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Sriram Nagaraj, Socratis Petrides, and Leszek F. Demkowicz

The Institute for Computational Engineering and Sciences
The University of Texas at Austin
Austin, Texas 78712

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Institute for Computational Engineering and Sciences
The University of Texas at Austin, Austin, TX 78712, USA

Abstract

The use of "ideal" optimal test functions in a Petrov-Galerkin scheme guarantees the discrete stability of the variational problem. However, in practice, the computation of the ideal optimal test functions is computationally intractable. In this paper, we study the effect of using approximate, "practical" test functions on the stability of the DPG (discontinuous Petrov-Galerkin) method and the change in stability between the "ideal" and "practical" cases is analyzed by constructing a Fortin operator.

We highlight the construction of an "optimal" DPG Fortin operator for H^1 and $H(\mathrm{div})$ spaces; the continuity constant of the Fortin operator is a measure of the loss of stability between the ideal and practical DPG methods. We take a two-pronged approach: first, we develop a numerical procedure to *estimate* an upper bound on the continuity constant of the Fortin operator in terms of the inf-sup constant γ_h of an auxiliary problem. Second, we construct a sequence of *approximate* Fortin operators and exactly compute the continuity constants of the approximate operators.

Our results shed light not only on the change in stability by using practical test functions, but also indicate how stability varies with the approximation order p and the enrichment order Δp. The latter has important ramifications when one wishes to pursue local hp-adaptivity.

1 Introduction

In the theory of Galerkin methods, one usually considers the variational formulation of boundary value problems. The variational formulation of second order equations is usually reduced to the following form [8]. Given a continuous bilinear form $b(\cdot,\cdot)$ defined on the product $U \times V$ of reflexive Banach spaces U, V (the "trial" and "test" spaces respectively), and a continuous linear form (functional) $l(\cdot)$ defined on V, we seek a solution $u \in U$ of the problem:

$$b(u, v) = l(v), \forall v \in V. \tag{1.1}$$

Here, in order to ensure well-posedness of the problem, we assume that the continuous inf-sup condition for the bilinear form $b(\cdot,\cdot)$ holds:

$$\gamma = \inf_{u \in U} \sup_{v \in V} \frac{|b(u, v)|}{\|u\|\|v\|} > 0, \tag{1.2}$$

where the constant γ is the inf-sup constant. The standard Petrov-Galerkin methodology is to introduce finite dimensional trial and test subspaces U_h, V_h of U, V respectively (with dim $U_h =$ dim V_h) and find $u_h \in U_h$ that solves the discrete problem:

$$b(u_h, v_h) = l(v_h), \forall v_h \in V_h. \tag{1.3}$$

By discretizing the continuous problem, we need to consider the effect of discretization on well-posedness (stability) as well as approximability. With regard to well-posedness of the discrete problem, Babuška's theorem [7] guarantees that if the *discrete* inf-sup condition holds, i.e., if

$$\gamma_h = \inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|u_h\|\|v_h\|} > 0, \tag{1.4}$$

then the discrete problem is well-posed. In addition, we also have the following error estimate that determines how much the discrete solution $u_h \in U_h$, differs from the continuous solution $u \in U$:

$$\|u - u_h\|_U \leq \frac{\|b\|}{\gamma_h} \inf_{w_h \in U_h} \|u - w_h\|. \tag{1.5}$$

Unfortunately, the inf-sup condition at the continuous level does not imply the inf-sup condition at the discrete level. Clearly, the continuous inf-sup condition is equivalent to:

$$\sup_{v \in V} \frac{|b(u_h, v)|}{\|v\|_V} > \gamma \|u_h\|_U, \tag{1.6}$$

and by taking the supremum over the smaller (finite dimensional) subspace $V_h \subset V$, we do NOT ensure the discrete inf-sup condition:

$$\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} > \gamma_h \|u_h\|_U, \tag{1.7}$$

so continuous inf-sup $\not\Rightarrow$ discrete inf-sup.

1.1 Operator Viewpoint

We note that the bilinear form $b(\cdot,\cdot)$ generates two operators $B : U \to V'$ and $B' : V \to U'$ as

$$< Bu, v >_{V' \times V} = b(u, v),\ v \in V, \tag{1.8}$$

$$< B'v, u >_{U' \times U} = b(u, v),\ u \in U, \tag{1.9}$$

and the variational formulation (1.1) implies the following operator equation $Bu = l$ must be solved for the solution $u \in U$. We note that the operator B maps the trial space U into the *dual* of the test space $V^{'}$, and the (continuous) inf-sup condition can be re-interpreted as asking for the operator B to be bounded below. Indeed, if we have the continuous inf-sup condition, then,

$$\|Bu_h\|_{V'} = \sup_{v \in V} \frac{|b(u_h, v)|}{\|v\|_V} > \gamma \|u_h\|_U, \tag{1.10}$$

and the boundedness below constant of B is the inf-sup constant of the bilinear form $b(\cdot,\cdot)$.

1.2 Optimal Test Functions

As we have seen, an arbitrary choice of the discrete test space may lead to lack of discrete stability. However, can one *choose* the test space in such a way to ensure the stability? In other words, given U_h, can we *construct* the space V_h in a way that we achieve the inf-sup condition? This question leads us to consider the so-called optimal test functions [3][4][1][5]. Henceforth, we restrict ourselves to the Hilbert space setting where U, V are Hilbert spaces. We begin with the observation that Bw is a functional on V and hence, by the Riesz representation theorem, there is a unique $\hat{w} \in V$ such that

$$< Bw, v >_{V' \times V} = (\hat{w}, v)_V. \tag{1.11}$$

We can thus define an operator $T : U \to V$ that assigns to each $w \in U$ the unique image under the Riesz map R_V of Bw, i.e.,

$$T = R_V^{-1} B, \tag{1.12}$$

where the Riesz map $R_V : V \to V'$ is an isometric isomorphism between V and V'. For obvious reasons, we refer to T as the "trial-to-test" operator. Clearly, we have $(Tu, v)_V = b(u, v)$. Returning to the discrete situation, given the trial space U_h, if we define the *optimal test space* $V_h^{\text{opt}} := T(U_h)$, and we consider the Petrov-Galerkin problem with the optimal test space, one can see that we have the discrete inf-sup condition *by construction*. Moreover, the corresponding discrete inf-sup constant γ_h obtained from the use of the optimal test space satisfies $\gamma_h \geq \gamma$ where γ is the continuous inf-sup constant. Moreover, it can be shown that, in the limit, $\gamma_h \to \gamma$ from above.

1.3 Ideal vs. Practial

The notion of optimal test functions we have considered can be referred to as the "ideal" optimal test functions, since they guarantee discrete stability for any choice of trial space. While desirable, the computation of ideal optimal test functions involves an infinite dimensional optimization problem which is computationally prohibitive. In order to obtain a more realistic set of optimal test functions, one is naturally led to approximate the trial-to-test operator T by another operator $T_r : U \to V_r$, where V_r is a large but finite dimensional subspace of V. T_r satisfies

$$(T_r u, r)_V = b(u, v_r),\ v_r \in V_r, \tag{1.13}$$

in other words, we are restricting the inversion of the Riesz operator to V_r to obtain $T_r = R_{V_r}^{-1} B$ [6][5]. The question of computing of T_r is now considered only over the finite dimensional subspace V_r and hence is a tractable quest. A natural question to ask at the moment is: how does the move from ideal to practical optimal test functions affect stability, and can this be quantified in some way? Answering these questions is the main aim of this paper.

1.4 Other Formulations

Before proceeding, it is important to note that our discussion of optimal test functions has two other equivalent points of view. The first is a minimum residual formulation and the second is a mixed formulation.

1.4.1 Minimum Residual Formulation

From this point of view, we consider the problem of finding $u_h \in U_h$ as that of minimizing a "residual" error [5]. Indeed, solving $Bu = l$ in U_h means that we make an error of $Bu_h - l$ (the "residual"). Thus, our discrete solution u_h is the one that minimizes the residual, i.e.,

$$u_h = \arg \min_{w_h \in U_h} \|Bw_h - l\|_{V'}. \tag{1.14}$$

1.4.2 Mixed Formulation

Another way of viewing our discussion is a mixed formulation [9]. Here, we solve for both $u_h \in U_h$ and $\psi \in V$ that satisfy:

$$\begin{cases} u_h, \psi & u_h \in U_h,\ \psi \in V \\ (\psi, v)_V - b(u_h, v) = -l(v), & v \in V \\ b(w_h, \psi) = 0. & w_h \in U_h \end{cases} \tag{1.15}$$

Clearly, the mixed formulation is equivalent to the minimum residual formulation, with ψ being the Riesz inverse of the residual, i.e., $\psi = R_V^{-1}(Bu_h - l)$. Again, in the "practical" version, the approximate residual ψ_r is computed by the approximate inversion of the Riesz map in V_r, i.e., $\psi_r = R_{V_r}^{-1}(Bu_h - l)$.

1.5 Fortin Operator

We now address the question of measuring the change in stability while moving from the ideal to practical notions of optimal test spaces. In order to do so, we introduce the idea of a Fortin operator [6]. A Fortin operator $\Pi : V \to V_r$ is a linear map that satisfies the conditions:

$$\begin{cases} \|\Pi v\|_V \leq C(r)\|v\|_V & 0 < C(r) < \infty,\, v \in V \\ b(w_h, \Pi v - v) = 0. & w_h \in U_h,\, v \in V \end{cases} \tag{1.16}$$

where $C(r)$, which we shall call the Fortin constant, is the operator norm of Π and depends, in particular, on the dimension of V_r. The second condition can be viewed as a b-orthogonality requirement on $\Pi v - v$. We then have the following relation ([6]) that shows how the stability is altered due to using the practical optimal test functions.

$$\|u - u_h\|_U \leq \frac{\|b\|}{\gamma_h} C(r) \inf_{w_h \in U_h} \|u - w_h\|. \tag{1.17}$$

It is therefore clear that we would like to have the constant $C(r)$ as close to unity as possible to ensure the least loss of stability. We also have the following illustrative diagram that shows the various spaces involved:

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[dr, "T_r"'] & V \arrow[d, "\Pi"] \\ & V_r \end{tikzcd}$$

However, the diagram in general is NOT commutative. At this juncture, we note that our approach of first constructing a Fortin operator Π, and then using the Fortin operator to study the change in stability between the ideal and practical cases is motivated by the fact that the Fortin operator is constructed elementwise (locally). However, it is conceivable that one may desire the above diagram to commute, i.e., $\Pi \circ T = T_r$, but in this case, the construction may no longer be local.

1.6 Aims and Scope

The main aim of this paper is to study the variation of the stability of the discontinuous Petrov-Galerkin method while changing from ideal to practical test functions using a suitable Fortin operator as a means to do so. As shown in [6, 2], the existence of a continuous Fortin operator ensures (in fact, is equivalent to) the discrete stability of the variational problem. We shall restrict ourselves to the two dimensional case, and we shall provide the construction for the H^1 and $H(\text{div})$ spaces using the Helmholtz and acoustic equations as motivation. Using discontinuous test functions and scaling arguments, we reduce our construction to be on a master triangular element and derive sufficient conditions to solve for the Fortin operator. We take a two-prong approach to the analysis of the Fortin operator we construct. First, we derive an *upper bound* on the Fortin operator's continuity constant $C(r)$ (which we shall call the *Fortin constant*) using the inf-sup constant γ_h associated with an *auxiliary* bilinear form. We are able to construct only an upper bound for the Fortin constant since a direct computation of the Fortin constant is not possible: evaluating the norm of the Fortin operator involves an infinite dimensional optimization problem. Based on this upper bound, we consider a numerical procedure that estimates γ_h and we thereby obtain an order of magnitude estimate on the Fortin constant $C(r)$. As a second line of analysis, we construct a sequence of *approximate* Fortin operators, each member of which is defined on an increasingly larger, yet finite dimensional subspace of the trial space. We then *exactly* compute the continuity constants of the *approximate* Fortin operators, which presumably converge to the *exact* Fortin constant of the *true* Fortin operator. This approach gives us a computable lower bound of the Fortin constant. In summary, our aim is to approximate the Fortin constant from above and below, thereby yielding a numerical range of how the overall stability of the DPG method is affected by using practical test functions.

In [6], the authors provide the construction of a Fortin operator arising from the Poisson problem, and show the corresponding stability of the discrete method, although an explicit value of the Fortin constant is not provided. As we have indicated, it is very desirable to have an order of magnitude estimate on the Fortin constant to conclude how well the optimal test functions are resolved. Also, the value of the Fortin constant, and especially its dependence on the order p, indicates the possibility of hp-adaptivity. In particular, we will be interested in the p-(in)dependence of the Fortin constant.

1.7 Organization of Paper

After this preliminary introduction, we detail our H^1 and $H(\mathrm{div})$ construction in the following two sections. In the fourth section, we outline a numerical procedure that is used to estimate an upper bound for $C(r)$ and we present our upper bound results. The next section is devoted to constructing and analyzing a sequence of approximate Fortin operators, and the means of exactly computing their continuity constants thereby obtaining a lower bound on the true Fortin constant. We conclude in the last section with brief remarks.

2 Construction of a DPG Fortin Operator for a General Second Order Problem

We shall construct two Fortin operators defined on broken H^1 and $H(\mathrm{div})$ spaces. To a large extent, the operators are problem independent, and are applicable to a general class of second order diffusion-convection-reaction problems and their primal and ultraweak variational formulations. To be more concrete, we shall use the Helmholtz equation and the corresponding first order system of linear acoustics as motivating examples for the two constructions respectively.

We shall now consider the construction of the DPG Fortin operator for the H^1 case.

2.1 $\mathbf{H^1}$ DPG Fortin Operator

Notation and Mesh Assumptions. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We shall consider the standard energy spaces:

$$H^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in (L^2(\omega))^2\} ,$$
$$H(\text{div}, \Omega) := \{u \in (L^2(\omega))^2 : \nabla \cdot u \in L^2(\omega)\} ,$$

where the operators ∇ and $\nabla\cdot$ are understood in the sense of distributions. Given a triangular mesh $\mathcal{T}_h$ covering Ω and an integer $r \geq 1$, we consider the standard Finite Element (FE) spaces:

$$W^p := \{u \in H^1(\Omega) : u|_K \in \mathcal{P}^p(K), K \in \mathcal{T}_h\} ,$$
$$RT^p := \{u \in H(\text{div}, \Omega) : u|_K \in (\mathcal{P}^{p-1}(K))^2 \bigoplus x\widetilde{\mathcal{P}}^{p-1}(K) \ , K \in \mathcal{T}_h\} ,$$

where $\mathcal{P}^p(K)$ denotes the polynomials of (total) order less than or equal to p, and $\widetilde{\mathcal{P}}^{p-1}(K)$ denotes the subspace of homogeneous polynomials of order p.

The H^1 and $H(\text{div})$ trace operators map W^p and RT^p onto the space of continuous polynomials $\mathcal{P}_c^p(\partial K)$ and the space of discontinuous polynomials $\mathcal{P}_d^{p-1}(\partial K)$,

$$\text{tr} : u \in W^p \to u|_{\partial K} \in \mathcal{P}_c^p(\partial K) ,$$
$$\text{tr} : u \in RT^p \to u|_{\partial K} \cdot n \in \mathcal{P}_d^{p-1}(\partial K).$$

Finally, by $H^1(\Omega_h)$ and $H(\text{div}, \Omega_h)$, we mean the broken counterparts of the $H^1(\Omega)$ and $H(\text{div}, \Omega)$ spaces that are discretized with the same elements of order $r > p$ but with no conformity requirements.

For our construction, we will assume the following assumption on the FE mesh: a structured 2D mesh consisting of identical triangles where each element K can be obtained by scaling the master triangle $\hat{K}$ with element size h.

Helmholtz Equation. To motivate our construction, we first consider the Helmholtz equation with Dirichlet and Neumann boundary conditions:

$$\begin{cases} -\Delta u - \omega^2 u = f & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_u, \\ \dfrac{\partial u}{\partial n} = t_0 & \text{on } \Gamma_t. \end{cases} \tag{2.18}$$

Here, Γ_u and Γ_t are two disjoint parts of the boundary, $\omega > 0$ denotes the (angular) frequency and f, u_0, t_0 are given data.

We proceed now with an elementary derivation of the primal DPG variational formulation. We multiply the Helmholtz equation with a test function v, integrate by parts over one element K, and sum over all elements to obtain:

$$\sum_K \{(\nabla u, \nabla v)_K - \omega^2(u,v) + < \frac{\partial u}{\partial n}, v >_{\partial K}\} = \sum_K (f,v)_K \,,$$

where $(\cdot,\cdot)_K$ and $<\cdot,\cdot>_{\partial K}$ denote the $L^2(K)$ product and $H^{-\frac{1}{2}}(\partial K) \times H^{\frac{1}{2}}(\partial K)$ duality pairing on ∂K respectively. In the standard case, we assume that the test function v is globally conforming, $v \in H^1(K)$, which reduces $\sum_K < \frac{\partial u}{\partial n}, v >_{\partial K}$ to $< \frac{\partial u}{\partial n}, v >_{\partial\Omega}$. We set $v = 0$ on Γ_u (i.e., do not test there) and replace $\frac{\partial u}{\partial n}$ with known boundary data t_0 on Γ_t, which is then moved to the right hand side.

In the DPG method, we test with discontinuous test functions $v \in H^1(\Omega_h)$, and make no additional assumptions about v on $\partial\Omega$. The normal derivative $\frac{\partial u}{\partial n}$ is identified as an extra unknown, the *flux* $\hat{t}$. More precisely, we take a field $t \in H(\text{div}, \Omega)$, restrict it to element K, so that $t|_K \in H(\text{div}, K)$, and consider its normal trace:

$$< \hat{t}, v >_{\Gamma_h} := \sum_K < t|_{\partial K} \cdot n_K, v >_{\partial K} \;.$$

The space of all such restrictions to the mesh skeleton $\Gamma_h = \bigcup_K \partial K$ is denoted by $H^{-\frac{1}{2}}(\Gamma_h)$ and is equipped with the quotient (minimum energy extension) norm:

$$\|\hat{t}\|_{H^{-\frac{1}{2}}(\Gamma_h)} := \inf_{t|_{\partial K} \cdot n_K = \hat{t}} \|t\|_{H(\text{div},\Omega)} \,,$$

where $t \in H(\text{div}, \Omega)$. Notice that, by construction, the flux is single valued. The final variational formulation is now obtained as:

$$\left\{ \begin{array}{c} u \in H^1(\Omega), \; u = u_0 \text{ on } \Gamma_u, \\ \hat{t} \in H^{-\frac{1}{2}}(\Gamma_h), \; \hat{t} = t_0 \text{ on } \Gamma_t, \\ (\nabla u, \nabla_h v) - \omega^2(u,v) + < \hat{t}, v >_{\Gamma_h} = (f,v) \text{ and } v \in H^1(\Omega_h). \end{array} \right. \tag{2.19}$$

The symbol h in $\nabla_h(\cdot)$ indicates that the gradient is computed elementwise.

Discretization. The two unknowns are discretized now with standard conforming elements:

$$u_h \in W^p \,, \hat{t}_h \in \text{tr}_{\Gamma_h} RT^p.$$

Note that the best approximation errors for the two unknowns are of the same order. If $b_K((u,\hat{t}),v)$ denotes the elementwise bilinear form corresponding to the variational formulation mentioned earlier, i.e.,

$$b_K((u,\hat{t}),v) = (\nabla u, \nabla_h v)_K - \omega^2(u,v)_K + < \hat{t}, v >_{\partial K} = (-\Delta u - \omega^2 u, v)_K + < \hat{t}, v >_{\partial K},$$

then the ideal DPG method minimizes the residual

$$(u_h, \hat{t}_h) = \arg\min(\sum_K (\sup_{v \in H^1(\Omega_h)} \frac{|b_K((u,\hat{t}),v)|}{\|v\|_{H^1(K}})^2)^{\frac{1}{2}},$$

with the $\arg\min$ taken over $u \in W^p$, $\hat{t} \in \mathrm{tr}_{\Gamma_h} RT^p$.

In the *practical* DPG method, the supremum of $v \in H^1(K)$ is replaced with a computable supremum of $v \in \mathcal{P}^r(K)$ with $r > p$ (refer sec 1.3).

Fortin Operator. Given the elementwise bilinear form $b_K(\cdot,\cdot)$ over all elements K, one can obtain the bilinear form $b(\cdot,\cdot)$ on Ω:

$$b((u,\hat{t}),v) = \sum_K b_K((u,\hat{t}),v).$$

In constructing a Fortin operator, we are looking for a linear operator

$$\Pi : H^1(\Omega_h) \to \mathcal{P}^{p+\Delta p}(\Omega_h)$$

that satisfies two conditions:

$$\begin{aligned} &\text{(i)} \quad b((u,\hat{t}), v - \Pi v) = 0 \quad \forall u \in W_p,\ \hat{t} \in \mathrm{Tr}_{\Gamma_h} RT^p \\ &\text{(ii)} \qquad\qquad \|\Pi v\|_{H^1(\Omega_h)} \le C \|v\|_{H^1(\Omega_h)} \end{aligned} \tag{2.20}$$

with a mesh independent constant C. Thus, in this case, $V_r = \mathcal{P}^{p+\Delta p}(\Omega_h) := \mathcal{P}^r(\Omega_h)$ and the index $r = p + \Delta p$.

Ideally, we would like the continuity constant C of Π to be independent of p. Now, while one may attempt to solve for Πv as stated, we will take a slightly different approach by augmenting $(2.20)_1$ by a set of stronger conditions:

$$\begin{aligned} &\int_K (v - \Pi v)\,\phi = 0 \quad \forall \phi \in \mathcal{P}^p(K)\,, \\ &\int_e (v - \Pi v)\,\phi = 0 \quad \forall \phi \in \mathcal{P}^{p-1}(e),\ e = 1,2,3\,. \end{aligned} \tag{2.21}$$

Clearly, the stronger conditions are sufficient to conclude the orthogonality which we sought originally, and while the stronger conditions may result in a more pessimistic estimate for C, they lead to a computationally tractable problem. Notice that conditions (2.21) are to a large extent *problem independent*. Indeed, the integration by parts argument that have led to them will work for a general class of second order operators as long as the material data are constant elementwise.

Ensuring Uniqueness. Although we have identified a possible candidate for Πv, due to the underdetermined nature of the constraints, we may have more than one element in $\mathcal{P}^r$ that satisfies our requirements.

In order to complete the definition of Π we request that in addition to the orthogonality conditions (2.21), the norm $\|\Pi v\|_{H^1(K)}$ to be minimal,

$$\|\Pi v\|_{H^1(K)} \to \min \ .$$

Thus, we can view our construction of Π as a constrained minimization problem:

$$\begin{cases} \Pi v = \text{argmin}\|v^*\|_{H^1(K)} & v^* \in \mathcal{P}^r,\, v \in V \\ (\phi, v^*)_{L^2(K)} = (\phi, v)_{L^2(K)}, & \phi \in \mathcal{P}^p(K) \\ < \phi_e, v^* >_{L^2(\partial K)} =< \phi_e, v >_{L^2(\partial K)} . & \phi_e \in \mathcal{P}_d^{p-1}(e),\, e = 1,2,3 \end{cases} \tag{2.22}$$

We can re-write this constrained minimization problem as a mixed (saddle-point) problem as follows:

$$\begin{cases} (v^*, \delta v^*)_{H^1(K)} + (\phi, \delta v^*)_{L^2(K)} + < \phi_e, \delta v^* >_{L^2(\partial K)} = 0 \quad \delta v^* \in \mathcal{P}^r, \\ (\delta\phi, v^*)_{L^2(K)} = (\delta\phi, v)_{L^2(K)},\ \delta\phi \in \mathcal{P}^p(K) \\ < \delta\phi_e, v^* >_{L^2(\partial K)} =< \delta\phi_e, v >_{L^2(\partial K)},\ \delta\phi_e \in \mathcal{P}^{p-1}(e),\, e = 1,2,3 \end{cases}$$

Thus, our construction of the Fortin operator is the same as solving the above mixed problem.

Use of Alternate $\mathbf{H^1}$ Norm. Given any $v \in H^1(\Omega_h)$, we can always split it into its average and the zero average parts,

$$v = \bar{v} + (v - \bar{v}), \quad \Pi v = \Pi\bar{v} + \Pi(v - \bar{v})\,. \tag{2.23}$$

We will denote by $H^1_{avg}(\Omega_h)$ the set of $H^1(\Omega_h)$ functions with zero average. In this context, we remark that the usual H^1 norm is equivalent to the following norm:

$$\|u\|^2_{H^1(\Omega_h)} = \sum_K \left(\|\nabla u\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(K)} \right) \tag{2.24}$$

where

$$\bar{u} := \frac{1}{|K|} \int_K u\, dK$$

is the average value of function u in element K. This fact is proved in the appendix. The use of this norm in place of the standard H^1 norm will be critical as we continue our analysis. Henceforth by H^1 norm we shall mean the above mentioned norm, as opposed to the usual H^1 norm.

Minimization of $\mathbf{H^1}$ Norm as Quotient Norm. To see our construction clearly, we define the *set*

$$\Pi_1 v = \{w \in \mathcal{P}^r \,|w \text{ satisfies the orthogonality conditions}\} \tag{2.25}$$

To see how minimizing the gradient of Πv is significant, we note that what we are defining Πv to be is basically $\arg\min_{w \in \Pi_1 v} \|w\|_{H^1}$. Indeed, if we define the kernel

$$W_0 = \{z \in \mathcal{P}^r | (z, \phi)_{L^2(K)} = 0 \text{ and } (z, \phi_e)_{L^2(\partial K)} = 0\, \forall \phi \in\ \mathcal{P}^p(K),\ \phi_e \in\ \mathcal{P}^{p-1}(e)\, e = 1,2,3\}, \tag{2.26}$$

then clearly, all the vectors $w \in \Pi_1 v$ differ only by elements in W_0, and therefore, picking the element of least H^1 norm in $\Pi_1 v$ amounts to choosing a single element in the quotient space $\mathcal{P}^r/W_0$ that satisfies the orthogonality conditions, so that Πv resides in $\mathcal{P}^r/W_0$. Since we are considering functions with zero average, it is clear that minimizing the gradient norm of $w \in \Pi_1 v$ is equivalent to choosing an equivalence class in the quotient space $\mathcal{P}^r/W_0$. This follows from the definition of the quotient norm. However, the quotient space $\mathcal{P}^r/W_0$ can be identified uniquely as the orthogonal complement $W_0^\perp$ of W_0 in $\mathcal{P}^r$, so that our Fortin operator Π can be viewed as a map into subspace $W_0^\perp$.

Local Construction. At this stage, we make a crucial remark. While dealing with a conforming mesh, we would need to deal with contributions from possibly the entire domain, whereas with broken test spaces, our computations are purely local. While the use of broken test spaces forces us to solve for an additional unknown (the term $\hat{t} = \frac{\partial u}{\partial n}$), the ability to do purely local computations allows us the great ability of parallelizing our computations. In particular, this means that the use of broken test spaces allows us, without loss of generality, to concentrate on the Fortin operator construction on a single element.

If we manage to construct an operator for a single element that satisfies elementwise version of conditions (2.21), with an element independent constant C, the global conditions will easily follow. We will thus focus on constructing Π on a single triangular element K, obtained by a simple scaling of the master triangle $\hat{K}$.

Scaling Arguments. Coming back to our construction, if we request operator Π to preserve constants, i.e. $\Pi\bar{v} = \bar{v}$, then we immediately see that conditions (2.21) are trivially satisfied. We can thus restrict our construction to functions with zero average, $v \in H^1_{avg}(K)$. Condition $(2.21)_1$ implies that Πv has a zero average as well, i.e. Π respects decomposition (2.23). For functions with zero average, our norm includes only the gradient part which remains invariant under the pullback. Consequently, we need to define only operator $\hat{\Pi}$ for the master element and use the pullback map to extend it to the physical element,

$$\widehat{\Pi u} := \hat{\Pi}\hat{u}$$

As the pullback map respects decomposition (2.23) as well, we obtain:

$$\begin{aligned}
\|\Pi v\|_{H^1(K)} &= \|\nabla \Pi v\|_{L^2(K)} = \|\hat{\nabla}\hat{\Pi}\hat{v}\|_{L^2(\hat{K})} \\
&\le \hat{C}\|\hat{\nabla}\hat{v}\|_{L^2(\hat{K})} = \hat{C}\|\nabla v\|_{L^2(K)} \\
&= \hat{C}\|v\|_{H^1(K)} .
\end{aligned}$$

This implies that it is sufficient to define and investigate the operator only for the master element.

Comparing Dimensions. We have to make sure that the space of polynomials from $\mathcal{P}^r(K)$ satisfying the constraints is non-empty. As the constraints are linearly independent, it is sufficient to compare the number of constraints with the dimension of involved spaces. Condition $(2.21)_2$ alone implies:

$$3(p+\Delta p) \geq 3p$$

and it is satisfied for any $\Delta p \geq 0$. The combined conditions imply that we must satisfy:

$$\dim \mathcal{P}^{p+\Delta p}(K) = \frac{(p+\Delta p+1)(p+\Delta p+2)}{2} \geq \dim \mathcal{P}^{p}(K) + 3\dim \mathcal{P}^{p-1}(e) = \frac{(p+1)(p+2)}{2} + 3p$$

which reduces to:

$$\Delta p(2p+3+\Delta p) \geq 6p\,.$$

In order to avoid an overdetermined system, we must therefore satisfy the following minimum conditions on Δp,

$$\begin{aligned} \Delta p &\geq 1 \quad \text{for } p = 1 \\ \Delta p &\geq 2 \quad \text{for } 2 \leq p \leq 5 \\ \Delta p &\geq 3 \quad \text{for } p \geq 6\,. \end{aligned} \tag{2.27}$$

This concludes the H^1 analysis for the h-version of the DPG Fortin operator construction. Of course, we do not know at this point how large the continuity constant is and how it depends upon $r = p + \Delta p$. This is the objective of the numerical experiments, which are detailed after the construction of the Fortin operator for the $H(\text{div})$ case.

2.2 $\mathbf{H}(\mathbf{div}, \boldsymbol{\Omega})$ DPG Fortin Operator

We now consider the construction of a DPG Fortin operator for the $H(\text{div}, \Omega)$ case. Our analysis will mirror that of the H^1 case, as will be seen shortly.

Recall that the Helmholtz problem is obtained by eliminating velocity u from the linear acoustics equations:

$$\begin{cases} i\omega p + \text{div}\, u = 0, & \text{in } \Omega \\ i\omega u + \nabla p = 0, & \text{in } \Omega \\ p = p_0, & \text{on } \Gamma_p \\ u \cdot n = u_0, & \text{on } \Gamma_u \end{cases} \tag{2.28}$$

The equations are obtained by linearizing the isentropic Euler equations around a hydrostatic solution $u = 0$, $p = p_0 =$ constant. The first equation represents conservation of mass, and the second one conservation of linear momentum. The equations have been non-dimensionalized to obtain a unit sound speed.

The so-called *ultraweak variational formulation* for the system is obtained by multiplying the first equation with a test function q, the second equation with a test function v, integrating by parts over an element K, and then summing up the element contributions. Similar to the primal method, the boundary terms of u and p are identified as new unknowns. The final formulation is:

$$
\begin{cases}
p \in L^2(\Omega), u \in (L^2(\Omega))^2 \\
\hat{p} \in H^{\frac{1}{2}}(\Gamma_h),\ \hat{p} = p_0 \quad \text{on } \Gamma_p \\
\hat{t} \in H^{-\frac{1}{2}}(\Gamma_h),\ \hat{t} = u_0 \quad \text{on } \Gamma_u \\
i\omega(p,q) - (u, \nabla_h q) + < \hat{t}, q >_{\Gamma_h} = 0, \quad q \in H^1(\Omega_h) \\
i\omega(u,v) - (p, \mathrm{div}_h\, v) + < \hat{p}, v \cdot n >_{\Gamma_h} = 0, \quad v \in H(\mathrm{div}, \Omega_h),
\end{cases}
\tag{2.29}
$$

where $\hat{p}$ is a trace of a global $p \in H^1(\Omega)$ to the mesh skeleton Γ_h, and

$$
< \hat{p}, q >_{\Gamma_h} = \sum_K < p, q >_{\partial K} .
$$

As in the case of $H^{-\frac{1}{2}}(\Gamma_h)$, $\hat{p}$ is measured in the quotient (minimum energy extention) norm:

$$
\|\hat{p}\|_{H^{\frac{1}{2}}(\Gamma_h)} := \inf_{p|_{\partial K} = \hat{p}} \|p\|_{H^1(\Omega)},
$$

where $p \in H^1(\Omega)$.

Discretization. Consistent with the exact sequence structure, the L^2 unknowns p, u are discretized with discontinuous polynomials of order p, traces $\hat{p}$ are discretized with the traces of W^p functions, i.e., continuous polynomials of order p on the mesh skeleton Γ_h, and traces $\hat{t}$ are approximated with traces of RT^p on Γ_h, i.e. discontinuous polynomials of order $p-1$.

The practical DPG method is based on minimizing the residuals in the norms dual to the $H^1(\Omega_h)$ and $H(\mathrm{div}, \Omega_h)$ approximated by taking the supremum with respect to discontinuous test spaces, i.e., $W^r(\Omega_h), RT^r(\Omega_h)$ with $r > p$.

As we did in the H^1 case, consider the element bilinear form $b_K(\cdot,\cdot)$:

$$
b_K((p,u,\hat{p},\hat{t}),(q,v)) := (i\omega p + \mathrm{div}\, u, q)_K + < \hat{t} - u \cdot n, q >_{\partial K} + (i\omega u + \nabla p, v)_K + < \hat{p} - p, v \cdot n >_{\partial K},
$$

and the practical DPG method is:

$$
(p_h, u_h, \hat{p}_h, \hat{t}_h) = \arg\min (\sum_K (\sup_{q \in \mathcal{P}^r, v \in RT^r} \frac{|b_K((p,u,\hat{p},\hat{t}),(q,v))|}{(\|q\|^2_{H^1(K)} + \|v\|^2_{H(\mathrm{div},K)})^{\frac{1}{2}}})^2)^{\frac{1}{2}},
$$

It is well known that in order to obtain correct scaling arguments for the Fortin operator Π, the operator must satisfy the commuting exact sequence property:

$$
\begin{tikzcd}
H(\mathrm{div}, K) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi"] & L^2(K) \arrow[d, "P"] \\
RT^r(K) \arrow[r, "\mathrm{div}"] & \mathcal{P}^{r-1}
\end{tikzcd}
$$

i.e.,

$$\text{div }(\Pi v) = P(\text{div } v)$$

Here, in principle, $P(\cdot)$ is any well-defined continuous operator but, in order to minimize the continuity constant for $P(\cdot)$, it is natural to assume that $P(\cdot)$ is the L^2 projection operator.

H(div) Construction. We now construct the operator on the master element $\hat{K}$ and subsequently develop scaling arguments for the generic element K. At the outset, it is clear we will enforce the orthogonality constraints on the element interior and boundary separately. As was the case with the H^1 construction, we will enforce stronger conditions which are sufficient for the orthogonality we seek. We see that in order for the orthogonality to hold,

$$\begin{aligned} \int_K \Psi \cdot (v - \Pi v) \; &= 0 \qquad \forall \Psi \in RT^p(K)\,, \\ \int_{\partial K} \phi\, [(v - \Pi v) \cdot n] \; &= 0 \quad \forall \phi \in \mathcal{P}_c^p(\partial K). \end{aligned} \tag{2.30}$$

are sufficient. However, for ease of analysis, we enforce the stronger conditions in (2.30). Moreover, choosing $\Psi = \nabla \eta$ in the first condition yields

$$\begin{aligned} 0 &= \int_K \Psi \cdot (v - \Pi v) \\ &= \int_K \nabla \eta \cdot (v - \Pi v) \\ &= -\int_K \eta \,\text{div}(v - \Pi v) + \int_{\partial K} \eta[(v - \Pi v) \cdot n] \end{aligned} \tag{2.31}$$

which, using the fact that

$$\int_{\partial K} \phi\, [(v - \Pi v) \cdot n] \; = 0 \,\forall\, \phi \in \mathcal{P}_c^p(\partial K),$$

implies the final condition

$$\int_K \eta \,\text{div }(v - \Pi v) = 0 \,\forall\, \eta \in \mathcal{P}^p(K).$$

However, we would like to ensure that the divergence of Πv yields the L^2 projection, i.e., we would like to have div $\Pi v = P(\text{div } v)$ where $P(\cdot)$ is the L^2 projection. Having this condition would greatly simplify our estimates on the total $H(\text{div})$ norm of Πv. Towards this end, we note that we are essentially asking for orthogonality upto $\mathcal{P}^{r-1}(K)$, which, combined with the fact that we already have orthogonality upto $\mathcal{P}^p(K)$, means we need to enforce the last condition for η coming from the space $\mathcal{P}^{r-1}(K)/\mathcal{P}^p(K)$. We thus see that our orthogonality requirement has given us three constraints:

$$\begin{aligned} \int_K \Psi \cdot (v - \Pi v) = 0 \qquad & \forall \Psi \in RT^p(K) \\ \int_K \eta \text{div}(v - \Pi v) = 0 \qquad & \forall \eta \in \mathcal{P}^{r-1}(K)/\mathcal{P}^p(K) \\ \int_{\partial K} \phi[(v - \Pi v) \cdot n] = 0 \quad & \forall \phi \in \mathcal{P}_c^p(\partial K). \end{aligned} \tag{2.32}$$

Norm Minimization and Mixed Formulation. As in the H^1 case, we have identified a possible candidate for Πv, but due to the underdetermined nature of the constraints, we need to identify a unique $\Pi v \in RT^r(K)$ that satisfies our requirements. We therefore request that in addition to the orthogonality conditions, the norm $\|\Pi v\|_{H(\text{div})}$ be minimal:

$$\begin{cases} \Pi v = \text{argmin}\|v^*\|_{H(\text{div})} & v^* \in RT^r,\, v \in V \\ (\Psi, v^*)_{L^2(K)} = (\Psi, v)_{L^2(K)}, & \Psi \in RT^p(K) \\ (\eta, \text{div}\, v^*)_{L^2(K)} = (\eta, \text{div}\, v)_{L^2(K)}, & \eta \in \mathcal{P}^{r-1}(K)/\mathcal{P}^p(K) \\ < \phi, v^* >_{L^2(\partial K)} =< \phi, v >_{L^2(\partial K)} . & \phi \in \mathcal{P}_c^p(\partial K). \end{cases} \tag{2.33}$$

As in the H^1 case, we can re-write this constrained minimization problem as a mixed (saddle-point) problem as follows:

$$\begin{cases} (v^*, \delta v^*)_{H(\text{div})} + (\Psi, \delta v^*)_{L^2(K)} + \\ \qquad (\eta, \text{div}\, \delta v^*)_{L^2(K)} + < \phi, \delta v^* >_{L^2(\partial K)} = 0,\, \delta v^* \in \mathcal{P}^r \\ (\delta\Psi, v^*)_{L^2(K)} = (\delta\Psi, v)_{L^2(K)},\, \delta\Psi \in RT^p(K) \\ (\delta\eta, \text{div}\, v^*)_{L^2(K)} = (\delta\eta, \text{div}\, v)_{L^2(K)},\, \delta\eta \in \mathcal{P}^{r-1}(K)/\mathcal{P}^p(K) \\ < \delta\phi, v^* >_{L^2(\partial K)} = (\delta\phi, v)_{L^2(\partial K)},\, \delta\phi \in \mathcal{P}_c^p(\partial K). \end{cases} \tag{2.34}$$

As in the H^1 case, the solution to the Fortin conditions coincides with the solution of the above mixed problem.

Scaling Arguments. We now extend the above construction of the Fortin operator as defined on the master element $\hat{K}$ to an arbitrary triangle element K obtained from $\hat{K}$ by scaling the master element with a parameter h. Recall the Piola map for the $H(\text{div})$ case:

$$\begin{aligned} & x = h\xi\,, \\ & u(x) = \hat{u}(\hat{\xi})\,, \\ & v_i = \frac{\partial x_i}{\partial \xi_j}\frac{\hat{v}_j}{J}\,, \end{aligned} \tag{2.35}$$

where the Jacobian $J = h^2$ when the dimension $d = 2$. Applying the Piola map, we obtain:

$$\begin{aligned} & v_i = h^{1-d}\hat{v}_i\,, \\ & \text{div}\, v_i = h^{-d}\hat{\text{div}}\, \hat{v}_i\,, \\ & \|f\|^2_{L^2(K)} = h^{-d}\|\hat{f}\|^2_{L^2(\hat{K})}\,. \end{aligned} \tag{2.36}$$

We now estimate the $H(\text{div})$ norm of Πv. First, consider the L^2 norm of Πv. We have:

$$
\begin{aligned}
\|\Pi v\|^2_{L^2(K)} &= h^{2-d}\|\hat{\Pi}\,\hat{v}\|^2_{L^2(\hat{K})} \\
&\leq h^{2-d}\|\hat{\Pi}\,\hat{v}\|^2_{H(\hat{\text{div}})} \\
&\leq h^{2-d}\,\hat{C}^2\,(\|\hat{v}\|^2_{L^2(\hat{K})} + \|\hat{\text{div}}\,\hat{v}\|^2_{L^2(\hat{K})}\,), \\
&= h^{2-d}\,\hat{C}^2\,(h^{d-2}\,\|v\|^2_{L^2(K)} + h^d\|\text{div}\,v\|^2_{L^2(K)})\,.
\end{aligned}
\tag{2.37}
$$

So $\|\Pi v\|^2_{L^2(K)} \leq \hat{C}^2\,(\|v\|^2_{L^2(K)} + h^2\|\text{div}v\|^2_{L^2(K)})$ and thus, with $h \leq 1$, we arrive at:

$$
\|\Pi v\|^2_{L^2(K)} \leq \hat{C}^2\|v\|^2_{H(\text{div})} \tag{2.38}
$$

Second, we now consider the L^2 norm of $\text{div}\Pi v$:

$$
\begin{aligned}
\|\text{div}\,\Pi v\|^2_{L^2(K)} &= h^{-d}\|\hat{\text{div}}\,\hat{\Pi}\hat{v}\|^2_{L^2(\hat{K})} \\
&= h^{-d}\|\hat{P}\hat{\text{div}}\,\hat{v}\|^2_{L^2(\hat{K})} \\
&\leq \|\hat{P}\|h^{-d}\|\hat{\text{div}}\,\hat{v}\|^2_{L^2(\hat{K})}\,, \\
&= h^{-d}h^d\|\text{div}\,v\|^2_{L^2(K)}.
\end{aligned}
\tag{2.39}
$$

So $\|\text{div}\,\Pi v\|^2_{L^2(K)} \leq \|\text{div}\,v\|^2_{L^2(K)}$. Finally, we can estimate the full $H(\text{div})$ norm of Πv as

$$
\begin{aligned}
\|\Pi v\|^2_{H(\text{div,K})} &\leq \hat{C}^2\|v\|^2_{L^2(K)} + (1 + (\hat{C}h)^2)\|\text{div}\,v\|^2_{L^2(K)} \\
&\leq \hat{C}^2(\|v\|^2_{L^2(K)} + (h^2 + \frac{1}{\hat{C}^2})\|\text{div}\,v\|^2_{L^2(K)}) \\
&\leq \hat{C}^2 M\,(\|v\|^2_{L^2(K)} + \|\text{div}\,v\|^2_{L^2(K)}) \\
&= C\|v\|^2_{H(\text{div,K})},
\end{aligned}
\tag{2.40}
$$

where $M = \max(1, h^2 + \frac{1}{\hat{C}^2})$.

This concludes our construction of the $H(\text{div}, \Omega)$ DPG Fortin Operator. We now turn to the task of estimating the continuity constant of Π.

3 Numerical Experiments to Estimate the Continuity Constants

Recall that $\Pi : V \to V_r$ is a linear, continuous map from the infinite dimensional space V to the finite dimensional subspace V_r. Thus, exactly computing the norm of Π is an infinite dimensional maximization problem, and is computationally intractable. In this section, we confine our attention to *estimating* an upper bound of $\|\Pi\|$ in both the H^1 and $H(\mathrm{div})$ case.

3.1 $\mathbf{H^1}$ case

The continuity constant,

$$C = \sup_{\|v\|_{H^1(K)} \leq 1} \|\Pi v\|_{H^1(K)} .$$

In order to obtain an upper bound on the constant, we consider a variational problem,

$$\left\{ \begin{array}{l} w \in \mathcal{P}^{p+\Delta p}(K) \\ \displaystyle\int_K w\phi = \int_K v\phi \quad \forall \phi \in \mathcal{P}^p(K) \\ \displaystyle\int_e w\phi_e = \int_e v\phi \quad \forall \phi_e \in \mathcal{P}^p(e),\ e = 1,2,3 \end{array} \right. \tag{3.41}$$

or, in a more concise form,

$$\left\{ \begin{array}{l} w \in \mathcal{P}^r(K) \\ \widetilde{b}(w, (\phi, \phi_e)) = \widetilde{b}(v, (\phi, \phi_e)) \quad \phi \in \mathcal{P}^p(K),\ \phi_e \in \mathcal{P}^p(e) \end{array} \right.$$

where

$$\widetilde{b}(w, (\phi, \phi_e)) = \int_K w\phi + \sum_{e=1}^{3} \int_e w\phi_e .$$

This new bilinear form $\widetilde{b}(\cdot,\cdot)$ is an auxiliary bilinear form obtained from the orthogonality conditions imposed by the definition of Π. Recall now the fundamental property of inf-sup condition, now applied to the auxiliary form $\widetilde{b}(\cdot,\cdot)$:

$$\inf_{(\phi,\phi_e)} \sup_{w} \frac{|\widetilde{b}(w, (\phi, \phi_e))|}{\|(\phi, \phi_e)\| \, \|w\|} = \inf_{[w]} \sup_{(\phi,\phi_e)} \frac{|\widetilde{b}(w, (\phi, \phi_e))|}{\|(\phi, \phi_e)\| \, \|[w]\|} =: \gamma = \gamma(p, \Delta p) \tag{3.42}$$

where $[w]$ denotes the equivalence class,

$$[w] = w + W_0, \quad W_0 := \{w : b(w, (\phi, \phi_e)) = 0 \quad \forall \phi, \phi_e\}$$

and $\|[w]\|$ the corresponding quotient norm. Notice that with w measured in the H^1 norm, $\|\Pi v\|$ coincides precisely with the quotient norm. We have thus:

$$\begin{aligned} \|\Pi v\| &\leq \frac{1}{\gamma} \sup_{(\phi,\phi_e)} \frac{|\widetilde{b}(\Pi v, (\phi, \phi_e))|}{\|(\phi, \phi_e)\|} = \frac{1}{\gamma} \sup_{(\phi,\phi_e)} \frac{|\widetilde{b}(v, (\phi, \phi_e))|}{\|(\phi, \phi_e)\|} \\ &\leq \frac{M}{\gamma} \|v\|_{H^1(K)} \end{aligned}$$

where M is the continuity constant for the bilinear form,

$$|\widetilde{b}(v,(\phi,\phi_e))| \leq M\|v\|_{H^1(K)}\,\|(\phi,\phi_e)\|\,.$$

If the choice of norm for w is clear, the choice of norm for (ϕ,ϕ_e) is not unique and, obviously, the inf-sup constant γ will depend upon this choice. We may start with simple L^2 norms,

$$\|(\phi,\phi_e)\|^2 := \int_K |\phi|^2 + \sum_{e=1}^{3}\int_e |\phi_e|^2\,, \tag{3.43}$$

but we may think of more sophisticated choices like,

$$\|(\phi,\phi_e)\|^2 := \|\phi\|^2_{(H^1(K))'} + \|\phi_e\|^2_{H^{-\frac{1}{2}}(\partial K)}\,. \tag{3.44}$$

3.2 H(div) case

We now consider the numerical experiment to estimate the continuity constant of the Fortin operator in the $H(\text{div})$ case. Towards this end, consider the problem of estimating

$$C = \sup_{\|v\|_{H(\text{div,K})}\leq 1} \|\Pi v\|_{H(\text{div,K})}. \tag{3.45}$$

As was the case with H^1, we shall design an experiment to estimate the value of C rather than directly computing it. We have the conditions defining operator Π in (2.32), which we can translate to a variational formulation:

$$\begin{aligned} \textstyle\int_K \Psi\cdot v &= \textstyle\int_K \Psi\cdot w && \forall\Psi\in RT^p(K)\,,\\ \textstyle\int_K \eta\,\text{div}\,v &= \textstyle\int_K \eta\,\text{div}\,w && \forall\eta\in\mathcal{P}^{r-1}(K)/\mathcal{P}^p(K)\,,\\ \textstyle\int_{\partial K}\phi\,(v\cdot n) &= \textstyle\int_{\partial K}\phi\,(w\cdot n) && \forall\phi\in\mathcal{P}^p_c(\partial K). \end{aligned} \tag{3.46}$$

In other words, we find, given $v\in H(\text{div},K)$ a vector $w\in RT^r(K)$ that satisfies the above conditions. For ease of notation, we can define again an auxiliary bilinear form $\widetilde{b}(\cdot,\cdot)$ as:

$$\widetilde{b}(w,(\Psi,\eta,\phi)) = \int_K \Psi\cdot w + \int_K \eta\,\text{div}\,w + \int_{\partial K}\phi\,(w\cdot n) \tag{3.47}$$

with $\Psi\in RT^p(K), \eta\in\mathcal{P}^{r-1}(K)/\mathcal{P}^p(K), \phi\in\mathcal{P}^p_c(\partial K)$ and simplify our formulation as that of finding $w\in RT^r(K)$ that satisfies

$$\widetilde{b}(v,(\Psi,\eta,\phi)) = \widetilde{b}(w,(\Psi,\eta,\phi)). \tag{3.48}$$

Inf-sup Argument. We now show how the inf-sup constant of the above bilinear form can be used to estimate the continuity constant C. We first define the subspace

$$W_0 = \{w\in RT^r(K)\,:\widetilde{b}(w,(\Psi,\eta,\phi)) = 0\,\forall(\Psi,\eta,\phi)\}, \tag{3.49}$$

and denote the equivalence class of w in RT^r/W_0 as $[w]$. From the fundamental properties of the inf-sup constant γ_h of the form $\widetilde{b}(\cdot,\cdot)$, we have

$$\gamma_h = \inf_{(\Psi,\eta,\phi)} \sup_{w} \frac{|\widetilde{b}(w,(\Psi,\eta,\phi))|}{\|(\Psi,\eta,\phi)\|\|w\|} = \inf_{[w]} \sup_{(\Psi,\eta,\phi)} \frac{|\widetilde{b}(w,(\Psi,\eta,\phi))|}{\|(\Psi,\eta,\phi)\|\|w\|}. \tag{3.50}$$

We can thus conclude that

$$\begin{aligned} &\|\Pi v\|_{H(\text{div,K}} \leq \tfrac{1}{\gamma_h} \sup_{(\Psi,\eta,\phi)} \tfrac{|\widetilde{b}(\Pi v,(\Psi,\eta,\phi))|}{\|(\Psi,\eta,\phi)\|} \\ &= \tfrac{1}{\gamma_h} \sup_{(\Psi,\eta,\phi)} \tfrac{|\widetilde{b}(v,(\Psi,\eta,\phi))|}{\|(\Psi,\eta,\phi)\|} \\ &\leq \tfrac{M}{\gamma_h}\|v\|_{H(\text{div})}, \end{aligned} \tag{3.51}$$

where the equality follows from the $\widetilde{b}(\cdot,\cdot)$-orthogonality of $v - \Pi v$ to (Ψ,η,ϕ). The constant M is the continuity constant of $\widetilde{b}(\cdot,\cdot)$, i.e.,

$$|\widetilde{b}(v,(\Psi,\eta,\phi)) \leq M\|v\|_{H(\text{div})}\|(\Psi,\eta,\phi)\|. \tag{3.52}$$

3.3 Numerical Experiments

Recall that the Fortin operator orthogonality conditions resulted in an underdetermined system of equations for the image $w = \Pi v$ in both the H^1 and $H(\text{div})$ cases. Therefore, in addition to the orthogonality conditions, we imposed an additional condition on the image $w = \Pi v$, namely, that it was of minimal norm in the appropriate space, in order to guarantee a unique solution to the constraint equations. We now derive a variational formulation for obtaining the optimal $w = \Pi v$ and thereby estimate the inf-sup constant γ_h. Since the bilinear form continuity constant M, i.e.,

$$|\widetilde{b}(v,u) \leq M\|v\|_V\|u\|_U \tag{3.53}$$

is $O(1)$ in case of both $V = H^1(\Omega_h)$ and $V = H(\text{div},\Omega_h)$ (with the corresponding spaces for U), and the Fortin continuity constant $C = \frac{M}{\gamma_h}$, we have that overall $C = O(\frac{1}{\gamma_h})$.

$\mathbf{H^1}$ case: First consider the H^1 case where we required Πv to satisfy the orthogonality conditions, as well as Πv having least H^1 norm. Recall that upon the use of the alternate H^1 norm based on the zero-average splitting of the H^1 space, minimization of the full H^1 norm is equivalent to the minimization of the H^1 semi-norm. Consider now the cost functional

$$J(w) = \frac{1}{2}\|\nabla w\|_{L^2}^2 - \widetilde{b}(w,(\phi,\phi_e)). \tag{3.54}$$

As before, setting the derivative of the perturbed cost, $\frac{dJ(w+\epsilon\delta w)}{d\epsilon}|_{\epsilon=0} = 0$, we arrive at the variational formulation

$$(\nabla w,\nabla\delta w)_{L^2} = \widetilde{b}(\delta w,(\phi,\phi_e)) \text{ i.e.,} \tag{3.55}$$

$$\begin{cases} w \in \mathcal{P}^r(K) \\ \displaystyle\int_K \nabla w \nabla \delta w = \int_K \phi \delta w + \sum_e \int_e \phi_e \delta w, \quad \delta w \in \mathcal{P}^{p+\Delta p}(K)\,. \end{cases}$$

Introducing operator $T : (\phi, \phi_e) \to w$, we have,

$$\gamma = \inf_{(\phi,\phi_e)} \frac{\|T(\phi,\phi_e)\|_{H^1(K)}}{\|(\phi,\phi_e)\|}\,.$$

Thus γ is the smallest eigenvalue corresponding to the generalized eigenvalue problem:

$$(T(\phi,\phi_e), T(\delta\phi,\delta\phi_e)) = \lambda((\phi,\phi_e),(\delta\phi,\delta\phi_e)) \quad \forall \delta\phi, \delta\phi_e\,.$$

H(div) case: We now consider the $H(\text{div})$ case where, as before, we required Πv to satisfy the orthogonality conditions, as well as having least $H(\text{div})$ norm, so that we are interested in

$$\Pi v = \arg\min_{w \in RT^r} \|w\|_{H(\text{div})}. \tag{3.56}$$

Consider now the cost functional

$$J(w) = \frac{1}{2}\|w\|^2_{H(\text{div})} - \widetilde{b}(w, (\Psi, \eta, \phi)). \tag{3.57}$$

Setting the derivative of the perturbed cost, $\frac{dJ(w+\epsilon\delta w)}{d\epsilon}|_{\epsilon=0} = 0$, we arrive at the variational formulation

$$(w, \delta w)_{H(\text{div})} = \widetilde{b}(\delta w, (\Psi, \eta, \phi)) \text{ i.e.,} \tag{3.58}$$

$$\begin{cases} w \in RT^r(K) \\ \displaystyle\int_K (w, \delta w)_{L^2} = \int_K \Psi \cdot \delta w + \int_K \eta \,\text{div} \delta w + \int_{\partial K} \phi \delta w\,. \end{cases}$$

As in the H^1 case, we introduce the operator $T : (\Psi, \eta, \phi) \to w$ and we have,

$$\gamma = \inf_{(\Psi,\eta,\phi)} \frac{\|T(\Psi,\eta,\phi)\|_{H(\text{div},K)}}{\|(\Psi,\eta,\phi)\|}\,.$$

and γ is the smallest eigenvalue corresponding to the generalized eigenvalue problem:

$$(T(\Psi,\eta,\phi), T(\delta\Psi,\delta\eta,\delta\phi)) = \lambda((\Psi,\eta,\phi),(\delta\Psi,\delta\eta,\delta\phi)) \quad \forall \delta\Psi, \delta\eta, \delta\phi\,.$$

Steps Involved in the Computation: In both cases, the computation of γ involves the following generic steps:

- Select a basis e_i for w, and a basis g_j for (ϕ, ϕ_e) or (Ψ, η, ϕ) as the case may be

$$w = \sum_j w_i e_i, \quad (\phi, \phi_e) \text{ or } (\Psi, \eta, \phi) = \sum_j \phi_j g_j \,,$$

- Compute matrix representation of operator T in those bases,

$$w_i = T_{ij} g_j \,,$$

- Compute the Gram matrices corresponding to the two inner products,

$$H_{kl} := (e_k, e_l)_{H^1(K)}, \quad M_{kl} := (g_k, g_l) \,,$$

- Solve the generalized eigenvalue problem,

$$(T^T H T)\phi = \lambda M \phi \,.$$

As we saw earlier, the γ we are interested in can be seen as the (square root of) the smallest eigenvalue of the generalized eigenvalue problem in the above steps.

4 Approximate Fortin Operators

We now take a different approach to the problem of computing the continuity constant of the Fortin operator. Recall that our difficulty is in exactly computing the constant due to the infinite dimensional nature of norm optimization. In this section, we construct a sequence of *approximate* Fortin operators, each member of which is defined on a large, yet finite dimensional subspace $V^{r+\Delta r}$ containing the enriched test space V^r. In other words, we construct a sequence $\Pi_{\Delta r} : V^{r+\Delta r} \to V^r$ indexed by Δr in place of the *exact* Fortin operator we constructed earlier, $\Pi : V \to V^r$. However, since each approximate Fortin operator $\|\Pi_{\Delta r}\|$ is defined on a finite dimensional subspace $V^{r+\Delta r}$ of V, we can *exactly* compute the continuity constant of $\|\Pi_{\Delta r}\|$. With increasing Δr, it is natural to expect $\|\Pi_{\Delta r}\| \to \|\Pi\|$.

We do this analysis for two reasons. First, we will be able to get a realistic view of how sharp our previous estimates of the *exact* H^1 and $H(\text{div})$ Fortin constants were. Since we were working with sufficient conditions, we know apriori that our previous estimates correspond to the worst case scenario. However, it may be the case that in practice, we always do better than the estimates' promise. The exact computation of $\|\Pi_{\Delta r}\|$ will shed light on how close we actually are to the estimate. In other words, the approximate Fortin constants will give a heuristic lower bound of the exact Fortin constant, for our particular construction. Thus, a different Fortin operator construction would naturally yield estimates and lower bounds corresponding to its construction, and thereby a different estimate on the change in stability while transitioning from ideal to practical test spaces.

4.1 Construction of Approximate Fortin Operators

The construction of the approximate H^1 and $H(\text{div})$ Fortin operators Π_Δ follows the exact case, with the only difference being that the domain of $\Pi_{\Delta r}$ is the finite dimensional space $V^{r+\Delta r}$ containing the enriched test space V^r instead of the infinite dimensional V. Note that the approximate Fortin operators $\Pi_{\Delta r}$ depend on the trial function approximation order p, the enrichment order Δp as well as the new (additional) Fortin enrichment order Δr. We can easily define the approximate Fortin operators using the constrained optimization definition of the exact Fortin operator we gave earlier.

Approximate $\mathbf{H^1}$ Fortin Operator

Recall the exact H^1 Fortin operator is defined via the solution of the following constrained optimization problem:

$$\begin{cases} \Pi v = \text{argmin} \|v^*\|_{H^1(K)} & v^* \in \mathcal{P}^r,\, v \in V \\ (\phi, v^*)_{L^2(K)} = (\phi, v)_{L^2(K)}, & \phi \in \mathcal{P}^p(K) \\ < \phi_e, v^* >_{L^2(\partial K)} =< \phi_e, v >_{L^2(\partial K)} . & \phi_e \in \mathcal{P}^{p-1}(e),\, e = 1,2,3 \end{cases} \tag{4.59}$$

To define the approximate Fortin operator, we simply restrict our space of constraints to lie in the finite dimensional subspace $V^{r+\Delta r}$ instead of the full space V:

$$\begin{cases} \Pi_{\Delta r} v = \text{argmin} \|v^*\|_{H^1(K)} & v^* \in \mathcal{P}^r,\, v \in \mathcal{P}r + \Delta r \\ (\phi, v^*)_{L^2(K)} = (\phi, v)_{L^2(K)}, & \phi \in \mathcal{P}^p(K) \\ < \phi_e, v^* >_{L^2(\partial K)} =< \phi_e, v >_{L^2(\partial K)} . & \phi_e \in \mathcal{P}^{p-1}(e),\, e = 1,2,3 \end{cases} \tag{4.60}$$

We can then derive a mixed (saddle-point) formulation analagous to the exact case. One can then form a matrix representation of $\Pi_{\Delta r}$ and compute the *exact* value of the continuity constant of $\Pi_{\Delta r}$ as:

$$\begin{cases} \|\Pi_{\Delta r}\| = \max \dfrac{\|\Pi v\|_{H^1}}{\|v\|_{H^1}}, & v \in V^{r+\Delta r} \\ \text{or, } (\Pi v, \Pi \delta v)_{H^1} = \lambda^2 (v, \delta v)_{H^1}, & \delta v \in V^{r+\Delta r} \\ \text{or, } (P^* G P) v = \lambda^2 G v, & \end{cases} \tag{4.61}$$

where P is the matrix representation of $\Pi_{\Delta r}$ and G is the Gram matrix corresponding to the alternate (zero-average split) H^1 inner product:

$$G = (\nabla v_i, \nabla v_j)_{L^2} \tag{4.62}$$

and $\text{span}\{v_i\} = V^{r+\Delta r}$. The maximum eigenvalue λ_m of the above generalized eigenvalue problem is the required norm of $\Pi_{\Delta r}$.

Approximate H(div) Fortin Operator

The definition of the approximate $H(\text{div})$ Fortin operator follows the H^1 case:

$$\begin{cases} \Pi_{\Delta r} v = \text{argmin} \|v^*\|_{H(\text{div})} & v^* \in RT^r,\ v \in RT^{r+\Delta r} \\ (\Psi, v^*)_{L^2(K)} = (\Psi, v)_{L^2(K)}, & \Psi \in RT^p(K) \\ (\eta, \text{div } v^*)_{L^2(K)} = (\eta, \text{div } v)_{L^2(K)}, & \eta \in \mathcal{P}^{r-1}(K)/\mathcal{P}^p(K) \\ < \phi, v^* >_{L^2(\partial K)} =< \phi, v >_{L^2(\partial K)} . & \phi \in \mathcal{P}^p_c(\partial K). \end{cases} \tag{4.63}$$

Again, we can compute $\|\Pi_{\Delta r}\|$ as:

$$\begin{cases} \|\Pi_{\Delta r}\| = \max \dfrac{\|\Pi v\|_{H(\text{div})}}{\|v\|_{H(\text{div})}}, & v \in RT^{r+\Delta r} \\ \text{or, } (\Pi v, \Pi \delta v)_{H(\text{div})} = \lambda^2 (v, \delta v)_{H(\text{div})}, & \delta v \in RT^{r+\Delta r} \\ \text{or, } (P^* G P) v = \lambda^2 G v, & \end{cases} \tag{4.64}$$

where P is the matrix representation of $\Pi_{\Delta r}$ and G is the Gram matrix corresponding to the $H(\text{div})$ inner product:

$$G = (v_i, v_j)_{H(\text{div})} \tag{4.65}$$

and $\text{span}\{v_i\} = RT^{r+\Delta r}$. The maximum eigenvalue λ_m of the above generalized eigenvalue problem is the required norm of $\Pi_{\Delta r}$.

Importance of Approximate Fortin Operator

The approximate Fortin constants give us a heuristic as to how good our upper bound for the exact Fortin constant is. In particular, it is expected that with increasing Δr, the approximate Fortin constants will converge to the exact Fortin constant. Moreover, it is not hard to see that the convergence is from below, and also, the sequence of approximate Fortin constants in non-decreasing. Thus, we can view the approximate Fortin constants as being a lower bound on the exact Fortin constant. Finally, one expects a tighter lower estimate via the approximate Fortin constants, i.e., the approximate Fortin constants are expected to be "closer" to the exact Fortin constant than the upper bound estimate we derived earlier.

We now give details of the numerical results of both the upper bound estimates as well as the approximate Fortin constants.

5 Numerical Results

We now discuss our computational results in detail, starting with the results of the upper bound estimates for the exact Fortin operator.

5.1 $\mathbf{H^1}$ Fortin operator upper bound

Figure 1: Upper bound of the H^1 DPG Fortin constant as a function of Δp for various p

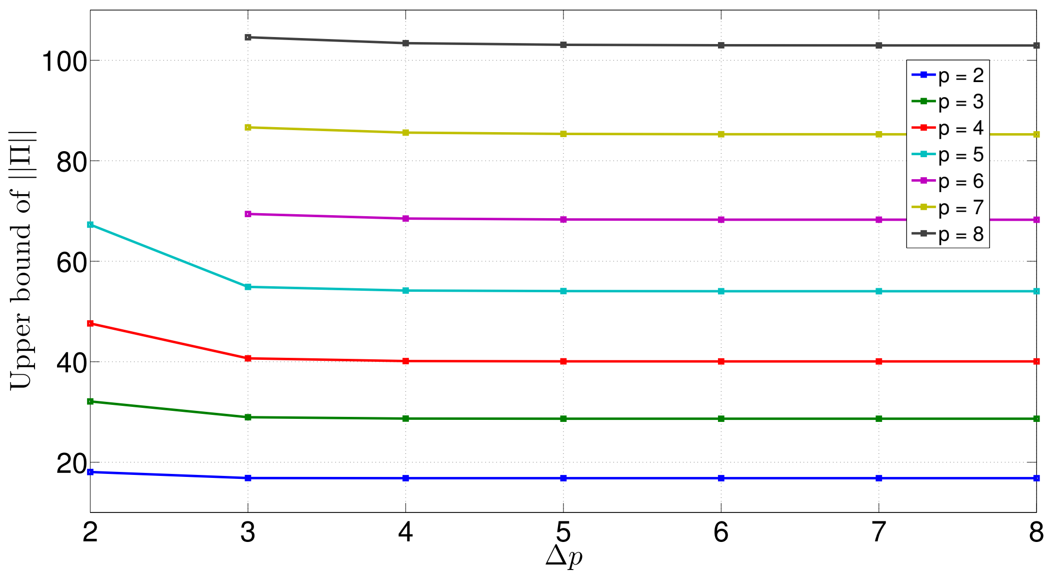
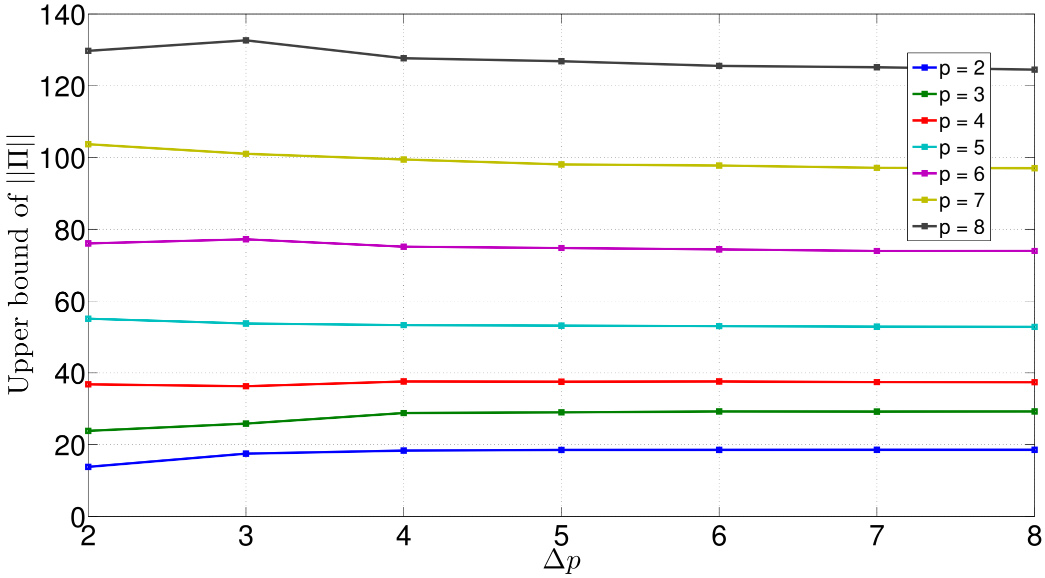


Figure 1 shows the upper bound of the H^1 DPG Fortin constant as a function of p and Δp. We observe a few interesting results. First, with increasing p, we find that the upper bound of the Fortin constant also increases. This directly translates to the fact that with higher p, we loose stability, which of course is to be expected since increasing p makes resolving the optimal test functions more difficult. Second, we find that increasing Δp only marginally increases stability, that too, only for low p. This means that we do not gain significant stability by an indiscriminate increase in Δp. Also, for fixed p, the Fortin constant is a decreasing function of Δp. This is expected, since we search for the minimum eigenvalue of the generalized eigenvalue problem over a larger space with increasing Δp. Finally, we see that we need atleast $\Delta p = 3$ for $p \geq 6$, for any reasonable stability, as was indicated by the theory.

5.2 $\mathbf{H(div)}$ Fortin operator upper bound

Figure 2 shows the upper bound of the $H(\text{div})$ DPG Fortin constant as a function ofΔp for various p. Here, some of the results are, at first glance, a bit anomalous. At the very outset, we see that the actual values of the Fortin constant are much larger than in the H^1 case. This is due to the fact that we impose a greater

Figure 2: Upper bound of the $H(\mathrm{div})$ DPG Fortin constant as a function of Δp for various p



number of constraints in the $H(\mathrm{div})$ case than the H^1 case. As in the H^1 case, with increasing p, we find that the upper bound of the Fortin constant also increases. Also, in the $H(\mathrm{div})$ case as well, increasing Δp only marginally increases stability, mainly for low p. However, for fixed p, we do not find monotonic decrease of the Fortin constant with increasing Δp, which may seem odd. However, this is clarified upon closer inspection of the constraints we have imposed. In order to obtain an L^2 projection on the divergence part of the $H(\mathrm{div})$ norm, we imposed a Δp dependent constraint, which means that increasing Δp changes the set of constraints, and therefore, the image of the Fortin operator. It is thus not reasonable to expect a monotonic decrease of the Fortin constant with increasing Δp in the $H(\mathrm{div})$ case. This is unavoidable with our construction, as we require the L^2 projection constraint on the divergence part of the Fortin operator in order to ensure that the scaling arguments can be used to reduce the construction to the master element.

5.3 Approximate $\mathbf{H}^1$ Fortin constant

Figure 3 shows the plots of the exact the Fortin constant of the *approximate* H^1 DPG Fortin operators. The plots display values of $\|\Pi_{\Delta r}\|$ after convergence with sufficiently large Δr as a function of Δp for various p. As we see from the plots, the Fortin constant is of order unity. Moreover, we get rapid convergence (for sufficiently large Δr) with increasing Δp. Finally, as expected, the convergence is from above, i.e., the values decay with increasing Δp. These results are very optimistic, and indicate that we have, in practice, no recognizable loss of stability in the H^1 case with the use of practical test functions.

Figure 3: Exact Fortin constant of approximate H^1 DPG Fortin operators. Plots show values of $\|\Pi_{\Delta r}\|$ after convergence with sufficiently large Δr as a function of Δp for various p

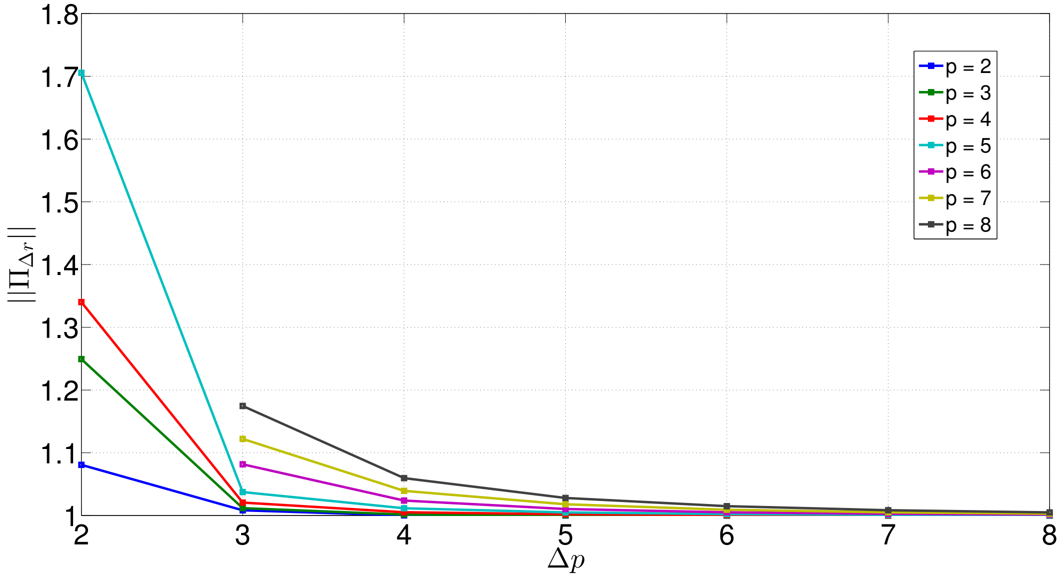
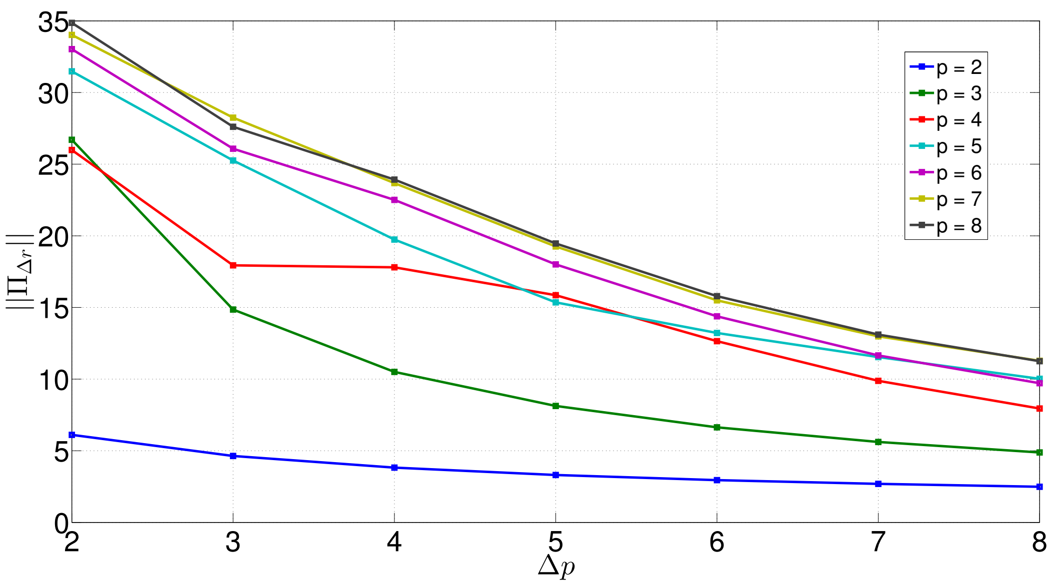


Figure 4: Exact Fortin constant of approximate $H(\text{div})$ DPG Fortin operators. Plots show values of $\|\Pi_{\Delta r}\|$ after convergence with sufficiently large Δr as a function of Δp for various p



5.4 Approximate H(div) Fortin constant

Figure 4 shows the plots of the exact the Fortin constant of the *approximate* $H(\text{div})$ DPG Fortin operators. Again, for ease of comparison with the earlier plots, we display values of $\|\Pi_{\Delta r}\|$ (after convergence with sufficiently large Δr) as a function of Δp for various p. In this case, the Fortin constants are larger than the H^1 values, being an order of magnitude larger. However, we do still see a decay with increasing Δp. The decay is non-monotonic due to the same reasons we observed in the $H(\text{div})$ estimates earlier: we imposed a Δp dependent constraint, which means that increasing Δp changes the set of constraints, and therefore, the image of the Fortin operator. Finally, although the values are larger than the H^1 case, we still have a more optimistic result than the $H(\text{div})$ estimates, which were almost two orders of magnitude larger.

5.5 Conclusions

In this paper, we analyzed the stability of a "practical" choice of optimal test functions in the DPG context. Towards this end, we constructed a DPG Fortin operator which allowed us to quantitatively measure the change in stability when one approximately inverts the Riesz map while computing the optimal test space. The stability loss is measured by the norm (continuity constant) of the Fortin operator. In particular, we considered the case of H^1 and $H(\text{div})$ spaces and restricted our attention, via the use of broken test functions and scaling arguments, to the master triangular element. We enforced a set of sufficient conditions in order to uniquely define the Fortin operator.

Our analysis was a two-pronged approach. First, we estimated an upper bound on the Fortin continuity constant using an inf-sup argument on an auxiliary bilinear form. This estimate gave us a rough idea of the "worst case" scenario of how much stability loss is theoretically possible. This was in part due to the fact that an exact computation of the Fortin constant is not feasible due to the infinite dimensional nature of the problem. Needless to say, the bound was rather loose and the resulting predicted loss was pessimistic. Second, we constructed a sequence of approximate Fortin operators, each defined on progressively larger, yet finite dimensional subspaces of the full test space. We then computed the exact Fortin constant of the approximate Fortin operators which gave a much more optimistic lower bound on the Fortin constant of the exact Fortin operator. The culmination of the two approaches led us to believe that, in practice, one is justified in using the "practical" optimal test functions as opposed to the ideal optimal test functions with a very slight loss of stability. This was the case in both the H^1 and $H(\text{div})$ spaces.

Finally, we note that our results, particularly the results pertaining to the approximate Fortin constants, indicate that one may pursue *hp-adaptivity* with practical optimal test functions. Indeed, local enrichment of the optimal test functions can be done due to the use of broken test functions, and the results of this paper indicate that due to only slight loss of stability, one can in fact adaptively enrich the practical optimal test function computation locally on the element level.

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6 Appendix

6.1 Another Charaterization of the Optimal Test Space

As we have seen, the optimal test space (for a given trial space) consists of the vectors that achieve the inf-sup condition. Another way to see how the space of optimal test functions guarantees stability is as follows.

The standard Petrov-Galerkin method *fixes* both U_h and V_h and hence, in order to ensure the inf-sup condition, we must have $B(U_h) = R_V(V_h)$. However, depending on the exact form of B and the subspaces U_h, V_h, this may not always be true. Indeed, in general, $B(U_h)$ is just an arbitrary finite dimensional subspace of V', and there is no reason for us to believe that the operator B restricted to U_h must map to $R_V(V_h)$. This discrepency between $B(U_h)$ and $R_V(V_h)$ is fundamental cause for lack of stability at the discrete level.

Now, we present yet another characterization of the optimal test space. We start with a lemma:

Lemma A1 Given a Hilbert space V and a closed subspace $A \subset V$, we have $R_V(A) = (A^\perp)^\circ$, where $(S)^\circ$ denotes the annihilator of a set $S \subset V$, i.e., $(S)^\circ = \{f \in V^{'} | f(s) = 0 \ \forall s \in S\}$.

Proof: Let $f \in R_V(A)$, and $a^\perp \in A^\perp$. Then, $f(a^\perp) = (R_V^{-1}(f), a^\perp)_V = 0$, since $R_V^{-1}(f) \in A$ and $a^\perp \in A^\perp$, so $R_V(A) \subset (A^\perp)^\circ$.

Next, let $f \in (A^\perp)^\circ$. Now, $f(a^\perp) = (R_V^{-1}(f), a^\perp) = 0$ for $a^\perp \in A^\perp$. Since $V = A \oplus A^\perp$, we have that $R_V^{-1}(f) \in A$, or, $f \in R_V(A)$. Thus, $R_V(A) = (A^\perp)^\circ$. $\square$

Let us denote $R_V(A)$ for a closed subspace of V as A'. We thus have the following situation. For an arbitrary choice of V_h, we may not have $V_h^{'} = B(U_h)$, and so may not have the discrete inf-sup condition. However, if we pick V_h as the optimal test space, i.e., $R_V^{-1} B(U_h) = V_h^{opt}$, then, by definition, $B(U_h) = (V_h^{opt})'$, ensuring the discrete inf-sup condition.

6.2 Proof of H^1 norm equivalence:

Choice of norm for the broken test space. We shall use the following norm (for the analysis):

$$\|u\|^2_{H^1(\Omega_h)} = \sum_K \left(\|\nabla u\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(K)} \right) \tag{6.66}$$

where

$$\bar{u} := \frac{1}{|K|} \int_K u \, dK$$

is the average value of function u in element K. The norm is equivalent with the standard broken H^1-norm with mesh independent equivalence constants. Indeed,

$$\begin{aligned} \|\bar{u}\|^2_{L^2(K)} &= \int_K |\bar{u}|^2 = |k|\,|\bar{u}|^2 = |K|^{-1}\,|\int_K u|^2 \\ &\le |K|^{-1} \int_K |u|^2\,|K| \qquad\qquad \text{(Schwartz inequality for } \int_K u\,1) \\ &= \|u\|^2_{L^2(K)}\,. \end{aligned}$$

Likewise, by Pythagoras theorem,

$$\|u\|^2_{L^2(K)} = \|u - \bar{u} + \bar{u}\|^2_{L^2(K)} = \|u - \bar{u}\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(K)}\,.$$

Function $u - \bar{u}$ has a zero average, $u - \bar{u} \in H^1_{avg}(K)$, and so does the corresponding pullback $\hat{u} - \overline{\hat{u}}$. Recalling the Poincaré inequality for the master element,

$$\|\hat{u}\|^2_{L^2(\hat{K})} \le \frac{\sqrt{2}}{\pi}\|\hat{\nabla}\hat{u}\|^2_{L^2(\hat{K})} \quad \forall \hat{u} \in H^1_{avg}(\hat{K})\,, \tag{6.67}$$

and applying the scaling argument, we get,

$$\|u\|^2_{L^2(K)} \le \frac{\sqrt{2}}{\pi}\,h^2\,\|\nabla u\|^2_{L^2(K)} \quad \forall u \in H^1_{avg}(K)\,. \tag{6.68}$$

In conclusion,

$$\|u\|^2_{L^2(K)} \le \frac{\sqrt{2}}{\pi}h^2\,\|\nabla u\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(K)}\,.$$

The first equivalence constant is one and, for small h, the second equivalence constant is very close to one, too.

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