dt}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\\&+\frac{1}{24\mu_f}\frac{d}{dt}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}-\frac{1}{12\mu_f}\langle p_c(t),\overline{\nabla}\cdot((w^{\frac{3}{2}})'(t)\boldsymbol{\zeta}(t))\rangle_{\mathcal{C}}-\frac{1}{12\mu_f}((w^{\frac{3}{2}})'(t)\overline{\nabla}(\rho_{f,r}g\eta),\boldsymbol{\zeta}(t))_{\mathcal{C}}\\&=(\tilde{q}(t),p'(t))+(\boldsymbol{f}'(t),\boldsymbol{u}'(t))-\langle\tilde{q}_W'(t),p_c(t)\rangle_{\mathcal{C}}+\frac{d}{dt}\langle\tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}},\end{aligned}\tag{3.13}$$

where we have passed the time derivative to the first factor in $\langle\tilde{q}_W(t),p_c'(t)\rangle_{\mathcal{C}}$. The term involving the time derivative of $w^{\frac{3}{2}}$ is written as follows:

$$\overline{\nabla}\cdot((w^{\frac{3}{2}})'(t)\boldsymbol{\zeta}(t))=\frac{(w^{\frac{3}{2}})'}{w^{\frac{3}{2}}}(t)\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t))+w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)\cdot\overline{\nabla}\Big(\frac{(w^{\frac{3}{2}})'}{w^{\frac{3}{2}}}(t)\Big),\tag{3.14}$$

and the factor $\frac{(w^{\frac{3}{2}})'}{w^{\frac{3}{2}}}$ is controlled via the following assumption, that complements Hypothesis 2.1.

Hypothesis 3.1. *The width function is the product of two positive functions*

$$\forall(\boldsymbol{x},t)\in\mathcal{C}\times]0,T[\,,\ w(\boldsymbol{x},t)=\varphi(\boldsymbol{x})\psi(t),\tag{3.15}$$

and there exists a constant C such that

$$\forall t\in[0,T]\,,\ |\frac{\psi'(t)}{\psi(t)}|\le C.\tag{3.16}$$

Under this assumption, we have sharper a priori estimates. To simplify, we do not specify the constant below.

Theorem 3.2. *We retain the assumptions of Theorem 3.1 and in addition, we suppose that w satisfies Hypothesis 3.1, the data satisfy $\tilde{q}_W\in H^1(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, $\boldsymbol{z}(0)\in L^2(\Omega\setminus\mathcal{C})^d$, $\boldsymbol{\zeta}(0)\in L^2(\mathcal{C})^{d-1}$, and $p_c(0)\in H^{\frac{1}{2}}(\mathcal{C})$. Then this solution satisfies the following a priori bound almost everywhere in $]0,T[$:*

$$\begin{aligned}&\Big(\frac{1}{M}+c_f\varphi_0\Big)\|p'\|^2_{L^2(\Omega\times]0,t[)}+2G\|\boldsymbol{\varepsilon}(\boldsymbol{u}')\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)}+\lambda\|\nabla\cdot(\boldsymbol{u}')\|^2_{L^2((\Omega\setminus\mathcal{C})\times]0,t[)}\\&\qquad+\frac{1}{2\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{1}{24\mu_f}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}\le C(\boldsymbol{K},\boldsymbol{f},\tilde{q},\tilde{q}_W,p(0),p_c(0),\boldsymbol{z}(0),\boldsymbol{\zeta}(0),\rho_{f,r}g\eta,t).\end{aligned}\tag{3.17}$$

Proof. Owing to the decomposition (3.15), we have

$$\frac{(w^{\frac{3}{2}})'}{w^{\frac{3}{2}}}(\boldsymbol{x},t) = \frac{(\psi^{\frac{3}{2}}(t))'}{\psi^{\frac{3}{2}}(t)},$$

that does not depend on $\boldsymbol{x}$. Consequently, on one hand, the product of this factor with p_c belongs to $H^{\frac{1}{2}}(\mathcal{C})$, and on the other hand, the second term in (3.14) vanishes. Hence, after an application of (3.16), (3.13) implies

$$\begin{aligned}
&\big(\frac{1}{M}+c_f\varphi_0\big)\|p'(t)\|^2_{L^2(\Omega)}+2G\|\boldsymbol{\varepsilon}(\boldsymbol{u}')(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}+\lambda\|\nabla\cdot(\boldsymbol{u}')(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{1}{2\mu_f}\frac{d}{dt}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\\
&+\frac{1}{24\mu_f}\frac{d}{dt}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})} \le \frac{C}{12\mu_f}|\langle p_c(t),\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)\rangle_{\mathcal{C}}|+\frac{C}{12\mu_f}\|w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta)\|_{L^2(\mathcal{C})}\|\boldsymbol{\zeta}(t)\|_{L^2(\mathcal{C})}\\
&+\|\tilde{q}(t)\|_{L^2(\Omega\setminus\mathcal{C})}\|p'(t)\|_{L^2(\Omega)}+\|\boldsymbol{f}'(t)\|_{L^2(\Omega\setminus\mathcal{C})}\|\boldsymbol{u}'(t)\|_{L^2(\Omega\setminus\mathcal{C})}+\|\tilde{q}_W'(t)\|_{H^{-\frac{1}{2}}(\mathcal{C})}\|p_c(t)\|_{H^{\frac{1}{2}}(\mathcal{C})}+\frac{d}{dt}\langle\tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}}\\
&= \sum_{i=1}^{6} T_i. \quad (3.18)
\end{aligned}$$

Indeed,

$$|\langle p_c(t),\frac{(w^{\frac{3}{2}})'}{w^{\frac{3}{2}}}(t)\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t))\rangle_{\mathcal{C}}| = |\langle p_c(t),\frac{(\psi^{\frac{3}{2}})'}{\psi^{\frac{3}{2}}}(t)\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t))\rangle_{\mathcal{C}}| = |\frac{(\psi^{\frac{3}{2}})'}{\psi^{\frac{3}{2}}}(t)\langle p_c(t),\overline{\nabla}\cdot(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t))\rangle_{\mathcal{C}}|.$$

Regarding T_1, we immediately derive from (3.9) that

$$\begin{aligned}
\frac{C}{12\mu_f}|\langle p_c(t),\overline{\nabla}\cdot\big(w^{\frac{3}{2}}(t)\boldsymbol{\zeta}(t)\big)\rangle_{\mathcal{C}}| &\le \frac{C}{12\mu_f}\Big(\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}+\|w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta)\|_{L^2(\mathcal{C})}\|\boldsymbol{\zeta}(t)\|_{L^2(\mathcal{C})}\Big)\\
&\le \frac{1}{12\mu_f}\Big(C\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}+\frac{\delta_1}{2}\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}+\frac{C^2}{2\delta_1}\|w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta)\|^2_{L^2\mathcal{C})}\Big).
\end{aligned}$$

The bounds for T_i, $2\le i\le 5$, are straightforward, with positive constants δ_j, $2\le j\le 5$:

$$\begin{aligned}
T_2 &\le \frac{1}{24\mu_f}\Big(\delta_2\|\boldsymbol{\zeta}(t)\|^2_{L^2(\mathcal{C})}+\frac{C^2}{\delta_2}\|w^{\frac{3}{2}}(t)\overline{\nabla}(\rho_{f,r}g\eta)\|^2_{L^2\mathcal{C})}\Big),\\
T_3 &\le \frac{1}{2}\Big(\delta_3\big(\frac{1}{M}+c_f\varphi_0\big)\|p'(t)\|^2_{L^2(\Omega)}+\frac{1}{\delta_3}\frac{1}{\frac{1}{M}+c_f\varphi_0}\|\tilde{q}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big),\\
T_4 &\le \frac{1}{2}\Big(2G\delta_4\|\boldsymbol{\varepsilon}(\boldsymbol{u}')(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{C^2}{2G\delta_4}\|\boldsymbol{f}'(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big),\\
T_5 &\le \frac{1}{2}\Big(\delta_5\|p_c(t)\|^2_{H^{\frac{1}{2}}(\mathcal{C})}+\frac{1}{\delta_5}\|\tilde{q}_W'(t)\|^2_{H^{-\frac{1}{2}}(\mathcal{C})}\Big).
\end{aligned}$$

We shall bound T_6 after an integration over $]0,t[$:

$$\begin{aligned}
\Big|\int_0^t \frac{d}{dt}\langle\tilde{q}_W(s),p_c(s)\rangle_{\mathcal{C}}\Big| &\le \Big|\langle\tilde{q}_W(t),p_c(t)\rangle_{\mathcal{C}}\Big|+\Big|\langle\tilde{q}_W(0),p_c(0)\rangle_{\mathcal{C}}\Big|\\
&\le \|\tilde{q}_W(t)\|_{H^{-\frac{1}{2}}(\mathcal{C})}\|p_c(t)\|_{H^{\frac{1}{2}}(\mathcal{C})}+\|\tilde{q}_W(0)\|_{H^{-\frac{1}{2}}(\mathcal{C})}\|p_c(0)\|_{H^{\frac{1}{2}}(\mathcal{C})}.
\end{aligned}$$

For the first term, we use formula (3.4) at time t. Thus

$$\|\tilde{q}_W(t)\|_{H^{-\frac{1}{2}}(\mathcal{C})}\|p_c(t)\|_{H^{\frac{1}{2}}(\mathcal{C})} \le \frac{1}{2}\Big(\delta_6\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}(t)\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{1}{\delta_6\beta^2}\|\boldsymbol{K}^{-\frac{1}{2}}\|^2_{L^\infty(\Omega\setminus\mathcal{C})}\|\tilde{q}_W(t)\|^2_{H^{-\frac{1}{2}}(\mathcal{C})}\Big)$$
$$+\frac{1}{2}\Big(\frac{\delta_7}{\beta}\big(\|p(t)\|^2_{L^2(\Omega)}+|\rho_{f,r}g\eta)|^2_{H^1(\Omega\setminus\mathcal{C})}\big)+\frac{1}{\delta_7\beta}\|\tilde{q}_W(t)\|^2_{H^{-\frac{1}{2}}(\mathcal{C})}\Big).$$

Finally, we substitute the bounds for T_i, $1 \le i \le 5$, into (3.18), we integrate on time over $]0,t[$, and we substitute the bound for T_6. A suitable choice of constants δ_i, $i = 3,4,6$ allows to absorb the terms involving $\boldsymbol{u}'$, p', and $\boldsymbol{z}$ into the left-hand side of (3.18). This yields (3.17), considering that all other terms are either data or terms that have been bounded by Theorem 3.1. □

The next corollary complements the bounds on $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ of Theorem 3.2.

Corollary 3.2. *Under the assumptions of Theorem 3.2, we have*

$$\|\nabla\cdot\boldsymbol{z}\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)} \le \mu_f\Big[\big(\frac{1}{M}+c_f\varphi_0\big)\|p'\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}+\alpha\|\nabla\cdot\boldsymbol{u}'\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}+\|\tilde{q}\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}\Big], \tag{3.19}$$

$$\|\overline{\nabla}\cdot(w^{\frac{3}{2}}\boldsymbol{\zeta})\|_{L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))} \le 12\mu_f\Big[C\|\boldsymbol{u}'\|_{L^2(0,T;\boldsymbol{Z})}+\frac{C}{\mu_f}\|\boldsymbol{z}\|_{L^2(0,T;\boldsymbol{Z})}+\|\tilde{q}_W\|_{L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))}\Big], \tag{3.20}$$

where C is the constant of the trace inequality

$$\forall \boldsymbol{q}\in\boldsymbol{Z}\,,\ \|[\boldsymbol{q}]_{\mathcal{C}}\cdot\boldsymbol{n}^+\|_{H^{-\frac{1}{2}}(\mathcal{C})} \le C\|\boldsymbol{q}\|_{\boldsymbol{Z}}. \tag{3.21}$$

Proof. Formula (3.19) follows directly from (2.31), and (3.20) follows from (2.32) and (3.21). □

Remark 3.1. The bounds (3.21) and (3.19) lead to an immediate a priori estimate for the leakage term:

$$\|\tilde{q}_L\|_{L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))} = \frac{1}{\mu_f}\|[\boldsymbol{z}]_{\mathcal{C}}\cdot\boldsymbol{n}^+\|_{L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))}$$
$$\le C\Big[\|\boldsymbol{z}\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}+\big(\frac{1}{M}+c_f\varphi_0\big)\|p'\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}+\alpha\|\nabla\cdot\boldsymbol{u}'\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}+\|\tilde{q}\|_{L^2((\Omega\setminus\mathcal{C})\times]0,T[)}\Big]. \tag{3.22}$$

□

The assumption 3.1 on the fracture width's growth in time is restrictive because it does not allow the fracture to propagate. It is a sufficient condition but we do not know if it is necessary.

3.3 Existence and uniqueness of solutions

The above estimates show that if problem (2.30)–(2.34) and (2.14) has a solution with the regularity stated in Theorem 3.2, then this solution is unique.

Regarding existence, rather than directly constructing a solution for the mixed formulation, let us use known existence results for Problem (Q) and the equivalence theorem 2.2, even though the assumptions may not be optimal. For instance, if $\boldsymbol{f} \in H^2(0,T;L^2(\Omega \setminus \mathcal{C})^d)$, $\tilde{q} \in L^2(\Omega\times]0,T[)$, $\tilde{q}_W \in H^1(0,T;L^2(\mathcal{C}))$, $p_0 \in Q$, and w satisfies Hypotheses 2.1 and 3.1, then the solution of problem (Q) satisfies $p \in H^1(0,T;L^2(\Omega \setminus \mathcal{C})) \cap L^\infty(0,T;Q)$, $\boldsymbol{u} \in H^1(0,T;\boldsymbol{V})$, $\tilde{q}_L \in L^2(0,T;H^1_w(\mathcal{C})')$ and is unique in these spaces, see [16]. Once this is known, additional regularity can be derived from the equations of Problem (Q). In particular, with the definition (2.18) of $\boldsymbol{z}$, the third line of (Q) implies that $\boldsymbol{z}$ is in $L^2(0,T;\boldsymbol{Z})$. In view of the sixth line of (Q), this means that $\tilde{q}_L$ belongs to $L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$. Then the fourth and seventh lines of (Q), and the definition (2.19) of $\boldsymbol{\zeta}$ imply that $\boldsymbol{\zeta}$ is in $L^2(0,T;\boldsymbol{Z}_{\mathcal{C}})$. Hence we are in the setting of Theorem 2.2, which yields existence of a solution of (2.30)–(2.34), (2.14), (2.10), and (2.29) with the above regularity. This is summarized in the next theorem.

Theorem 3.3. *Let the data $\boldsymbol{f}$, $\tilde{q}$, $\tilde{q}_W$ and p_0 be respectively given in $H^2(0,T;L^2(\Omega \setminus \mathcal{C})^d)$, $L^2(\Omega\times]0,T[)$, $H^1(0,T;L^2(\mathcal{C}))$, and Q, and let w satisfies Hypotheses 2.1 and 3.1. Then the mixed problem* (2.30)*–*(2.34)*,* (2.14)*,* (2.10)*, and* (2.29) *has one and only one solution p, $\boldsymbol{u}$, $\tilde{q}_L$, $\boldsymbol{z}$, and $\boldsymbol{\zeta}$ respectively in $H^1(0,T;L^2(\Omega \setminus \mathcal{C})) \cap L^\infty(0,T;Q)$, $H^1(0,T;\boldsymbol{V})$, $L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, $L^2(0,T;\boldsymbol{Z})$, and $L^2(0,T;\boldsymbol{Z}_{\mathcal{C}})$.*

4 Discretization

In this section, we study a space–time discretization of the linearized mixed problem (2.30)–(2.34) and (2.14), with a backward Euler scheme in time and finite elements in space that are conforming for the displacement $\boldsymbol{u}$ and velocity variables $\boldsymbol{z}$ and $\boldsymbol{\zeta}$. In order to avoid handling curved elements or approximating curved surfaces, we assume that both $\partial\Omega$ and the fracture $\mathcal{C}$ are polygonal or polyhedral surfaces.

4.1 General discrete spaces

Let $\mathcal{T}_h$ be a regular family of conforming triangulation of $\overline{\Omega}$, made of triangles or quadrilaterals in 2D and tetrahedra or hexahedra in 3D. To simplify, we assume that $\mathcal{T}_h$ triangulates Ω^+ and Ω^-, i.e. $\mathcal{C}$ does not cross the elements of $\mathcal{T}_h$.

Let $N \geq 1$ be a fixed integer, $\Delta t = T/N$ the time step, and $t_i = i\Delta t$, $0 \leq i \leq N$, the discrete time points.

In each element, if the element is a simplex, the functions are approximated by polynomials P_k of total degree k, and if the element is a quadrilateral or hexahedon, the functions are approximated by images of tensor product polynomials Q_k of degree k in each variable. The displacement, velocity and pressure finite element spaces on any physical element E are defined, respectively, via the vector transformation

$$\boldsymbol{v} \leftrightarrow \hat{\boldsymbol{v}} : \boldsymbol{v} = \hat{\boldsymbol{v}} \circ F_E^{-1},$$

via the Piola transformation

$$\boldsymbol{z} \leftrightarrow \hat{\boldsymbol{z}} : \boldsymbol{z} = \frac{1}{J_E} \mathbb{DF}_E \hat{\boldsymbol{z}} \circ F_E^{-1}, \tag{4.1}$$

and via the scalar transformation

$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1},$$

where F_E denotes a mapping from the reference element $\hat{E}$, unit square or cube according to the dimension, to the physical element E, $\mathbb{DF}_E$ is the Jacobian of F_E, and J_E is its determinant. The advantage of the Piola transformation is that it preserves the divergence and the normal components of the velocity vectors on the sides or faces [15, Ch.III,4.4] in the following sense:

$$(\nabla \cdot \boldsymbol{v}, w)_E = (\hat{\nabla} \cdot \hat{\boldsymbol{v}}, \hat{w})_{\hat{E}} \quad \text{and} \quad (\boldsymbol{v} \cdot \boldsymbol{n}_e, w)_e = (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}_{\hat{e}}, \hat{w})_{\hat{e}}. \tag{4.2}$$

This is used in constructing the $H(\text{div}; \Omega)$-conforming velocity space $\boldsymbol{Z}_h$ defined below. On $\mathcal{T}_h$, the finite element spaces $\boldsymbol{V}_h$ for the displacement $\boldsymbol{u}_h$, $\boldsymbol{Z}_h$ for the velocity $\boldsymbol{z}_h$, and Q_h for the pressure p_h are given by

$$\begin{aligned}
\boldsymbol{V}_h &= \left\{ \boldsymbol{v} \in \boldsymbol{V} \,;\, \boldsymbol{v}|_E = \hat{\boldsymbol{v}} \circ F_E^{-1},\ \hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}(\hat{E}), \quad \forall E \in \mathcal{T}_h \right\}, \\
\boldsymbol{Z}_h &= \left\{ \boldsymbol{z} \in \boldsymbol{Z} \,;\, \boldsymbol{z}|_E = \frac{1}{J_E} \mathbb{DF}_E \hat{\boldsymbol{z}} \circ F_E^{-1},\ \hat{\boldsymbol{z}} \in \hat{\boldsymbol{Z}}(\hat{E}), \quad \forall E \in \mathcal{T}_h \right\}, \\
Q_h &= \left\{ q \in L^2(\Omega) \,;\, q|_E = \hat{q} \circ F_E^{-1},\ \hat{q} \in \hat{Q}(\hat{E}), \quad \forall E \in \mathcal{T}_h \right\},
\end{aligned} \tag{4.3}$$

where $\hat{\boldsymbol{V}}(\hat{E})$, $\hat{\boldsymbol{Z}}(\hat{E})$ and $\hat{Q}(\hat{E})$ are suitable finite element spaces on the reference element $\hat{E}$. In particular, we suppose that $\hat{\boldsymbol{Z}}(\hat{E})$ and $\hat{Q}(\hat{E})$ are compatible pairs such as the Raviart-Thomas pairs of elements on simplices or enhanced BDM pairs of elements on quadrilaterals and hexahedra. The enhanced BDM pairs are used on quadrilaterals and hexahedra as described in Section 4.3.
By definition, the functions of $\boldsymbol{V}_h$ and $\boldsymbol{Z}_h$ are conforming in $\boldsymbol{V}$ and $\boldsymbol{Z}$ respectively. Moreover, we assume that the conformity holds also on the boundary of $\mathcal{C}$, i.e., the functions of $\boldsymbol{V}_h$ as well as the normal components of functions of $\boldsymbol{Z}_h$ have no jump on $\partial\mathcal{C}$.

4.2 Discretization in the fracture

Let $\mathcal{C}_h$ denote the trace of $\mathcal{T}_h$ on $\mathcal{C}$. Since $\mathcal{C}$ is assumed to be polygonal or polyhedral, we can map each line segment or plane face of $\mathcal{C}$ onto a segment in the x_1 line (when $d = 2$) or a polygon in the $x_1 - x_2$ plane (when $d = 3$) by a rigid-body motion that preserves both surface gradient and divergence, maps the normal $\boldsymbol{n}^+$ into a unit vector along x_3, for exemple $-\boldsymbol{e}_3$, and whose Jacobian is one. After this change in variable, all operations on this line segment or plane face can be treated as the same operations on the x_1 axis or $x_1 - x_2$ plane. To simplify, we do not use a particular notation for this change in variable, and work as if the line segments or plane faces of $\mathcal{C}$ lie on the x_1 line or $x_1 - x_2$ plane. Let $\mathcal{S}_i$, $1 \leq i \leq I$, denote the line segments or plane faces of $\mathcal{C}$; to simplify, we drop the index i.

Again, to simplify the analysis, we take the trace of $\mathcal{T}_h$ on $\mathcal{S}$, say $\mathcal{T}_{\mathcal{S},h}$ as partition of $\mathcal{S}$. Let e denote a generic element of $\mathcal{T}_{\mathcal{S},h}$, with reference element $\hat{e}$, and let the scalar and Piola transforms be defined by the same formula as above, but with respect to e instead of E. Then we define the finite element spaces on $\mathcal{C}$ by:

$$\begin{aligned} \boldsymbol{Z}_{\mathcal{C},h} &= \left\{\boldsymbol{\mu} \in \boldsymbol{Z}_{\mathcal{C}}\,;\, \boldsymbol{\mu}|_{\mathcal{S}_i} \in \boldsymbol{Z}_{\mathcal{S}_i,h}, 1 \leq i \leq I\right\},\\ \Theta_{\mathcal{C},h} &= \left\{q \in L^2(\mathcal{C})\,;\, q|_{\mathcal{S}_i} \in \Theta_{\mathcal{S}_i,h}, 1 \leq i \leq I\right\}, \end{aligned} \tag{4.4}$$

with

$$\boldsymbol{Z}_{\mathcal{S},h} = \left\{\boldsymbol{\mu} \in \boldsymbol{Z}_{\mathcal{C}}\,;\, \boldsymbol{\mu}|_e \leftrightarrow \hat{\boldsymbol{\mu}},\ \hat{\boldsymbol{\mu}} \in \hat{\boldsymbol{Z}}_{\mathcal{C}}(\hat{e}), \quad \forall e \in \mathcal{T}_{\mathcal{S},h}\right\},$$

$$\Theta_{\mathcal{S},h} = \left\{q \in L^2(\mathcal{C})\,;\, q|_e \leftrightarrow \hat{q},\ \hat{q} \in \hat{\Theta}_{\mathcal{C}}(\hat{e}), \quad \forall e \in \mathcal{T}_{\mathcal{S},h}\right\},$$

where $\hat{\boldsymbol{Z}}_{\mathcal{C}}(\hat{e})$ and $\hat{\Theta}_{\mathcal{C}}(\hat{e})$ are finite element spaces on the reference element $\hat{e}$. Again, we assume that they are compatible pairs like the Raviart-Thomas pairs on triangles or enhanced BDM pairs on quadrlaterals. This implies that the functions $\boldsymbol{q}_{c,h}$ of $\boldsymbol{Z}_{\mathcal{S},h}$ belong globally to $H(\mathrm{div};\mathcal{S}) \cap L^\infty(\mathcal{S})^{d-1}$. By expanding $\overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{q}_{c,h})$ and using the assumption (2.15) on w, we easily derive that $\overline{\nabla} \cdot (w^{\frac{3}{2}} \boldsymbol{q}_{c,h})$ belongs to $L^2(\mathcal{S})$. This allows to take the discrete pressure $p_{c,h}$ in $L^2(\mathcal{C})$ instead of $H^{\frac{1}{2}}(\mathcal{C})$.

4.3 Elements on convex quadrilaterals and hexahedra

In the case of convex quadrilaterals, $\hat{E}$ is the unit square with vertices $\hat{\boldsymbol{r}}_1 = (0,0)^T$, $\hat{\boldsymbol{r}}_2 = (1,0)^T$, $\hat{\boldsymbol{r}}_3 = (1,1)^T$, and $\hat{\boldsymbol{r}}_4 = (0,1)^T$. Denote by $\boldsymbol{r}_i$, $1 \leq i \leq 4$, the corresponding vertices of E. In this case, F_E is the bilinear mapping given as

$$F_E(\hat{x},\hat{y}) = \boldsymbol{r}_1(1-\hat{x})(1-\hat{y}) + \boldsymbol{r}_2\hat{x}(1-\hat{y}) + \boldsymbol{r}_3\hat{x}\hat{y} + \boldsymbol{r}_4(1-\hat{x})\hat{y};$$

the space for the displacement is

$$\hat{\boldsymbol{V}}(\hat{E}) = Q_1(\hat{E})^2,$$

and the space for the flow is the lowest order BDM_1 [7] space

$$\hat{\boldsymbol{Z}}(\hat{E}) = P_1(\hat{E})^2 + r\ \mathbf{curl}(\hat{x}^2\hat{y}) + s\ \mathbf{curl}(\hat{x}\hat{y}^2)\ ,\ \hat{Q}(\hat{E}) = P_0(\hat{E}),$$

where r and s are real constants.

In the case of hexahedra, $\hat{E}$ is the unit cube but the element E can have non-planar faces. The vertices of $\hat{E}$ are $\hat{\boldsymbol{r}}_1 = (0,0,0)^T$, $\hat{\boldsymbol{r}}_2 = (1,0,0)^T$, $\hat{\boldsymbol{r}}_3 = (1,1,0)^T$, $\hat{\boldsymbol{r}}_4 = (0,1,0)^T$, $\hat{\boldsymbol{r}}_5 = (0,0,1)^T$, $\hat{\boldsymbol{r}}_6 = (1,0,1)^T$, $\hat{\boldsymbol{r}}_7 = (1,1,1)^T$, and $\hat{\boldsymbol{r}}_8 = (0,1,1)^T$. Denote by $\boldsymbol{r}_i = (x_i,y_i,z_i)^T, 1 \leq i \leq 8$, the eight corresponding vertices of E. In this case F_E is a trilinear mapping given by

$$\begin{aligned} F_E(\hat{x},\hat{y},\hat{z}) =& \boldsymbol{r}_1(1-\hat{x})(1-\hat{y})(1-\hat{z}) + \boldsymbol{r}_2\hat{x}(1-\hat{y})(1-\hat{z}) + \boldsymbol{r}_3\hat{x}\hat{y}(1-\hat{z}) + \boldsymbol{r}_4(1-\hat{x})\hat{y}(1-\hat{z})\\ &+ \boldsymbol{r}_5(1-\hat{x})(1-\hat{y})\hat{z} + \boldsymbol{r}_6\hat{x}(1-\hat{y})\hat{z} + \boldsymbol{r}_7\hat{x}\hat{y}\hat{z} + \boldsymbol{r}_8(1-\hat{x})\hat{y}\hat{z}, \end{aligned}$$

the space for the displacement is defined by

$$\hat{\boldsymbol{V}}(\hat{E}) = Q_1(\hat{E})^3,$$

the space for the flow is an enhanced BDDF_1 spaces [18]:

$$\begin{aligned}\hat{\boldsymbol{Z}}(\hat{E}) &= \mathrm{BDDF}_1(\hat{E}) + s_2\mathbf{curl}(0,0,\hat{x}^2\hat{z})^T + s_3\mathbf{curl}(0,0,\hat{x}^2\hat{y}\hat{z})^T + t_2\mathbf{curl}(\hat{x}\hat{y}^2,0,0)^T \\ &\quad + t_3\mathbf{curl}(\hat{x}\hat{y}^2\hat{z},0,0)^T + w_2\mathbf{curl}(0,\hat{y}\hat{z}^2,0)^T + w_3\mathbf{curl}(0,\hat{x}\hat{y}\hat{z}^2,0)^T, \\ \hat{Q}(\hat{E}) &= P_0(\hat{E}),\end{aligned} \tag{4.5}$$

where the $\mathrm{BDDF}_1(\hat{E})$ space is defined as [8]:

$$\begin{aligned}\mathrm{BDDF}_1(\hat{E}) = P_1(\hat{E})^3 &+ s_0\mathbf{curl}(0,0,\hat{x}\hat{y}\hat{z})^T + s_1\mathbf{curl}(0,0,\hat{x}\hat{y}^2)^T + t_0\mathbf{curl}(\hat{x}\hat{y}\hat{z},0,0)^T \\ &+ t_1\mathbf{curl}(\hat{y}\hat{z}^2,0,0)^T + w_0\mathbf{curl}(0,\hat{x}\hat{y}\hat{z},0)^T + w_1\mathbf{curl}(0,\hat{x}^2\hat{z},0)^T.\end{aligned}$$

In the above equations, s_i, t_i, w_i, $0 \le i \le 3$, are real constants. In all cases the degrees of freedom (DOF) for the displacements are chosen as Lagrangian nodal point values. The velocity DOF are chosen to be the normal components at the d vertices on each face. The dimension of the space is dn_v, where $d = 2, 3$ is the dimension and n_v is the number of vertices in E. Note that, although the original BDDF_1 spaces have only three DOF on square faces, these spaces have been enhanced in [18] to have four DOF on square faces. This special choice is needed in the reduction to a cell-centered pressure stencil in a pure Darcy flow problem as described later in this section.

4.4 The fully discrete equations

The assumptions on the data are: $\boldsymbol{f} \in H^1(0,T;L^2(\Omega\setminus\mathcal{C})^d)$, $\tilde{q} \in \mathcal{C}^0([0,T];L^2(\Omega\setminus\mathcal{C}))$, $\tilde{q}_W \in H^1(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, $p(0) = p_0 \in Q$ with $p_c(0)$ the trace of p_0 on $\mathcal{C}$, and in addition to Hypothesis 2.1, w is continuous in time.
For each n and for almost every $\boldsymbol{x} \in \Omega^+ \cup \Omega^-$ or Ω, we set

$$\boldsymbol{f}^n(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}, t_n)\ ,\ \tilde{q}^n(\boldsymbol{x}) = \tilde{q}(\boldsymbol{x}, t_n), \tag{4.6}$$

and for almost every $s \in \mathcal{C}$

$$w^n(s) = w(s, t_n)\ ,\ \tilde{q}_W^n(s) = \tilde{q}_W(s, t_n). \tag{4.7}$$

To simplify, we denote the first backward difference in time of any function v (continuous in time) as follows,

$$\delta v^n = v^n - v^{n-1}. \tag{4.8}$$

We propose the following fully discrete implicit coupled mixed scheme called Problem (D-M); it is assumed that the finite element functions are sufficiently smooth to give meaning to all integrals below.
• At time $t = 0$, let $p_h^0 = r_h(p_0)$, where r_h is the local L^2 projection on each element E of $\mathcal{T}_h$, with values in Q_h. By assumption $p_0 \in Q$ and therefore $p_{c,0}$, its trace on $\mathcal{C}$ belongs to

$H^1_w(\mathcal{C}) \subset H^{\frac{1}{2}}(\mathcal{C})$. We take $p^0_{c,h} = r_{\mathcal{C},h}(p_{c,0})$, where $r_{\mathcal{C},h}$ is the local L^2 projection on each element e of $\mathcal{C}_h$, with values in $\Theta_{\mathcal{C},h}$.
Once p^0_h and $p^0_{c,h}$ are known, $\boldsymbol{u}(p^0_h)$ is approximated by discretizing the elasticity equation (2.30) in $\Omega \setminus \mathcal{C}$: Find $\boldsymbol{u}^0_h \in \boldsymbol{V}_h$ solution of

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^0_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla\cdot \boldsymbol{u}^0_h, \nabla\cdot \boldsymbol{v}_h\big) = \alpha\big(p^0_h, \nabla\cdot \boldsymbol{v}_h\big) - \int_{\mathcal{C}} p^0_{c,h}[\boldsymbol{v}_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+ + \big(\boldsymbol{f}^0, \boldsymbol{v}_h\big). \tag{4.9}$$

Similarly, $\boldsymbol{z}^0_h$ and $\boldsymbol{\zeta}^0_h$ are approximated by discretizing respectively (2.33) and (2.34):

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \big(\boldsymbol{K}^{-1}\boldsymbol{z}^0_h, \boldsymbol{q}_h\big) = (p^0_h, \nabla\cdot \boldsymbol{q}_h) - \int_{\mathcal{C}} p^0_{c,h}[\boldsymbol{q}_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+ + \big(\nabla(\rho_{f,r} g\eta), \boldsymbol{q}_h\big), \tag{4.10}$$

$$\forall \boldsymbol{q}_{c,h} \in \boldsymbol{Z}_{\mathcal{C},h}\,,\ \int_{\mathcal{C}} \boldsymbol{\zeta}^0_h\cdot \boldsymbol{q}_{c,h} = \int_{\mathcal{C}} p^0_{c,h}\overline{\nabla}\cdot\big((w^0)^{\frac{3}{2}}\boldsymbol{q}_{c,h}\big) + \int_{\mathcal{C}} (w^0)^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r} g\eta)\cdot \boldsymbol{q}_{c,h}. \tag{4.11}$$

• For any n, $1 \le n \le N$, $\boldsymbol{u}^n_h$, p^n_h, $p^n_{c,h}$, $\boldsymbol{z}^n_h$, and $\boldsymbol{\zeta}^n_h$ are approximated by discretizing (2.30)–(2.34): Knowing $\boldsymbol{u}^{n-1}_h$, p^{n-1}_h, find $\boldsymbol{u}^n_h \in \boldsymbol{V}_h$, $p^n_h \in Q_h$, $p^n_{c,h} \in \Theta_{\mathcal{C},h}$, $\boldsymbol{z}^n_h \in \boldsymbol{Z}_h$, and $\boldsymbol{\zeta}^n_h \in \boldsymbol{Z}_{\mathcal{C},h}$ solutions of

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^n_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla\cdot \boldsymbol{u}^n_h, \nabla\cdot \boldsymbol{v}_h\big) - \alpha\big(p^n_h, \nabla\cdot \boldsymbol{v}_h\big) + \int_{\mathcal{C}} p^n_{c,h}[\boldsymbol{v}_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+ = \big(\boldsymbol{f}^n, \boldsymbol{v}_h\big)\,, \tag{4.12}$$

$$\forall \theta_h \in Q_h\,,\ \Big(\big(\frac{1}{M} + c_f\varphi_0\big)\frac{1}{\Delta t}\delta p^n_h + \frac{\alpha}{\Delta t}\nabla\cdot \delta \boldsymbol{u}^n_h, \theta_h\Big) + \frac{1}{\mu_f}\big(\nabla\cdot \boldsymbol{z}^n_h, \theta_h\big) = \big(\tilde{q}^n, \theta_h\big), \tag{4.13}$$

$$\forall \theta_{c,h} \in \Theta_{\mathcal{C},h}\,,\ -\frac{1}{\Delta t}\int_{\mathcal{C}} \delta([\boldsymbol{u}^n_h]_{\mathcal{C}})\cdot \boldsymbol{n}^+\theta_{c,h} + \frac{1}{12\mu_f}\int_{\mathcal{C}} \big(\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}\boldsymbol{\zeta}^n_h)\theta_{c,h} - \frac{1}{\mu_f}\int_{\mathcal{C}} [\boldsymbol{z}^n_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+\theta_{c,h} = \langle \tilde{q}^n_W, \theta_{c,h}\rangle_{\mathcal{C}}, \tag{4.14}$$

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \big(\boldsymbol{K}^{-1}\boldsymbol{z}^n_h, \boldsymbol{q}_h\big) = (p^n_h, \nabla\cdot \boldsymbol{q}_h) - \int_{\mathcal{C}} p^n_{c,h}[\boldsymbol{q}_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+ + \big(\nabla(\rho_{f,r} g\eta), \boldsymbol{q}_h\big), \tag{4.15}$$

$$\forall \boldsymbol{q}_{c,h} \in \boldsymbol{Z}_{\mathcal{C},h}\,,\ \int_{\mathcal{C}} \boldsymbol{\zeta}^n_h\cdot \boldsymbol{q}_{c,h} = \int_{\mathcal{C}} p^n_{c,h}\overline{\nabla}\cdot\big((w^n)^{\frac{3}{2}}\boldsymbol{q}_{c,h}\big) + \int_{\mathcal{C}} (w^n)^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r} g\eta)\cdot \boldsymbol{q}_{c,h}. \tag{4.16}$$

4.4.1 Existence and uniqueness of the discrete solution

Problem (D-M) is a square system of linear equations in finite dimension. Therefore, to show existence of a solution, it suffices to prove that, at each time step, if all data are zero (including the values at the preceding step) then the only solution is the zero solution. Existence and uniqueness of p^n_h, $\boldsymbol{u}^n_h$, $\boldsymbol{z}^n_h$, and $\boldsymbol{\zeta}^n_h$ follow immediately from the following stability equality, obtained by testing (4.13) with p^n_h, (4.14) with $p^n_{c,h}$, (4.15) with $\boldsymbol{z}^n_h$, (4.16) with $\boldsymbol{\zeta}^n_h$, (4.12) with $\delta\boldsymbol{u}^n_h$, multiplying everything by $\Delta\, t$, and combining the resulting equations:

$$\frac{1}{2}\big(\frac{1}{M}+c_f\varphi_0\big)\Big(\delta\big(\|p_h^n\|^2_{L^2(\Omega)}\big)+\|\delta\, p_h^n\|^2_{L^2(\Omega)}\Big)+G\Big(\delta\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n)\|^2_{L^2(\Omega\setminus\mathcal{C})}\big)+\|\delta\,\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big)$$
$$+\frac{\lambda}{2}\Big(\delta\big(\|\nabla\cdot\boldsymbol{u}_h^n\|^2_{L^2(\Omega\setminus\mathcal{C})}\big)+\|\delta(\nabla\cdot\boldsymbol{u}_h^n)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big)+\frac{\Delta\,t}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{z}_h^n\|^2_{L^2(\Omega\setminus\mathcal{C})}+\frac{\Delta\,t}{12\mu_f}\|\boldsymbol{\zeta}_h^n\|^2_{L^2(\mathcal{C})}$$
$$=\Delta\,t(\tilde{q}^n,p_h^n)_\Omega+(\boldsymbol{f}^n,\delta\,\boldsymbol{u}_h^n)+\Delta\,t\langle\tilde{q}_W^n,p_{c,h}^n\rangle_{\mathcal{C}}+\frac{\Delta\,t}{\mu_f}(\nabla(\rho_{f,r}g\eta),\boldsymbol{z}_h^n)+\frac{\Delta\,t}{12\mu_f}\int_{\mathcal{C}}(w^n)^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r}g\eta)\boldsymbol{\zeta}_h^n. \tag{4.17}$$

From here, existence and uniqueness of $p_{c,h}^n$ will be a consequence of the discrete inf-sup condition established in the next section.

4.5 Discrete inf-sup condition for $p_{c,h}$

As in the exact problem, we need an inf-sup condition to control the discrete surface pressure $p_{c,h}$ on $\mathcal{C}$. However, the argument used in deriving Lemma 3.1 does not carry over to the discrete case because the Raviart-Thomas interpolant R_h, which is the most obvious candidate for discretizing $\boldsymbol{q}$, is not defined in $H(\text{div};\Omega)$. We shall use instead an interior argument that creates a smoother function. In addition, we suppose the following compatibility condition on the finite element spaces.

Hypothesis 4.1. *There exists an approximation operator $R_h \in \mathcal{L}(\boldsymbol{Z}\cap H^s(\Omega^+\cup\Omega^-)^d;\boldsymbol{Z}_h)$ for some $s>0$, such that for all $\boldsymbol{q}\in\boldsymbol{Z}\cap H^s(\Omega^+\cup\Omega^-)^d$,*

$$\begin{gathered}\forall E\subset\Omega^\star,\star=+,-\,,\ \forall\theta_h\in Q_h\,,\ \int_E\theta_h\nabla\cdot(\boldsymbol{q}-R_h(\boldsymbol{q}))=0\ ,\\ \forall e\subset\mathcal{C}\,,\ \forall\theta_{c,h}\in\Theta_{\mathcal{C},h}\,,\ \langle\theta_{c,h},[\boldsymbol{q}-R_h(\boldsymbol{q}))]_{\mathcal{C}}\cdot\boldsymbol{n}^+\rangle_e=0,\end{gathered}\tag{4.18}$$

and there exists a constant C independent of h such that for all element E of $\mathcal{T}_h$

$$\begin{gathered}\forall\boldsymbol{q}\in H^s(E)^d\,,\ \|\boldsymbol{q}-R_h(\boldsymbol{q})\|_{L^2(E)}\le C\,h^s|\boldsymbol{q}|_{H^s(E)},\\ \forall\boldsymbol{q}\in H(\text{div};E)\cap H^s(E)^d\,,\ \|\text{div}(\boldsymbol{q}-R_h(\boldsymbol{q}))\|_{L^2(E)}\le\|\text{div}\,\boldsymbol{q}\|_{L^2(E)}.\end{gathered}\tag{4.19}$$

These assumptions are satisfied by the Raviart-Thomas RT_k finite elements pairs of degree $k\ge 0$, i.e., $H(\text{div})$ velocity with incomplete degree $k+1$ and discontinuous pressure with degree k. They are also satisfied, for instance, by the enhanced BDM_1 elements pairs described in Section 4.3, associated with piecewise constant pressures.
In view of the first part of (4.18), the compatibility between Q_h and $\boldsymbol{Z}_h$ implies that $\nabla\cdot R_h(\boldsymbol{q})$ is the projection of $\nabla\cdot\boldsymbol{q}$ onto Q_h.

Lemma 4.1. *Under Hypothesis 4.1, there exists a constant $\beta_1^*>0$, independent of h, such that*

$$\forall\theta_{c,h}\in\Theta_{\mathcal{C},h}\,,\ \sup_{\boldsymbol{q}_h\in\boldsymbol{Z}_h}\frac{\int_{\mathcal{C}}\theta_{c,h}[\boldsymbol{q}_h]_{\mathcal{C}}\cdot\boldsymbol{n}^+}{\|\boldsymbol{q}_h\|_{\boldsymbol{Z}}}\ge\beta_1^*\|\theta_{c,h}\|_{L^2(\mathcal{C})}. \tag{4.20}$$

Proof. The idea is to construct an adequately smooth function $\boldsymbol{q}$ in $\boldsymbol{Z}$ whose normal jump on $\mathcal{C}$ coincides with $\theta_{c,h}$, and to which R_h can be applied. We proceed in two steps.

1) As we only consider the L^2 norm, we extend $\theta_{c,h}$ by zero to $\partial\Omega^+$ (without changing its notation) and we consider the unique solution $\varphi^+ \in H^1(\Omega^+)$ of the following Laplace equation with Neumann boundary conditions:

$$-\Delta\,\varphi^+ = -\frac{1}{|\Omega^+|}\int_{\mathcal{C}} \theta_{c,h} \quad \text{in } \Omega^+,$$
$$\frac{\partial}{\partial \boldsymbol{n}}\varphi^+ = \theta_{c,h} \quad \text{on } \partial\Omega^+.$$

Note that the interior and boundary data are compatible; therefore this problem has a unique solution φ^+ and

$$\|\varphi^+\|_{H^1(\Omega^+)} \le C\|\theta_{c,h}\|_{L^2(\mathcal{C})}, \tag{4.21}$$

with a constant C that depends only on Ω^+ and $\mathcal{C}$.
Then we choose $\boldsymbol{q}^+ = \nabla\,\varphi^+$ in Ω^+ and $\boldsymbol{q}^- = \boldsymbol{0}$ in Ω^-. By construction, $\boldsymbol{q}$ belongs to $H(\text{div};\Omega^+ \cup \Omega^-)$,

$$\nabla\cdot\boldsymbol{q} = \begin{cases} \frac{1}{|\Omega^+|}\int_{\mathcal{C}} \theta_{c,h} & \text{in } \Omega^+, \\ 0 & \text{in } \Omega^-, \end{cases}$$

and

$$[\boldsymbol{q}\cdot\boldsymbol{n}^+]_{\mathcal{C}} = \theta_{c,h} \quad , \quad [\boldsymbol{q}\cdot\boldsymbol{n}^+]_{\Gamma\setminus\mathcal{C}} = 0 \quad , \quad \boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\Omega} = 0,$$

so that $\boldsymbol{q}$ belongs to $\boldsymbol{Z}$. Since the extended function $\theta_{c,h}$ belongs to $L^2(\partial\Omega^+)$, the regularity of the Laplace equation with Neumann boundary conditions imply that φ^+ is in $H^{\frac{3}{2}}(\Omega^+)$ (cf. [19]) with continuous dependence on $\theta_{c,h}$. Therefore $\boldsymbol{q} \in H^{\frac{1}{2}}(\Omega^+ \cup \Omega^-)^d$ and there exists a constant C depending only on Ω, Γ and $\mathcal{C}$ such that

$$\|\boldsymbol{q}\|_{H^{\frac{1}{2}}(\Omega^+\cup\Omega^-)} \le C\|\theta_{c,h}\|_{L^2(\mathcal{C})}. \tag{4.22}$$

2) The regularity (4.22) of $\boldsymbol{q}$ and Hypothesis 4.1 allow to define $\boldsymbol{q}_h = R_h(\boldsymbol{q})$. As $\nabla\cdot\boldsymbol{q}$ is constant in each subdomain, and Q_h contains at least the constant functions, the first part of (4.18) implies trivially that $\nabla\cdot R_h(\boldsymbol{q}) = \nabla\cdot\boldsymbol{q}$. Thus

$$\|R_h(\boldsymbol{q})\|_{\boldsymbol{Z}} \le \|R_h(\boldsymbol{q}) - \boldsymbol{q}\|_{\boldsymbol{Z}} + \|\boldsymbol{q}\|_{\boldsymbol{Z}} \le \|R_h(\boldsymbol{q}) - \boldsymbol{q}\|_{L^2(\Omega^+)} + \|\boldsymbol{q}\|_{\boldsymbol{Z}}.$$

In view of the first part of (4.19) with $s = \frac{1}{2}$, and (4.22),

$$\|R_h(\boldsymbol{q}) - \boldsymbol{q}\|_{L^2(\Omega^+)} \le C_1 h^{\frac{1}{2}}|\boldsymbol{q}|_{H^{\frac{1}{2}}(\Omega^+)} \le C_2 C_1 h^{\frac{1}{2}}\|\theta_{c,h}\|_{L^2(\mathcal{C})},$$

where C_1 and C_2 are the constant of (4.19) and (4.22) respectively. Similarly,

$$\|\boldsymbol{q}\|_{\boldsymbol{Z}} \le \Big(\|\boldsymbol{q}\|^2_{L^2(\Omega)} + \frac{|\mathcal{C}|}{|\Omega^+|}\|\theta_{c,h}\|^2_{L^2(\mathcal{C})}\Big)^{\frac{1}{2}} \le \Big(C_3^2 + \frac{|\mathcal{C}|}{|\Omega^+|}\Big)^{\frac{1}{2}}\|\theta_{c,h}\|_{L^2(\mathcal{C})},$$

where C_3 is the constant of (4.21). Combining these two inequalities, we have on one hand

$$\|R_h(\boldsymbol{q})\|_{\boldsymbol{Z}} \le C_4\|\theta_{c,h}\|_{L^2(\mathcal{C})}. \tag{4.23}$$

On the other hand, the second part of (4.18) yields for all e in $\mathcal{C}$

$$\int_e \theta_{c,h}[R_h(\boldsymbol{q})]_{\mathcal{C}} \cdot \boldsymbol{n}^+ = \langle \theta_{c,h}, [\boldsymbol{q}]_{\mathcal{C}} \cdot \boldsymbol{n}^+ \rangle_e = \int_e \big(\theta_{c,h}\big)^2 = \|\theta_{c,h}\|^2_{L^2(e)}. \tag{4.24}$$

Then (4.23) and (4.24) imply

$$\frac{1}{\|R_h(\boldsymbol{q})\|_{\boldsymbol{Z}}} \int_{\mathcal{C}} \theta_{c,h}[R_h(\boldsymbol{q})]_{\mathcal{C}} \cdot \boldsymbol{n}^+ \geq \frac{1}{C_4} \|\theta_{c,h}\|_{L^2(\mathcal{C})},$$

whence (4.20) with $\beta_1^* = \frac{1}{C_4}$. □

Then we have the analogue of Corollary 3.1 with the same proof. The operator B is replaced by B_h defined on $\boldsymbol{Z}_h$ as

$$\forall \theta_{c,h} \in \Theta_{\mathcal{C},h}\,,\ \langle B_h \boldsymbol{q}_h, \theta_{c,h} \rangle = \int_{\mathcal{C}} \theta_{c,h}[\boldsymbol{q}_h]_{\mathcal{C}} \cdot \boldsymbol{n}^+.$$

The operator B_h is linear and since we are in finite dimension where all norms are equivalent, B_h is continuous on $\boldsymbol{Z}_h$ (albeit the continuity constant is not expected to be bounded as h tends to zero). The kernel of B_h in $\boldsymbol{Z}_h$ is:

$$\mathrm{Ker}(B_h) = \{\boldsymbol{q}_h \in \boldsymbol{Z}_h\,;\, \forall \theta_{c,h} \in \Theta_{\mathcal{C},h}, \int_{\mathcal{C}} \theta_{c,h}[\boldsymbol{q}_h]_{\mathcal{C}} \cdot \boldsymbol{n}^+ = 0\}.$$

Corollary 4.1. *Let $\boldsymbol{z}_h \in \boldsymbol{Z}_h$ and $p_h \in Q_h$ be such that*

$$\forall \boldsymbol{q}_h \in \mathrm{Ker}(B_h)\,,\ (p_h, \nabla \cdot \boldsymbol{q}_h) - \big(\boldsymbol{K}^{-1}\boldsymbol{z}_h, \boldsymbol{q}_h\big) + \big(\nabla(\rho_{f,r} g \eta), \boldsymbol{q}_h\big) = 0. \tag{4.25}$$

Then there exists a unique $p_{c,h} \in \Theta_{\mathcal{C},h}$ such that $p_{c,h}$, p_h, and $\boldsymbol{z}_h$ satisfy (4.15), *and*

$$\|p_{c,h}\|_{L^2(\mathcal{C})} \leq \frac{1}{\beta_1^*}\Big(\big(\|p_h\|^2_{L^2(\Omega)} + |\rho_{f,r} g \eta|^2_{H^1(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}} + \|\boldsymbol{K}^{-1}\boldsymbol{z}_h\|_{L^2(\Omega\setminus\mathcal{C})}\Big), \tag{4.26}$$

where β_1^* is the constant of (4.20).

4.6 Error estimates

Here we assume that the solution and data are sufficiently smooth in time and space, as needed. It is convenient to split the scheme's error into a time consistency error and a spatial discretization error.

4.6.1 Time consistency error

The time consistency error measures the difference between the divided difference in time and the time derivative. More precisely, for $\theta_h \in Q_h$ and $\theta_{c,h} \in \Theta_{\mathcal{C},h}$, we define

$$\begin{aligned} E_n(\theta_h, \theta_{c,h}) =& \big(\frac{1}{M} + c_f \varphi_0\big) \int_\Omega \big(\frac{1}{\Delta t}\delta p(t_n) - p'(t_n)\big)\theta_h + \int_{\Omega\setminus\mathcal{C}} \alpha\big(\nabla \cdot (\frac{1}{\Delta t}\delta \boldsymbol{u}(t_n) - \boldsymbol{u}'(t_n))\big)\theta_h \\ & - \int_{\mathcal{C}} \big([\frac{1}{\Delta t}\delta \boldsymbol{u}(t_n) - \boldsymbol{u}'(t_n)]_{\mathcal{C}} \cdot \boldsymbol{n}^+\big)\theta_{c,h}. \end{aligned} \tag{4.27}$$

In view of the Taylor expansion valid for all functions v in $W^{2,1}(t_{n-1},t_n)$:

$$\frac{1}{\Delta t}\delta v(t_n) = v'(t_n) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})v''(s), \tag{4.28}$$

the expression for $E_n(\theta_h,\theta_{c,h})$ becomes:

$$\begin{aligned}E_n(\theta_h,\theta_{c,h}) = &-\big(\frac{1}{M}+c_f\varphi_0\big)\frac{1}{\Delta t}\int_\Omega\int_{t_{n-1}}^{t_n}(s-t_{n-1})p''(s)\theta_h - \frac{\alpha}{\Delta t}\int_{\Omega\setminus\mathcal{C}}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\nabla\cdot\boldsymbol{u}''(s)\theta_h\\ &+\frac{1}{\Delta t}\int_{\mathcal{C}}\int_{t_{n-1}}^{t_n}(s-t_{n-1})([\boldsymbol{u}''(s)]_{\mathcal{C}}\cdot\boldsymbol{n}^+)\theta_{c,h}.\end{aligned} \tag{4.29}$$

4.6.2 Full discretization error

Formulas (4.27), (4.13), and (4.14) lead to the following error equality:

$$\begin{aligned}&\frac{1}{\Delta t}\Big(\big(\frac{1}{M}+c_f\varphi_0\big)\delta(p_h^n-p(t_n))+\alpha\nabla\cdot\delta(\boldsymbol{u}_h^n-\boldsymbol{u}(t_n)),\theta_h\Big) - \frac{1}{\Delta t}\Big([\delta(\boldsymbol{u}_h^n-\boldsymbol{u}(t_n))]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\Big)_{\mathcal{C}}\\ &+\frac{1}{\mu_f}\Big(\nabla\cdot(\boldsymbol{z}_h^n-\boldsymbol{z}(t_n)),\theta_h\Big)+\frac{1}{12\mu_f}\langle\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}(\boldsymbol{\zeta}_h^n-\boldsymbol{\zeta}(t_n)),\theta_{c,h}\rangle_{\mathcal{C}} - \frac{1}{\mu_f}\langle[\boldsymbol{z}_h^n-\boldsymbol{z}(t^n)]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\rangle_{\mathcal{C}}\\ &= -E_n(\theta_h,\theta_{c,h}).\end{aligned} \tag{4.30}$$

In addition to r_h, R_h, and $r_{\mathcal{C},h}$, we need an approximation operator from $\boldsymbol{V}$ into $\boldsymbol{V}_h$, such as a Scott–Zhang approximation operator [24] or a Lagrange interpolation operator [12], and an approximation operator $R_{\mathcal{C},h}$ from $\boldsymbol{Z}_{\mathcal{C}}\cap H^s(\mathcal{C})^{d-1}$ into $\boldsymbol{Z}_{\mathcal{C},h}$ for some $s>0$. Then by adding and subtracting $I_h(\boldsymbol{u})$, $r_h(p)$, $r_{\mathcal{C},h}(p_c)$, $R_h(\boldsymbol{z})$, and $R_{\mathcal{C},h}(\boldsymbol{\zeta})$, and by denoting the discretization errors and interpolation errors respectively by:

$$\begin{aligned}&e_p^n = p_h^n - r_h(p(t_n))\ ,\ e_{c,p}^n = p_{c,h}^n - r_{\mathcal{C},h}(p_c(t_n))\ ,\ e_{\boldsymbol{u}}^n = \boldsymbol{u}_h^n - I_h(\boldsymbol{u}(t_n))\ ,\ e_{\boldsymbol{z}}^n = \boldsymbol{z}_h^n - R_h(\boldsymbol{z}(t_n)),\\ &e_{\boldsymbol{\zeta}}^n = \boldsymbol{\zeta}_h^n - R_{\mathcal{C},h}(\boldsymbol{\zeta}(t_n))\ ,\ a_p^n = r_h(p(t_n)) - p(t_n)\ ,\ a_{c,p}^n = r_{\mathcal{C},h}(p_c(t_n)) - p_c(t_n)\ ,\ a_{\boldsymbol{u}}^n = I_h(\boldsymbol{u}(t_n)) - \boldsymbol{u}(t_n),\\ &a_{\boldsymbol{z}}^n = R_h(\boldsymbol{z}(t_n)) - \boldsymbol{z}(t_n)\ ,\ a_{\boldsymbol{\zeta}}^n = R_{\mathcal{C},h}(\boldsymbol{\zeta}(t_n)) - \boldsymbol{\zeta}(t_n),\end{aligned} \tag{4.31}$$

we derive the pressure error equation for any $\theta_h\in Q_h$ and $\theta_{c,h}\in\Theta_{\mathcal{C},h}$:

$$\begin{aligned}&\big(\frac{1}{M}+c_f\varphi_0\big)\frac{1}{\Delta t}\big(\delta(e_p^n),\theta_h\big)+\frac{\alpha}{\Delta t}\big(\nabla\cdot\delta(e_{\boldsymbol{u}}^n),\theta_h\big)+\frac{1}{\mu_f}\big(\nabla\cdot e_{\boldsymbol{z}}^n,\theta_h\big)-\frac{1}{\Delta t}\big([\delta(e_{\boldsymbol{u}}^n)]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\big)_{\mathcal{C}}\\ &+\frac{1}{12\mu_f}\big(\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}e_{\boldsymbol{\zeta}}^n),\theta_{c,h}\big)_{\mathcal{C}}-\frac{1}{\mu_f}\big([e_{\boldsymbol{z}}^n]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\big)_{\mathcal{C}}\\ &= -\big(\frac{1}{M}+c_f\varphi_0\big)\frac{1}{\Delta t}\big(\delta(a_p^n),\theta_h\big)-\frac{\alpha}{\Delta t}\big(\nabla\cdot\delta(a_{\boldsymbol{u}}^n),\theta_h\big)-\frac{1}{\mu_f}\big(\nabla\cdot a_{\boldsymbol{z}}^n,\theta_h\big)+\frac{1}{\Delta t}\big([\delta(a_{\boldsymbol{u}}^n)]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\big)_{\mathcal{C}}\\ &-\frac{1}{12\mu_f}\langle\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}a_{\boldsymbol{\zeta}}^n),\theta_{c,h}\rangle_{\mathcal{C}}+\frac{1}{\mu_f}\langle[a_{\boldsymbol{z}}^n]_{\mathcal{C}}\cdot\boldsymbol{n}^+,\theta_{c,h}\rangle_{\mathcal{C}}-E_n(\theta_h,\theta_{c,h}).\end{aligned} \tag{4.32}$$

Note that the above right-hand side simplifies because both $\big(\nabla\cdot a_{\boldsymbol{z}}^n, \theta_h\big)$ and $\langle [a_{\boldsymbol{z}}^n]_{\mathcal{C}}\cdot \boldsymbol{n}^+, \theta_{c,h}\rangle_{\mathcal{C}}$ vanish owing to (4.18). Likewise, the poro-elastic displacement error equations are, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$:

$$
\begin{aligned}
&2G\big(\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^n), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) + \lambda\big(\nabla\cdot e_{\boldsymbol{u}}^n, \nabla\cdot \boldsymbol{v}_h\big) - \alpha\big(e_p^n, \nabla\cdot\boldsymbol{v}_h\big) + \big(e_{c,p}^n, [\boldsymbol{v}_h]_{\mathcal{C}}\cdot \boldsymbol{n}^+\big)_{\mathcal{C}} \\
&= -2G\big(\boldsymbol{\varepsilon}(a_{\boldsymbol{u}}^n), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) - \lambda\big(\nabla\cdot a_{\boldsymbol{u}}^n, \nabla\cdot\boldsymbol{v}_h\big) + \alpha\big(a_p^n, \nabla\cdot\boldsymbol{v}_h\big) - \big(a_{c,p}^n, [\boldsymbol{v}_h]_{\mathcal{C}}\cdot\boldsymbol{n}^+\big)_{\mathcal{C}}.
\end{aligned}
\tag{4.33}
$$

The fluid velocity error equations in the reservoir reduce to

$$
\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \big(\boldsymbol{K}^{-1}e_{\boldsymbol{z}}^n, \boldsymbol{q}_h\big) - \big(e_p^n, \nabla\cdot\boldsymbol{q}_h\big) + \big(e_{c,p}^n, [\boldsymbol{q}_h]_{\mathcal{C}}\cdot\boldsymbol{n}^+\big)_{\mathcal{C}} = -\big(\boldsymbol{K}^{-1}a_{\boldsymbol{z}}^n, \boldsymbol{q}_h\big), \tag{4.34}
$$

because the choice of r_h and $r_{\mathcal{C},h}$ (local L^2 projections) and the compatibility between the spaces imply that

$$
\big(a_p^n, \nabla\cdot\boldsymbol{q}_h\big) = 0\ ,\ \ \big(a_{c,p}^n, [\boldsymbol{q}_h]_{\mathcal{C}}\cdot\boldsymbol{n}^+\big)_{\mathcal{C}} = 0.
$$

In the fracture, the fluid velocity error equations read

$$
\forall \boldsymbol{q}_{c,h} \in \boldsymbol{Z}_{\mathcal{C},h}\,,\ \big(e_{\boldsymbol{\zeta}}^n, \boldsymbol{q}_{c,h}\big)_{\mathcal{C}} - \big(e_{c,p}^n, \overline{\nabla}\cdot((w^n)^{\frac{3}{2}}\boldsymbol{q}_{c,h})\big)_{\mathcal{C}} = -\big(a_{\boldsymbol{\zeta}}^n, \boldsymbol{q}_{c,h}\big)_{\mathcal{C}} + \big(a_{c,p}^n, \overline{\nabla}\cdot((w^n)^{\frac{3}{2}}\boldsymbol{q}_{c,h})\big)_{\mathcal{C}}. \tag{4.35}
$$

On one hand, (4.34) and the inf-sup condition (4.20) imply the estimate

$$
\|e_{c,p}^n\|_{L^2(\mathcal{C})} \le \frac{1}{\beta_1^\star} E_{p,\boldsymbol{z}}^n, \tag{4.36}
$$

where

$$
E_{p,\boldsymbol{z}}^n = \big(\|e_p^n\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{K}^{-1}e_{\boldsymbol{z}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}} + \|\boldsymbol{K}^{-1}a_{\boldsymbol{z}}^n\|_{L^2(\Omega\setminus\mathcal{C})}. \tag{4.37}
$$

On the other hand, by testing (4.32) with $\theta_h = e_p^n$ and $\theta_{c,h} = e_{c,p}^n$, (4.33) with $\boldsymbol{v}_h = \delta(e_{\boldsymbol{u}}^n)$, (4.34) with $\boldsymbol{q}_h = e_{\boldsymbol{z}}^n$, and (4.35) with $\boldsymbol{q}_{c,h} = e_{\boldsymbol{\zeta}}^n$, multiplying everything by $\Delta\, t$, and summing the resulting equations, we derive:

$$
\begin{aligned}
&\frac{1}{2}\Big(\frac{1}{M} + c_f\varphi_0\Big)\Big(\delta\big(\|e_p^n\|^2_{L^2(\Omega)}\big) + \|\delta e_p^n\|^2_{L^2(\Omega)}\Big) + G\Big(\delta\big(\|\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^n)\|^2_{L^2(\Omega\setminus\mathcal{C})}\big) + \|\boldsymbol{\varepsilon}(\delta e_{\boldsymbol{u}}^n)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big) \\
&+ \frac{\lambda}{2}\Big(\delta\big(\|\nabla\cdot e_{\boldsymbol{u}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})}\big) + \|\nabla\cdot\delta e_{\boldsymbol{u}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big) + \frac{\Delta\, t}{\mu_f}\|\boldsymbol{K}^{-\frac{1}{2}}e_{\boldsymbol{z}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{\Delta\, t}{12\mu_f}\|e_{\boldsymbol{\zeta}}^n\|^2_{L^2(\mathcal{C})} \\
&= -\Delta\, t\, E_n(e_p^n, e_{c,p}^n) - A_{\boldsymbol{u},p}^n - A_{\boldsymbol{u}}^n + A_{\delta,\boldsymbol{u}}^n + \frac{\Delta\, t}{\mu_f}\Big(- A_{p,\boldsymbol{z},\boldsymbol{\zeta}}^n - A_{\boldsymbol{z},\boldsymbol{\zeta}}^n + \frac{1}{12}A_{\boldsymbol{\zeta}}^n\Big),
\end{aligned}
\tag{4.38}
$$

where

$$
A_{\boldsymbol{u},p}^n = \Big(\frac{1}{M} + c_f\varphi_0\Big)\big(\delta(a_p^n), e_p^n\big) + \alpha\big(\nabla\cdot\delta(a_{\boldsymbol{u}}^n), e_p^n\big), \tag{4.39}
$$

$$
A_{p,\boldsymbol{z},\boldsymbol{\zeta}}^n = \big(\boldsymbol{K}^{-1}a_{\boldsymbol{z}}^n, e_{\boldsymbol{z}}^n\big) + \frac{1}{12}\big(a_{\boldsymbol{\zeta}}^n, e_{\boldsymbol{\zeta}}^n\big), \tag{4.40}
$$

$$
A_{\boldsymbol{u}}^n = 2G\big(\boldsymbol{\varepsilon}(a_{\boldsymbol{u}}^n), \boldsymbol{\varepsilon}(\delta(e_{\boldsymbol{u}}^n))\big) + \lambda\big(\nabla\cdot a_{\boldsymbol{u}}^n, \nabla\cdot(\delta e_{\boldsymbol{u}}^n)\big) - \alpha\big(a_p^n, \nabla\cdot\delta(e_{\boldsymbol{u}}^n)\big) + \big(a_{c,p}^n, [\delta(e_{\boldsymbol{u}}^n)]_{\mathcal{C}}\cdot\boldsymbol{n}^+\big)_{\mathcal{C}}, \tag{4.41}
$$

$$
A_{\delta,\boldsymbol{u}}^n = \big(e_{c,p}^n, [\delta(a_{\boldsymbol{u}}^n)]_{\mathcal{C}}\cdot\boldsymbol{n}^+\big)_{\mathcal{C}}, \tag{4.42}
$$

$$
A_{\boldsymbol{z},\boldsymbol{\zeta}}^n = \frac{1}{12}\langle\overline{\nabla}\cdot\big((w^n)^{\frac{3}{2}}a_{\boldsymbol{\zeta}}^n\big), e_{c,p}^n\rangle_{\mathcal{C}}, \tag{4.43}
$$

$$A_{\boldsymbol{\zeta}}^n = \big(\overline{\nabla}\cdot\big((w^n)^{\frac{3}{2}} e_{\boldsymbol{\zeta}}^n\big), a_{c,p}^n\big)_{\mathcal{C}}. \tag{4.44}$$

Therefore, we must derive bounds for these six quantities. First, $A_{p,\boldsymbol{z},\boldsymbol{\zeta}}^n$ has a straightforward bound:

$$\frac{\Delta t}{\mu_f}|A_{p,\boldsymbol{z},\boldsymbol{\zeta}}^n| \le \frac{\Delta t}{\mu_f}\Big[\|\boldsymbol{K}^{-\frac{1}{2}} e_{\boldsymbol{z}}^n\|_{L^2(\Omega\setminus\mathcal{C})}\|\boldsymbol{K}^{-\frac{1}{2}} a_{\boldsymbol{z}}^n\|_{L^2(\Omega\setminus\mathcal{C})} + \frac{1}{12}\|e_{\boldsymbol{\zeta}}^n\|_{L^2(\mathcal{C})}\|a_{\boldsymbol{\zeta}}^n\|_{L^2(\mathcal{C})}\Big]. \tag{4.45}$$

Next, considering that for example

$$\|\delta(r_h(p(t_n)) - p(t_n))\|_{L^2(\Omega)} \le \sqrt{\Delta t}\|r_h(p') - p'\|_{L^2(\Omega\times]t_{n-1},t_n[)},$$

we find a straightforward bound for $A_{\boldsymbol{u},p}^n$:

$$|A_{\boldsymbol{u},p}^n| \le \sqrt{\Delta t}\|e_p^n\|_{L^2(\Omega)}\Big[\big(\frac{1}{M}+c_f\varphi_0\big)\|r_h(p')-p'\|_{L^2(\Omega\times]t_{n-1},t_n[)}+\alpha\|\nabla\cdot(I_h(\boldsymbol{u}')-\boldsymbol{u}')\|_{L^2((\Omega\setminus\mathcal{C})\times]t_{n-1},t_n[)}\Big]. \tag{4.46}$$

Similarly, applying (4.36), the trace inequality (3.21), and (4.37), $A_{\delta,\boldsymbol{u}}^n$ is bounded by

$$|A_{\delta,\boldsymbol{u}}^n| \le \frac{\sqrt{\Delta t}}{\beta_1^\star}C\Big[\big(\|e_p^n\|^2_{L^2(\Omega\setminus\mathcal{C})}+\|\boldsymbol{K}^{-1}e_{\boldsymbol{z}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}}+\|\boldsymbol{K}^{-1}a_{\boldsymbol{z}}^n\|_{L^2(\Omega\setminus\mathcal{C})}\Big]\|I_h(\boldsymbol{u}')-\boldsymbol{u}'\|_{L^2(t_{n-1},t_n;\boldsymbol{V})}, \tag{4.47}$$

with the constant C of (3.21). Now we proceed with $A_{\boldsymbol{u}}^n$. As it involves factors of the form $\delta(e_{\boldsymbol{u}}^n)$ that cannot be absorbed by the left-hand side, and considering that the whole expression needs to be summed over n, we use a summation by parts that switches the difference to the first factor:

$$\sum_{m=1}^{n} a^m(\delta b^m) = -\sum_{m=1}^{n-1}(\delta a^{m+1})b^m + a^n b^n - a^1 b^0. \tag{4.48}$$

This gives

$$\begin{aligned}
|\sum_{m=1}^{n} A_{\boldsymbol{u}}^m| \le \sum_{m=1}^{n-1}\Big[&2G\|\boldsymbol{\varepsilon}(\delta(a_{\boldsymbol{u}}^{m+1}))\|_{L^2(\Omega\setminus\mathcal{C})}\|\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^m)\|_{L^2(\Omega\setminus\mathcal{C})} + \lambda\|\nabla\cdot\delta(a_{\boldsymbol{u}}^{m+1})\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^m\|_{L^2(\Omega\setminus\mathcal{C})}\\
&+ \alpha\|\delta(a_p^{m+1})\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^m\|_{L^2(\Omega\setminus\mathcal{C})} + C\|\delta(a_{c,p}^{m+1})\|_{L^2(\mathcal{C})}\|e_{\boldsymbol{u}}^m\|_{\boldsymbol{V}}\Big]\\
&+ 2G\|\boldsymbol{\varepsilon}(a_{\boldsymbol{u}}^n)\|_{L^2(\Omega\setminus\mathcal{C})}\|\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^n)\|_{L^2(\Omega\setminus\mathcal{C})} + \lambda\|\nabla\cdot a_{\boldsymbol{u}}^n\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^n\|_{L^2(\Omega\setminus\mathcal{C})}\\
&+ \alpha\|a_p^n\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^n\|_{L^2(\Omega\setminus\mathcal{C})} + C\|a_{c,p}^n\|_{L^2(\mathcal{C})}\|e_{\boldsymbol{u}}^n\|_{\boldsymbol{V}}\\
&+ 2G\|\boldsymbol{\varepsilon}(a_{\boldsymbol{u}}^1)\|_{L^2(\Omega\setminus\mathcal{C})}\|\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^0)\|_{L^2(\Omega\setminus\mathcal{C})} + \lambda\|\nabla\cdot a_{\boldsymbol{u}}^1\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^0\|_{L^2(\Omega\setminus\mathcal{C})}\\
&+ \alpha\|a_p^1\|_{L^2(\Omega\setminus\mathcal{C})}\|\nabla\cdot e_{\boldsymbol{u}}^0\|_{L^2(\Omega\setminus\mathcal{C})} + C\|a_{c,p}^1\|_{L^2(\mathcal{C})}\|e_{\boldsymbol{u}}^0\|_{\boldsymbol{V}}.
\end{aligned} \tag{4.49}$$

There remains to examine $A_{\boldsymbol{z},\boldsymbol{\zeta}}^n$ and $A_{\boldsymbol{\zeta}}^n$. Let us start with $A_{\boldsymbol{\zeta}}^n$; it involves a factor that cannot be absorbed by the left-hand side. We cannot use directly the compatibility properties of the spaces on $\mathcal{C}$ and the projection properties because of the variable factor $(w^n)^{\frac{3}{2}}$. By expanding the divergence, we write

$$A_{\boldsymbol{\zeta}}^n = \big((w^n)^{\frac{3}{2}}\overline{\nabla}\cdot e_{\boldsymbol{\zeta}}^n, a_{c,p}^n\big)_{\mathcal{C}} + \big(\overline{\nabla}(w^n)^{\frac{3}{2}}\cdot e_{\boldsymbol{\zeta}}^n, a_{c,p}^n\big)_{\mathcal{C}}. \tag{4.50}$$

Now, let $\pi_0(w^n)^{\frac{3}{2}}$ denote the average of $(w^n)^{\frac{3}{2}}$ in each e:

$$\pi_0(w^n)^{\frac{3}{2}} = \frac{1}{|e|}\int_e (w^n)^{\frac{3}{2}}.$$

Then the projection property of $r_{\mathcal{C},h}$ and the fact that $\pi_0(w^n)^{\frac{3}{2}}$ is a constant in each e yield

$$\begin{aligned}\big((w^n)^{\frac{3}{2}}\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, a^n_{c,p}\big)_{\mathcal{C}} =& \big(\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, \big((w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\big)a^n_{c,p}\big)_{\mathcal{C}} + \big(\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, \pi_0(w^n)^{\frac{3}{2}}a^n_{c,p}\big)_{\mathcal{C}}\\ =& \big(\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, \big((w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\big)a^n_{c,p}\big)_{\mathcal{C}}.\end{aligned} \tag{4.51}$$

Moreover,

$$\|(w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\|_{L^4(e)} \le Ch_e\|\overline{\nabla}((w^n)^{\frac{3}{2}})\|_{L^4(e)}.$$

Therefore, by applying a local inverse inequality in each e, we deduce that

$$\big|\big(\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, \big((w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\big)a^n_{c,p}\big)_e\big| \le C\|\overline{\nabla}((w^n)^{\frac{3}{2}})\|_{L^4(e)}\|e^n_{\boldsymbol{\zeta}}\|_{L^2(e)}\|a^n_{c,p}\|_{L^4(e)}.$$

Hence summing over all e in $\mathcal{C}_h$, we obtain

$$\big|\big(\overline{\nabla}\cdot e^n_{\boldsymbol{\zeta}}, \big((w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\big)a^n_{c,p}\big)_{\mathcal{C}}\big| \le C\|\overline{\nabla}((w^n)^{\frac{3}{2}})\|_{L^4(\mathcal{C})}\|e^n_{\boldsymbol{\zeta}}\|_{L^2(\mathcal{C})}\|a^n_{c,p}\|_{L^4(\mathcal{C})},$$

where the first factor is bounded in view of (2.15). We can easily check that the second term in (4.50) has the same bound. Therefore

$$\frac{\Delta t}{12\mu_f}|A^n_{\boldsymbol{\zeta}}| \le \frac{\Delta t}{12\mu_f}C|(w^n)^{\frac{3}{2}}|_{W^{1,4}(\mathcal{C})}\|e^n_{\boldsymbol{\zeta}}\|_{L^2(\mathcal{C})}\|a^n_{c,p}\|_{L^4(\mathcal{C})}. \tag{4.52}$$

A similar argument can be applied to $A^n_{\boldsymbol{z},\boldsymbol{\zeta}}$. Indeed, considering the compatibility of the finite element spaces on $\mathcal{C}$, we have:

$$\big(\overline{\nabla}\cdot a^n_{\boldsymbol{\zeta}}, \pi_0(w^n)^{\frac{3}{2}}e^n_{c,p}\big)_{\mathcal{C}} = 0.$$

Therefore, by writing

$$A^n_{\boldsymbol{z},\boldsymbol{\zeta}} = \frac{1}{12}\Big[\big(\big((w^n)^{\frac{3}{2}} - \pi_0(w^n)^{\frac{3}{2}}\big)\overline{\nabla}\cdot a^n_{\boldsymbol{\zeta}}, e^n_{c,p}\big)_{\mathcal{C}} + \big(\overline{\nabla}((w^n)^{\frac{3}{2}})\cdot a^n_{\boldsymbol{\zeta}}, e^n_{c,p}\big)_{\mathcal{C}}\Big],$$

we obtain

$$\big|A^n_{\boldsymbol{z},\boldsymbol{\zeta}}\big| \le \frac{1}{12}\|e^n_{c,p}\|_{L^2(\mathcal{C})}|(w^n)^{\frac{3}{2}}|_{W^{1,4}(\mathcal{C})}\Big(\|a^n_{\boldsymbol{\zeta}}\|_{L^4(\mathcal{C})} + Ch\|\overline{\nabla}\cdot a^n_{\boldsymbol{\zeta}}\|_{L^4(\mathcal{C})}\Big).$$

Then by substituting (4.36) and (4.37) in the above inequality, we derive

$$\begin{aligned}\frac{\Delta t}{\mu_f}|A^n_{\boldsymbol{z},\boldsymbol{\zeta}}| \le& \frac{\Delta t}{\mu_f}\frac{1}{12\beta_1^\star}\Big[\big(\|e^n_p\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{K}^{-1}e^n_{\boldsymbol{z}}\|^2_{L^2(\Omega\setminus\mathcal{C})}\big)^{\frac{1}{2}} + \|\boldsymbol{K}^{-1}a^n_{\boldsymbol{z}}\|_{L^2(\Omega\setminus\mathcal{C})}\Big]\\ &\times |(w^n)^{\frac{3}{2}}|_{W^{1,4}(\mathcal{C})}\Big(\|a^n_{\boldsymbol{\zeta}}\|_{L^4(\mathcal{C})} + Ch\|\overline{\nabla}\cdot a^n_{\boldsymbol{\zeta}}\|_{L^4(\mathcal{C})}\Big).\end{aligned} \tag{4.53}$$

The next theorem collects these results and concludes with a basic error bound. The proof is skipped, as it is a straightforward consequence of repeated applications of Young's inequality with suitable coefficients and a discrete Gronwall's Lemma.

Theorem 4.1. *Let the data $\boldsymbol{f}$, $\tilde{q}$, $\tilde{q}_W$ and $p(0)$ be sufficiently smooth and let Hypotheses 2.1 and 4.1 hold. Suppose that problem (2.30)–(2.34) and (2.14) has a sufficiently smooth solution. Then the sequence of solutions $(\boldsymbol{u}_h^n, p_h^n, p_{c,h}^n, \boldsymbol{z}_h^n, \boldsymbol{\zeta}_h^n)$ of (4.12)–(4.16) with starting values $(p_h^0, p_{c,h}^0, \boldsymbol{u}_h^0, \boldsymbol{z}_h^0, \boldsymbol{\zeta}_h^0)$, $\boldsymbol{u}_h^0$, $\boldsymbol{z}_h^0$, $\boldsymbol{\zeta}_h^0$ being computed respectively by (4.9), (4.10), (4.11), satisfies the following error bounds for any integer n, $1 \le n \le N$:*

$$\begin{aligned}
&\Big(\frac{1}{M} + c_f\varphi_0\Big)\Big(\|e_p^n\|^2_{L^2(\Omega)} + \sum_{m=1}^n \|\delta e_p^m\|^2_{L^2(\Omega)}\Big) + 2G\Big(\|\boldsymbol{\varepsilon}(e_{\boldsymbol{u}}^n)\|^2_{L^2(\Omega\setminus\mathcal{C})} + \sum_{m=1}^n \|\boldsymbol{\varepsilon}(\delta e_{\boldsymbol{u}}^m)\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big)\\
&+ \lambda\Big(\|\nabla\cdot e_{\boldsymbol{u}}^n\|^2_{L^2(\Omega\setminus\mathcal{C})} + \sum_{m=1}^n \|\nabla\cdot\delta e_{\boldsymbol{u}}^m\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big) + \frac{1}{\mu_f}\sum_{m=1}^n \Delta t\Big(\|\boldsymbol{K}^{-\frac{1}{2}} e_{\boldsymbol{z}}^m\|^2_{L^2(\Omega\setminus\mathcal{C})} + \frac{1}{12}\|e_{\boldsymbol{\zeta}}^m\|^2_{L^2(\mathcal{C})}\Big)\\
&\le C\Big[(\Delta t)^2\big(\|p''\|^2_{L^2(\Omega\times]0,t_n[)} + \|\boldsymbol{u}''\|^2_{L^2(0,t_n;\boldsymbol{V})}\big) + \|e_{\boldsymbol{u}}^0\|^2_{\boldsymbol{V}}\\
&+ \|I_h(\boldsymbol{u}) - \boldsymbol{u}\|^2_{H^1(0,t_n;\boldsymbol{V})} + \|r_h(p) - p\|^2_{H^1(0,t_n;L^2(\Omega\setminus\mathcal{C}))} + \|r_{\mathcal{C},h}(p_c) - p_c\|^2_{H^1(0,t_n;L^2(\mathcal{C}))}\\
&+ \|R_h(\boldsymbol{z}) - \boldsymbol{z}\|^2_{\mathcal{C}^0(0,t_n;L^2(\Omega\setminus\mathcal{C})^d)} + \|w^{\frac{3}{2}}\|_{\mathcal{C}^0(0,t_n;W^{1,4}(\mathcal{C}))}\big(\|r_{\mathcal{C},h}(p_c) - p_c\|^2_{\mathcal{C}^0(0,t_n;L^4(\mathcal{C}))}\\
&+ \|R_{\mathcal{C},h}(\boldsymbol{\zeta}) - \boldsymbol{\zeta}\|^2_{\mathcal{C}^0(0,t_n;L^4(\mathcal{C})^{d-1})} + h\|\overline{\nabla}\cdot(R_{\mathcal{C},h}(\boldsymbol{\zeta}) - \boldsymbol{\zeta})\|^2_{\mathcal{C}^0(0,t_n;L^4(\mathcal{C}))}\big)\Big]\exp(t_n),
\end{aligned} \tag{4.54}$$

with a constant C independent of n, h, and Δt, and

$$\sum_{m=1}^n \Delta t\|e_{c,p}^m\|^2_{L^2(\mathcal{C})} \le \frac{2}{(\beta_1^\star)^2}\sum_{m=1}^n \Delta t\Big(\|e_p^m\|^2_{L^2(\Omega)} + \|\boldsymbol{K}^{-1}e_{\boldsymbol{z}}^m\|^2_{L^2(\Omega\setminus\mathcal{C})} + \|\boldsymbol{K}^{-1}(R_h(\boldsymbol{z}(t_m)) - \boldsymbol{z}(t_m))\|^2_{L^2(\Omega\setminus\mathcal{C})}\Big), \tag{4.55}$$

with the constant $\beta_1^\star$ of (4.20).

Remark 4.1. Further error bounds, in the spirit of the estimates derived in Section 3.2, are more delicate. On one hand, Hypothesis 3.1 is quite restrictive, and on the other hand, the choice of the fracture's discrete spaces in (4.9)–(4.16) is not consistent with the theoretical setting because the relevant space for the fracture's pressure p_c should be $H^{\frac{1}{2}}(\mathcal{C})$ instead of $L^2(\mathcal{C})$. We use L^2 pressures because they are locally mass conservative and by taking advantage of the finite dimension, they lead to the basic estimates of Section 4.6, but it is not clear that they lead to additional satisfactory estimates and in particular to a useful bound for the discrete leakage term. If we want complete estimates, we can modify the scheme so that it matches the setting of (2.30)–(2.34), and in particular uses continuous pressures in the fracture. For instance, we can choose

$$\Theta_{\mathcal{C},h} = \left\{q \in \mathcal{C}^0(\mathcal{C})\,;\, q|_{\mathcal{S}_i} \in \Theta_{\mathcal{S}_i,h}, 1 \le i \le I\right\},$$

with

$$\Theta_{\mathcal{S},h} = \left\{q \in \mathcal{C}^0(\mathcal{S})\,;\, q|_e \leftrightarrow \hat{q},\ \hat{q} \in \hat{\Theta}_{\mathcal{C}}(\hat{e}), \quad \forall e \in \mathcal{T}_{\mathcal{S},h}\right\},$$

without changing the other spaces. We can prove that $p_{c,h}$ satisfies an inf-sup condition in $H^{\frac{1}{2}}(\mathcal{C})$ by exploiting the fact that if the functions of $\Theta_{\mathcal{C},h}$ are continuous and piecewise polynomials, then they belong to $H^1(\mathcal{C})$. The proof is more complex than that of Lemma 4.1, but it still requires (4.19). This hypothesis holds if we raise the degree of the polynomials, which may not be desirable. □

5 Fixed stress splitting

We shall use the following fixed stress splitting algorithm for decoupling the computation of the mechanics from that of the flow. To simplify, we describe it at the exact level and we denote the time derivative by ∂_t. It proceeds in two steps. First the flow problem in the reservoir and fracture is solved in a monolithic manner:

- Step (a)
  Given $\boldsymbol{u}^n$, we solve for $p^{n+1}, \boldsymbol{z}^{n+1}, p_c^{n+1}, \boldsymbol{\zeta}^{n+1}$ such that

$$\Big(\frac{1}{M} + c_f\varphi_0 + \frac{\alpha^2}{\lambda}\Big)\partial_t p^{n+1} + \frac{1}{\mu_f}\nabla\cdot\boldsymbol{z}^{n+1} = \frac{\alpha^2}{\lambda}\partial_t p^n - \alpha\nabla\cdot\partial_t\boldsymbol{u}^n + \tilde{q} \ \text{ in } \Omega\setminus\mathcal{C}, \quad (5.56)$$

$$\boldsymbol{z}^{n+1} = -\boldsymbol{K}\nabla\big(p^{n+1} - \rho_{f,r}g\eta\big), \quad (5.57)$$

$$\gamma_c\partial_t p_c^{n+1} + \partial_t w^n + \frac{1}{12\mu_f}\overline{\nabla}\cdot\big((w^n)^{\frac{3}{2}}\boldsymbol{\zeta}^{n+1}\big) = \gamma_c\partial_t p_c^n + \tilde{q}_W + \frac{1}{\mu_f}[\boldsymbol{z}^{n+1}]_{\mathcal{C}}\cdot\boldsymbol{n}^+ \ \text{ in } \mathcal{C}, \quad (5.58)$$

$$\boldsymbol{\zeta}^{n+1} = -(w^n)^{\frac{3}{2}}\overline{\nabla}\big(p_c^{n+1} - \rho_{f,r}g\eta\big). \quad (5.59)$$

$$w^n = -[\boldsymbol{u}^n]_{\mathcal{C}}\cdot\boldsymbol{n}^+.$$

  Once the flow is computed, we update the displacement solution.

- Step (b)
  Given $p^{n+1}, \boldsymbol{z}^{n+1}, p_c^{n+1}, \boldsymbol{\zeta}^{n+1}$, we solve for $\boldsymbol{u}^{n+1}$ satisfying

$$-\operatorname{div}\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}^{n+1}, p^{n+1}) = \boldsymbol{f} \ \text{ in } \Omega\setminus\mathcal{C}, \quad (5.60)$$

$$(\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}^{n+1}, p^{n+1}))^\star\boldsymbol{n}^\star = -p_c^{n+1}\boldsymbol{n}^\star\ ,\ \star = +,- \ \text{ on } \mathcal{C}, \quad (5.61)$$

$$\text{where} \qquad \boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}^{n+1}, p^{n+1}) = \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}) - \alpha\, p^{n+1}\boldsymbol{I} \ \text{ in } \Omega\setminus\mathcal{C}. \quad (5.62)$$

The stabilizing terms $\frac{\alpha^2}{\lambda}\partial_t p^{n+1}$ and $\gamma_c\partial_t p_c^{n+1}$ are added to the left-hand sides of (5.56) and (5.58) respectively, with similar terms on the right-hand sides of the equations for the sake of consistency. The first term is a standard addition in fixed stress splitting, see [22]. Motivated by this, we add a similar term to the fracture equation with an adjustable coefficient γ_c.
The following definition of the volumetric mean stress:

$$\sigma_v = \sigma_{v,0} + \lambda\nabla\cdot\boldsymbol{u} - \alpha(p - p_0), \quad (5.63)$$

where $\sigma_{v,0}$ denotes the initial volumetric stress, justifies the name of the algorithm. Indeed, as $\sigma_{v,0}$ and p_0 are constant in time, we have

$$-\frac{\alpha}{\lambda}\partial_t\sigma_v^n = \frac{\alpha^2}{\lambda}\partial_t p^n - \alpha\nabla\cdot\partial_t\boldsymbol{u}^n, \quad (5.64)$$

and we recognize the first two terms in the right-hand side of (5.56).
The variational form of the algorithm reads as follows:

- Step (a) find $p^{n+1} \in L^\infty(0,T;L^2(\Omega))$, $p_c^{n+1} \in L^2(0,T;H^{\frac{1}{2}}(\mathcal{C}))$, $\boldsymbol{z}^{n+1} \in L^2(0,T;\boldsymbol{Z})$, and $\boldsymbol{\zeta}^{n+1} \in L^2(0,T;\boldsymbol{Z}_\mathcal{C})$ such that for all $t \in]0,T[$,

$$\forall \theta \in L^2(\Omega)\,,\ \Big(\big(\frac{1}{M}+c_f\varphi_0+\frac{\alpha^2}{\lambda}\big)\partial_t p^{n+1},\theta\Big)+\frac{1}{\mu_f}\big(\nabla\cdot \boldsymbol{z}^{n+1},\theta\big)=\Big(-\frac{\alpha}{\lambda}\partial_t\sigma_v^n,\theta\Big)+\big(\tilde{q},\theta\big), \tag{5.65}$$

$$\forall \theta_c \in H^{\frac{1}{2}}(\mathcal{C})\,,\ \gamma_c\big(\partial_t p_c^{n+1},\theta_c\big)_\mathcal{C}+\frac{1}{12\mu_f}\big(\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}\boldsymbol{\zeta}^{n+1}),\theta_c\big)_\mathcal{C}$$
$$-\frac{1}{\mu_f}\big([\boldsymbol{z}^{n+1}]_\mathcal{C}\cdot\boldsymbol{n}^+,\theta_c\big)_\mathcal{C}=\Big(\gamma_c\partial_t p_c^n,\theta_c\Big)_\mathcal{C}+\big(\partial_t[\boldsymbol{u}^n]_\mathcal{C}\cdot\boldsymbol{n}^+,\theta_c\big)_\mathcal{C}+\big(\tilde{q}_W,\theta_c\big)_\mathcal{C}, \tag{5.66}$$

$$\forall \boldsymbol{q} \in \boldsymbol{Z}\,,\ \big(\boldsymbol{K}^{-1}\boldsymbol{z}^{n+1},\boldsymbol{q}\big)=(p^{n+1},\nabla\cdot\boldsymbol{q})-\big(p_c^{n+1},[\boldsymbol{q}]_\mathcal{C}\cdot\boldsymbol{n}^+\big)_\mathcal{C}+\big(\nabla(\rho_{f,r}g\eta),\boldsymbol{q}\big), \tag{5.67}$$

$$\forall \boldsymbol{q}_c \in \boldsymbol{Z}_\mathcal{C}\,,\ \big(\boldsymbol{\zeta}^{n+1},\boldsymbol{q}_c\big)_\mathcal{C}=\big(p_c^{n+1},\overline{\nabla}\cdot((w^n)^{\frac{3}{2}}\boldsymbol{q}_c)\big)_\mathcal{C}+\big((w^n)^{\frac{3}{2}}\overline{\nabla}(\rho_{f,r}g\eta),\boldsymbol{q}_c\big)_\mathcal{C}, \tag{5.68}$$

with the initial condition, independent of n,

$$p^{n+1}(0)=p_0\ ,\ p_c^{n+1}(0)=p_0|_\mathcal{C}. \tag{5.69}$$

- Step (b) Given $p^{n+1},\boldsymbol{z}^{n+1},p_c^{n+1},\boldsymbol{\zeta}^{n+1}$, find $\boldsymbol{u}^{n+1} \in L^\infty(0,T;V)$ such that for all $t \in (0,T)$,

$$\forall \boldsymbol{v} \in V\,,\ 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n+1}),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)+\lambda\big(\nabla\cdot\boldsymbol{u}^{n+1},\nabla\cdot\boldsymbol{v}\big)-\alpha\big(p^{n+1},\nabla\cdot\boldsymbol{v}\big)+\big(p_c^{n+1},[\boldsymbol{v}]_\mathcal{C}\cdot\boldsymbol{n}^+\big)_\mathcal{C}=\big(\boldsymbol{f},\boldsymbol{v}\big). \tag{5.70}$$

We have seen that (5.70) defines $\boldsymbol{u}^{n+1}(0)$ in terms of p_0 and p_c^0, and in turn $w^{n+1}(0) = -[\boldsymbol{u}^{n+1}(0)]_\mathcal{C}\cdot\boldsymbol{n}^+$, all quantities being independent of n. To begin the iteration, for $n=0$, we assign as initial condition $p^0 = p_0$, $p_c^0 = p_0|_\mathcal{C}$, $\boldsymbol{u}^0$ is computed from p^0 and p_c^0 by (2.30) at time $t=0$, and $w^0 = -[\boldsymbol{u}^0]_\mathcal{C}\cdot\boldsymbol{n}^+$. More specific details can be found in Ganis et al [11]. Notice that in (5.65), the right-hand side has been re-written in terms of the volumetric mean total stress as defined in (5.64). As such, the convergence of this algorithm is an open problem. With a suitable choice of parameter γ_c in terms of the material parameters, constants of the trace and the Korn's inequalities, convergence of a simplified version is established in Girault et al [13].

6 Solution Algorithm

A fixed stress splitting scheme is employed as described in section 5 for which a flowchart is provided in 6.1. Here we iterate between the flow solution assuming a fixed stress field and the mechanics solution assuming fixed pressure and saturation fields. For mechanics we apply a Galerkin finite element with continuous piecewise linears and for flow a mixed finite element (MFMFE) is used as described in section 4.3. The simulations were performed using the coupled flow and geomechanics reservoir simulator IPARS (Integrated Parallel Accurate Reservoir Simulator). IPARS is capable of handling complex subsurface flow descriptions such as two-phase, black oil and compositional flow along with chemical equilibrium and kinetic type reactions.

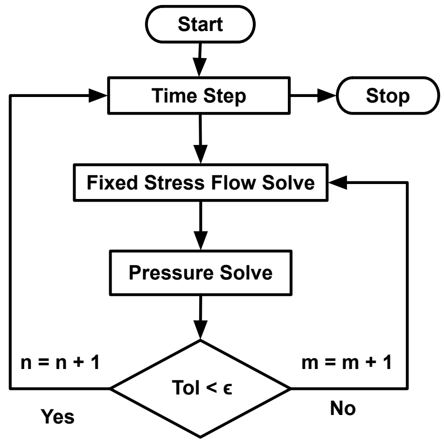

Figure 6.1: Flowchart for iteratively coupled flow and poroelasticity in IPARS

6.1 Numerical Results

In this numerical experiment, we show the stress and displacement fields in a poroelastic domain with two orthogonal fractures. Figure 6.2 shows a schematic of the problem along with boundary conditions and location of the fractures. A square domain $\Omega = (0, 250\,\text{ft}) \times (0, 250\,\text{ft})$ is considered with two orthogonal fractures along the axes $\{y = 125\}$ ft and $\{z = 150\}$ ft, each 50 ft in length with one end point at $(125\,\text{ft}, 62.5\,\text{ft})$ and $(100\,\text{ft}, 150\,\text{ft})$, respectively. A no flow ($\boldsymbol{z} = \mathbf{0}$) boundary condition is specified on all the edges allowing the pressure in the domain to rise with time. A zero displacement ($\boldsymbol{u} = \mathbf{0}$) boundary condition is specified for the left and bottom edges whereas normal stresses ($\boldsymbol{\sigma}^{por}\boldsymbol{n}$) of $(-6300, 0)$ psi and $(-6400, 0)$ psi are specified at the right and top edges, respectively as shown in Figure 6.2. Further, an initial condition of 500 psi for pressure is specified both in the poroelastic domain (Ω) and on the fracture ($\mathcal{C}$). Fluid is injected into the middle of each fracture at 5000 psi.

A homogeneous porosity value of 0.2 and homogeneous and isotropic permeability tensor of 50 mD is assumed. The fluid is assumed to be slightly compressible with density 62.4 lbm/ft^3 and compressibility 1×10^{-6} psi^{-1}. The Young's modulus and Poisson's ratio of the poroelastic medium are 7.3×10^6 psi and 0.2, respectively.

The domain is discretized into 80×80 structured hexahedral elements with a uniform mesh width of 3.125 ft in both y and z directions. Figures 6.3, 6.4, 6.5, 6.6, 6.7 show the pressure, stress and displacement profiles in y and z directions, respectively at $T = 0.0$, 0.05 and 0.1 days.

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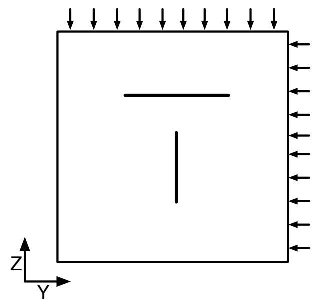

Figure 6.2: Problem Schematic

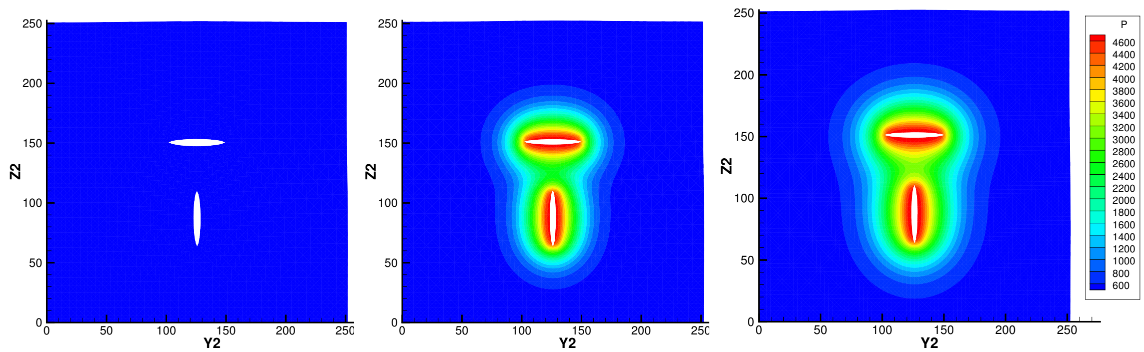

Figure 6.3: Pressure profiles at T = 0.0, 0.05 and 0.1 day.

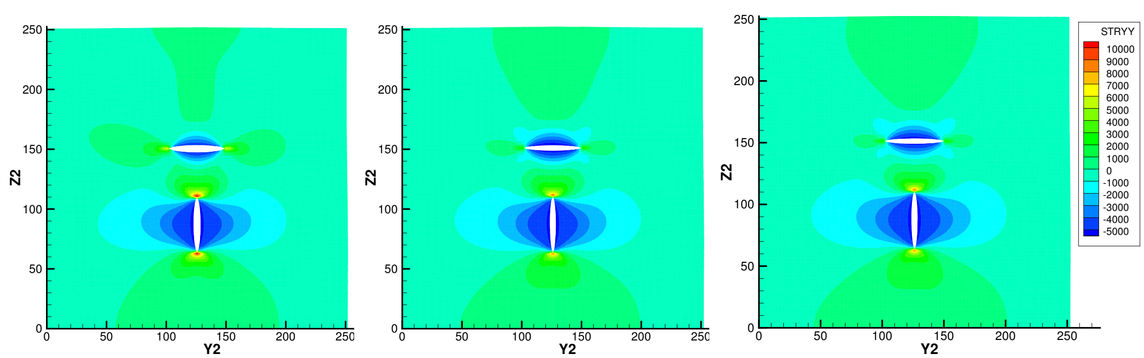

Figure 6.4: Stress (σ_{yy}) profiles at T = 0.0, 0.05 and 0.1 day.

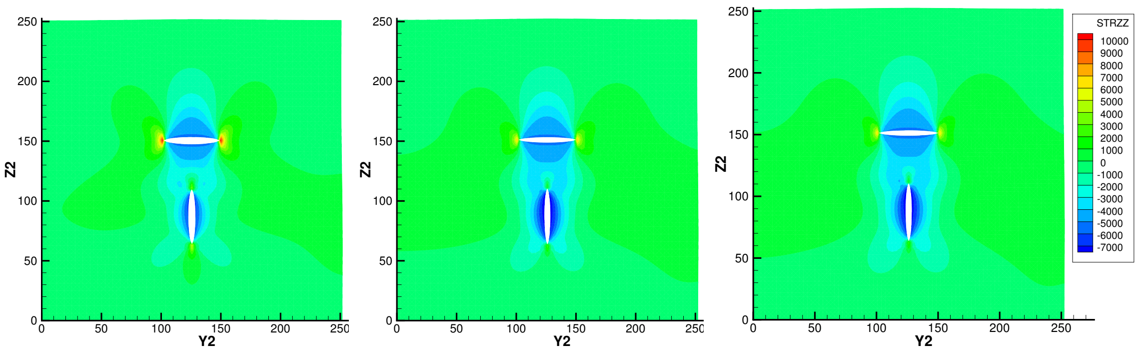

Figure 6.5: Stress (σ_{zz}) profiles at T = 0.0, 0.05 and 0.1 day.

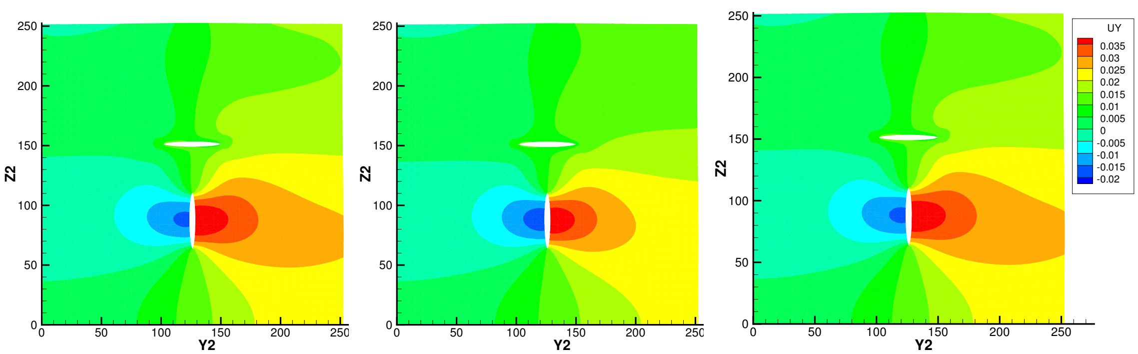

Figure 6.6: Y-direction displacement profiles at T = 0.0, 0.05 and 0.1 day.

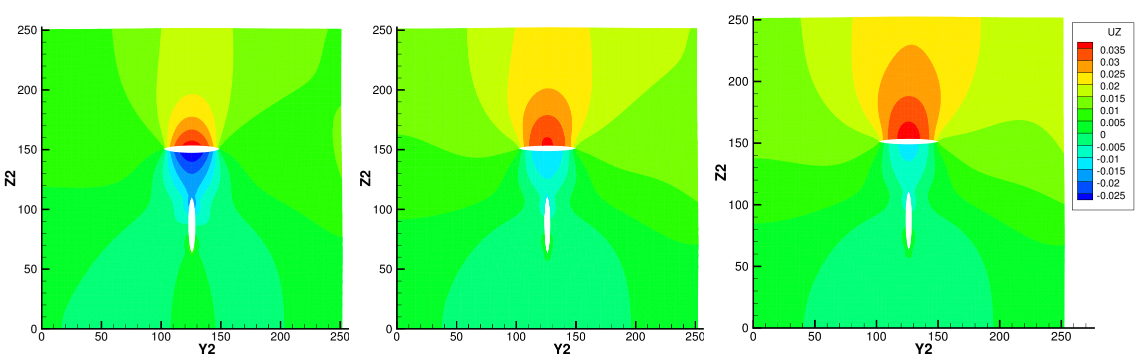

Figure 6.7: Z-direction displacement profiles at T = 0.0, 0.05 and 0.1 day.

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