The dome and the ring: Verification of an old mathematical model for the design of a stiffened shell roof

by

Juhani Pitkaranta, Ivo Babuska and Barna Szabo
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Juhani Pitkäranta, Ivo Babuška and Barna Szabó

Department of Mathematics and Systems Analysis,
School of Science, Aalto University, Espoo, Finland
Institute for Computational Engineering and Sciences,
The University of Texas at Austin
Department of Mechanical Engineering,
Washington University, St. Louis

Abstract

We study an old mathematical model, developed before the computer era,
for analyzing the strength of a stiffened shell roof. The specific problem con-
sidered is a textbook example presented in K. Girkmann: Flächentragwerke,
3rd edition, 1954. Here the roof consists of a spherical dome and a stiffening
ring of rectangular cross section attached to the edge of the dome. The prob-
lem is to compute the resultant force and moment acting at the junction of
the dome and the ring. We approach the old model for solving the problem
in two different ways. First we carry out a historical study, where we look for
possible improvements of the old model while limiting ourselves to manual
computations only. We find a variant of the model which, despite being about
as simple as the original one, is considerably more accurate in comparison
with recent numerical solutions based on FEM and axisymmetric 3D elastic
formulation of the problem. The second approach in our study is to carry
out an a posteriori error analysis of our refined old model. The analysis is
based on variational methods and on the Hypercircle theorem of the linear
theory of elasticity. The error analysis confirms, and largely also explains, the
observed — rather high — accuracy of the refined old mathematical model.
1 Introduction

The modeling of shell structures, such as shell roofs, is traditionally one of the most challenging tasks of engineering mathematics. Shells are the sensitive "primadonnas" of structures, both from the viewpoint of engineering design and mathematical modeling.

Sixty years ago the design of shell structures was still based largely on parametrized classical solutions and manual computing. From old textbooks like [1, 2] one can get a general idea of this rather advanced engineering science before the era of computers. The basis of the manual computational models was the classical shell theory, which was well developed already 60 years ago. The classical shell theory reduces the 3D linear elastic laws to 2D equations, so called shell equations, along the middle surface of the shell. Shell equations are still partial differential equations, and even worse, with variable coefficients, so they are not solvable by analytic means in general. Under special geometric or symmetry assumptions, however, the shell equations may be reduced further to ordinary differential equations in one space dimension. The old engineering shell theory covers a collection of such special situations. In most of these cases, further simplification of the 1D shell equations is still needed to allow a classical solution in terms of elementary functions.

The engineer faces a 1D shell problem, e.g., when he wants to certify that a dome-shaped shell roof, as designed, carries its own weight. In this paper we travel backwards in time to see how the engineer handled such a problem in the pre-computer era. The shell roof to be considered is taken from a textbook example presented in [1]. (The same example is found also in [3].) The example is named here the Girkmann problem according to its original reference. The roof consists here of a thin spherical dome and a stiffening footring connected to it at the meridional angle $\alpha = 40^\circ$. The ring is of rectangular cross section and connected to the dome along its edge. The material of the whole structure is concrete, assumed homogeneous and linearly elastic in the mathematical model. An equilibrium support at the base of the ring is assumed to balance the weight of the structure. In this setting, the computational problem to be solved is specified as: Find the values of the horizontal force ($R$) and moment ($M$) by which the dome and the ring act on each other at their intersection. Both $R$ and $M$ are reactions to be evaluated per unit length of the junction line. A more detailed description of the Girkmann problem is given in Section 2 below.

As stated, the Girkmann problem is part of the certification of the strength of the roof: Knowing the reaction force and moment acting at the edge of the dome, the engineer can compute further the stresses in the dome according to shell theory. In particular, he can evaluate the maximal bending stress in the vicinity of the junction — the most critical quantity concerning the strength of the roof. Bending stresses are due to the so called edge effect that
is characteristic to shell deformations near edges or interfaces.

In [1] it is demonstrated how an approximate solution to the Girkmann problem is found manually. First the classical shell theory applied to the spherical dome is simplified to an approximate engineering shell theory. The latter consists of the so called \textit{membrane theory} (M) and \textit{bending theory} (B) for the shell, each valid approximately under specific loading and edge conditions. For the ring the classical engineering \textit{ring theory} (R) is assumed. Upon combining the engineering shell and ring theories and imposing kinematic continuity constraints at the junction, one obtains the traditional simplified model for determining the two unknown quantities $R$ and $M$. We refer to this classical textbook model here as the M-B-R model. In the end the M-B-R model reduces to a $2 \times 2$ linear system for the unknowns, with given algebraic expressions for the coefficients of the system. Using such a model, a trained engineer of the old generation probably needed only a pencil, logarithmic and trigonometric tables, a back of an envelope, and half an hour to solve the problem for a given design.

But how accurate is such a simple model? — We should be able to answer such a question now, assuming that the 'exact' solution obeys the 3D laws of linear elasticity with the given material parameters of the problem. In cylindrical or spherical coordinates, with the rotational symmetry taken into account, the mathematical problem actually reduces to a 2D linear elastic problem on the vertical cross section of the roof. For the engineers of today, now working with a laptop computer and a FEM code, an accurate numerical solution of such a problem should be routine.

A recent test, however, tells a different story. In [4], the Girkmann problem was announced as a benchmark test for the expert users of finite element software products. The participants were asked to solve the problem using their favourite code and to verify that the error in the computed values of $R$ and $M$ was no more than 5\%. The results received from 15 respondents were summarized in [5, 6]. The desired accuracy was achieved in only 6 of the 15 solutions. In another 6 solutions the error in $M$ exceeded 100\% and in one solution, $R$ was about 20 times and $M$ about 500 times too large and even the sign of $M$ was wrong [6].

In a later contribution to the Girkmann benchmark test, different finite element approaches based on open software were tested, and this time quite accurate results were obtained consistently [7]. What then caused the wide scattering of the results in the earlier test remains largely conjectural. In any case, the Girkmann problem challenge succeeds in underlining the importance of \textit{verification} of numerical results even in the context of relatively simple-looking problems. In general, both verification and \textit{validation} (V&V, see [8, 9]) of numerical and mathematical models is of growing importance now that more and more complex problems are becoming numerically solvable and engineering curricula no longer cover classical methods in sufficient detail.
But let us return to the question posed above concerning the accuracy of the traditional manual solution to the Girkmann problem. This was the question that actually inspired the first finite element benchmarking on the problem in [10], but so far this original question has remained unresolved. Our aim here is to close the case and give a precise answer. In the V&V terminology, our aim is to carry out the full verification of the classical model when solving the Girkmann problem. The 2D formulation of the problem assumed in [4]–[7] (originally due to [10]) is considered here as ‘exact’.

The outline of the paper is as follows. In Section 2 we give the precise formulation of the Girkmann problem as a 2D linear elastic problem. In Sections 3 and 4 we approach the classical model for solving the problem in two quite different ways. Section 3 is a historical expedition back to the derivation of the model. Our aim is to find out, to what extent it is possible to improve the classical model so as to make it more accurate without sacrificing its simplicity. In Section 4 we focus on the model variant that we find experimentally to be the most accurate one. We attack this model by methods of mathematical error analysis, with the aim to both certify and explain the observed accuracy of the model. Finally, in Section 5 we present the summary and conclusions of our paper, together with some historical remarks.

In what follows we present first an extended introduction that gives a more detailed outline of the contents of Sections 3 and 4.

**Study of the old model (Section 3)**

In the old literature little or no attention is given to possible variations of the basic M-B-R model as found in textbooks. In the true accuracy test that we perform here, however, the fine tuning of the M-B-R model turns out to have a significant effect on our final conclusions. Therefore, to uncover the hidden capabilities of the classical model, we first time travel back to the derivation of the model, searching for possible ways to improve the model without sacrificing its simplicity. We come up with a number of alternative models, each about as simple as the basic model and suitable for manual computation. — We underline that all we needed in our computations was a simple pocket calculator.

The highlight of our time travel is the accuracy test at the end, where the 2D reference values of $R$ and $M$ are taken from [5] (p-FEM, axisymmetric solid). We test first the accuracy of the basic M-B-R model of [1]. We need first to adjust this model, since it turns out that the support assumed at the base of the ring in our 2D formulation of the Girkmann problem does not fully conform to the assumptions made in [1]. After the adjustment we find that the gap between the M-B-R model and the 2D model is about 2% in $R$ and about 90% in $M$. Thus the old mathematical model, as presented in
textbooks, solves the Girkmann problem fairly accurately for $R$ but not so for $M$.

Also when studying variations of the M-B-R model we find that $M$ is the leading error indicator in sensitivity. In the best of the variations we propose three improvements to the M-B-R model. The first two improvements are to replace the usual membrane and bending theories of the dome by what we call a *bending-corrected* membrane model (MB) and a *sloping-corrected* bending model (BS). The former model takes into account a small edge effect due to bending (ignored by the usual membrane theory), and the latter model improves the bending theory by taking into account the sloping of the edge of the shell when the shell is not hemispherical. These corrections are both made within the classical shell theory. As will be confirmed numerically, the two corrections almost cancel the effect of the simplifications in the usual engineering membrane and bending theories, so that the resulting MB-BS theory of the dome is very close to the classical shell theory.

The third improvement of the basic model is made in the ring theory. Here there are many possible variations of the basic theory, of which we choose to consider two. In the first improvement we determine the loading of the ring more precisely than in the basic theory, taking into account the details of the geometry of the ring and the junction as assumed in our 2D formulation. Otherwise we still rely on the usual ring theory as presented in [1]. We name this improvement of the standard theory as the *load-corrected* ring theory (RL). In the second improvement of the basic theory we take a completely different approach. Here we use directly the kinematic assumption of ring theory stating that the cross section of the ring remains undeformed when the ring is deformed. This leaves only two possible displacement modes for the cross section: rigid radial deflection and rotation. When taking these as the degrees of freedom and applying the energy principle we obtain what we call the *minimal-energy* model of the ring (RE). — Obviously this is nothing more than the simplest finite element approximation where the ring cross section acts as a single element with two degrees of freedom, and indeed, this was one of the approaches taken in [7].

Upon combining the basic and improved options of the shell membrane, shell bending and ring theories in different ways we come up with 12 different mathematical models. All of these are about equal in their simplicity, leading in the end to a 2 × 2 linear system for the unknowns $R$ and $M$, with slightly different coefficients in each case.

We are now ready for the final round of our man vs. computer race: We choose the best of our 12 models to challenge the model based on the 2-dimensional linear elastic formulation of the problem. As to be expected, the winner among the manual computational models is the MB-BS-RE model that combines the bending-corrected shell membrane theory, the sloping-corrected shell bending theory and the minimal-energy ring theory. For this
model we find that the gap with respect to the 2D model is less than 0.1% in $R$ and 1.7% in $M$, so compared with the basic M-B-R model the gap is reduced by factor about 20 for $R$ and about 50 for $M$.

As a summary of our historical expedition we must conclude that the old manual computational model, when carefully tuned as was found possible, is not just fairly accurate in comparison with the 2D elastic model. It is surprisingly accurate, beating clearly not only the old textbook version of the model but even quite many of the recent attempts to solve the problem in a modern way using existing 2D or 3D finite element software [6].

**Error analysis (Section 4)**

In the error analysis we focus on the best of our variants of the old model, the MB-BS-RE model, which we rename from this on as the simplified model (S). The numerical experiments so far leave us confronted with a mathematical problem: Can we certify, and possibly also explain, the observed accuracy of our S-model by mathematical error analysis? — At least we should take an effort to rule out the possibility of plain luck, since our experiments only tested the accuracy of the models when approximating two numbers.

We look for a mathematical explanation for the success of the simplified model by performing an *a posteriori* error analysis of this model with respect to the 2D elastic model. The advantage of the a posteriori approach, as compared with the *a priori* approach, is that no regularity assumptions on the unknown 2D solution are required. We need only to know the solution according to our simplified S-model.

The aim of our a posteriori error analysis is to derive explicit (computable) bounds for $|R - R_S|$ and $|M - M_S|$, where $R_S, M_S$ are the resultants according to the S-model and $R, M$ their exact counterparts according to the 2D model. We want also the bounds to be sharp, so that we can certify (if possible) the observed accuracy of the S-model. The main tool of our analysis is the classical *Hypercircle theorem* of the linear theory of elasticity [12]. The theorem states that when approximating the unknown stress field ($\sigma$) in a given linear elastic problem by the field $\frac{1}{2}(\sigma^s + \sigma^k)$, where $\sigma^s$ is statically admissible and $\sigma^k$ kinematically admissible for the problem, the error equals one half of the gap $\sigma^s - \sigma^k$ when measured in the energy norm.

When applying the Hypercircle theorem the first step is to express $R - R_S$ and $M - M_S$ in terms of functionals involving the fields $\sigma, \sigma^s$ and $\sigma^k$. These functionals are similar to the *extraction functionals* often used in the post-processing of finite element solutions [10]. Given the appropriate functional expressions, the usual idea is to use the Hypercircle theorem to drop the unknown field $\sigma$ from the bounds. Here we succeed to do this only partly, so that the resulting bounds still contain an unknown field component $\sigma^X$. This could only be determined numerically by solving an auxiliary problem.
about as difficult as the original 2D problem. Our way out is to estimate directly the contribution \((R^X, M^X)\) of \(\sigma^X\) to the resultants \(R\) and \(M\). This we can do, albeit only qualitatively, using a specific approximation process.

After the use of the Hypercircle theorem we are thus left with bounds involving the stress fields \(\sigma^s, \sigma^k\) and \(\sigma^X\). Assuming that \(\sigma^X\) can be taken care of, the remaining problem is to construct the two fields \(\sigma^s\) and \(\sigma^k\) in such a way that they are as close as possible to each other. We actually need the construction only in the dome, since the unknown field \(\sigma^X\) is defined so as to take care of the ring. In the dome the fields \(\sigma^s\) and \(\sigma^k\) are both associated to the known solution according to the S-model. To simplify our analysis we make no distinction between the S-model and the classical shell theory in the dome. (Numerical evidence presented in Section 3 supports the simplification.) After a rather tricky construction we are finally able to bring the two fields close enough, so that in the absence of the unknown field \(\sigma^X\), our bounds for \(|R - R_S|\) and \(|M - M_S|\) are of the same order of magnitude as the observed values.

Finally we estimate the contribution of \(\sigma^X\) to our error bounds. The main idea here is to first approximate the displacement gap at the junction caused by the genuinely 2–dimensional deformation of the ring (not captured by the simplified model) and then use the simplified model to approximate the additional reactions \(R^X\) and \(M^X\) needed to close the gap. At the first step we utilize idealized analytic solutions of plane elasticity theory from [11]. We postulate that the right orders of magnitude of \(R^X\) and \(M^X\) are found in this way. Modulo this uncertainty we conclude that the contribution of \(\sigma^X\) does not change the order of magnitude in our error bounds.

We have thus certified (at least ’almost’) that the high accuracy of the S-model is not just a coincidence. Our error analysis also largely explains, why the S-model is so accurate. — We note that a priori, suspicions concerning the accuracy of any simple model could arise because the 2D stress field \(\sigma\) of the problem is known to be rather complicated. For example, the re-entrant corners of the roof profile at the junction of the dome and the ring cause stresses at these points to be unbounded due to corner singularities [10]. Why a simple model that completely ignores such a local behaviour can be so accurate is apparently because such features of \(\sigma\) are concentrated in the component \(\sigma^X\). This component may be significant in the energy norm, but as our analysis indicates, its contribution to the resultants \(R\) and \(M\) is small.

The conclusion of our error analysis is thus that, modulo the mentioned slight uncertainty arising from the approximation of the unknown stress field \(\sigma^X\), we have mathematically certified our numerical observations, and we also succeeded in largely explaining, why our simple new-old model is so strikingly accurate.
2 The Girkmann problem

We consider a mathematical model of a shell roof consisting of a spherical dome and a stiffening footring attached to it. The geometric and physical specifications of the problem are taken from a textbook example presented in [1], here named as the Girkmann problem. The example (found also in [3]) aims to demonstrate, how a simplified mathematical model derived from linear elasticity theory and classical solutions of differential equations can be applied to certify the strength of the roof under the assumed loading. In [10] some of the missing details of the original problem formulation were specified so as to interpret the Girkmann problem as an axisymmetric 2D linear elastic problem posed on the cross section of the roof. This newer formulation of the problem will be our starting point. In the problem specifications and numerical calculations that follow we preserve the physical units of the original reference, so that the length unit below is cm = 10^{-2} m and the force unit is \( G = \) gravity force acting on one kg of mass (denoted by 'kg' in [1]).

The cross-sectional profile of the dome shell and the footring in the Girkmann problem are shown in Fig. 1 below. The larger scale refers to the original problem formulation in [1], where the inner radius of the stiffening ring is \( \rho_0 = 1500 \) cm and the cross section is a rectangle of width \( a = 60 \) cm and depth \( b = 50 \) cm. The dome, a spherical shell of thickness \( d = 6 \) cm, is attached to the stiffener along its vertex line. The opening meridional angle of the dome is \( \alpha = 40^\circ \), so that the radius of the spherical shell equals \( r_0 = \rho_0 / \sin \alpha = 2333.6 \) cm. In the zoomed picture the geometry at the junction of the dome and the ring is specified in more detail, following the interpretation given in [10]. Here \( \rho_0 \) is measured to the midpoint of the junction line \( AE \), \( r_0 \) is interpreted as the radius of the shell at the midsurface, and the cross section of the ring is reduced from a rectangle to the pentagon \( ABCDE \).

In the mathematical model of [1] the material of the roof (concrete) was assumed linearly elastic with given Young modulus \( E \) and Poisson ratio \( \nu = 0 \) (the value of \( E \) does not matter). The dead load of the structure was assumed to be balanced by a vertical support at the base of the ring. No kinematic constraints were imposed. In the dome the gravity load was further idealized to a surface load on the shell midsurface with surface density \( g = 0.02 \) G/cm$^2$, and the support at the base of the ring was assumed such that no moment on the ring arises when the force acting at the junction of the dome and the ring is tangential to the shell and balances the weight of the dome [1]. Since the assumed support cancels the effect of the ring gravity in the model of [1], the ring was formally assumed weightless.

As the "exact" mathematical model for the above problem we take the 2D axisymmetric laws of linear elasticity. We keep the original Girkmann assumptions concerning the material of the structure and the load in the
Figure 1: Geometry of the Girkmann problem: $\rho_0 = 1500$ cm, $\alpha = 40^\circ$, $r_0 = 2333.6$ cm, $d = 6$ cm, $a = 60$ cm, $b = 50$ cm.

Dome. Concerning the ring we add the gravity (volume) load. Finally we impose the equilibrium boundary condition by assuming a uniform normal pressure at the base of the ring, as in [10].

When writing the 2D linear elastic laws we adopt the spherical coordinates $(r, \theta, \varphi)$ for the dome and the cylindrical coordinates $(\rho, z, \varphi)$ for the ring, with $z = 0$ at the foot of the ring. In the dome the non-vanishing components of the stress tensor $\sigma$ are $\sigma_\theta$, $\sigma_r$, $\sigma_\varphi$ and $\tau_{r\theta}$, and the homogeneous equilibrium equations are (cf. [13])

$$\begin{align*}
\frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \sigma_\theta \right) + \frac{\sin \theta}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} - \frac{\cos \theta}{r} \sigma_\varphi &= 0 \\
\frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \tau_{r\theta} \right) + \frac{\sin \theta}{r^2} \frac{\partial (r^2 \sigma_r)}{\partial r} - \frac{\sin \theta}{r} (\sigma_\theta + \sigma_\varphi) &= 0
\end{align*}$$

(2.1)

These equations hold away from the mid-surface of the dome, i.e. for $0 < \theta < \alpha$ and for $r_0 - d/2 < r < r_0 + d/2$, $r \neq r_0$. At the mid-surface the assumed concentrated load implies the jump conditions

$$\begin{align*}
(\sigma_\theta^+ - \sigma_\theta^-)(r_0, \theta) &= g \cos \theta, \\
(\tau_{r\theta}^+ - \tau_{r\theta}^-)(r_0, \theta) &= -g \sin \theta,
\end{align*}$$

(2.2)

whereas the outer and inner surfaces are traction–free:

$$\sigma_r(r, \theta) = \tau_{r\theta}(r, \theta) = 0 \quad \text{at} \quad r = r_0 \pm d/2.$$

(2.3)

In the ring we assume cylindrical coordinates $(\rho, z, \varphi)$, so the non-vanishing stress components are $\sigma_\rho$, $\sigma_z$, $\sigma_\varphi$ and $\tau_{\rho z}$, and the equilibrium equations on
the ring cross section are

\[
\begin{align*}
\frac{1}{\rho} \frac{\partial (\rho \sigma)}{\partial \rho} + \frac{\partial \tau_{\rho z}}{\partial z} - \frac{1}{\rho} \sigma_{\varphi} &= 0 \\
\frac{1}{\rho} \frac{\partial (\rho \tau_{\rho z})}{\partial \rho} + \frac{\partial \sigma_z}{\partial z} &= f
\end{align*}
\]

(2.4)

Here \( f \) stands for the gravity load density. Consistently with the assumed idealized gravity load on the dome we set \( f \) to the constant value

\[ f = g/d. \]

(2.5)

The boundary line \( ABCDE \) of the ring (see Fig. 1) is free except for the bottom line where a constant normal pressure \( p \) (chosen to balance the weight of the structure) is imposed. The boundary conditions are thus

\[
\begin{align*}
\sigma_{\rho} &= \tau_{\rho z} = 0 \quad \text{(lines } AB, CD) \\
\sigma_z &= \tau_{\rho z} = 0 \quad \text{(line } DE) \\
\sigma_z &= -p, \quad \tau_{\rho z} = 0 \quad \text{(line } BC)
\end{align*}
\]

(2.6)

Finally, at the junction line \( AE \) connecting the dome and the ring, the normal stress and the shear stress must be continuous. We formulate the continuity conditions by requiring that for \( r_0 - d/2 < r < r_0 + d/2 \)

\[
\begin{align*}
\sigma_{\theta}(r, \alpha) &= \sigma_{\rho}(\rho_r, z_r) \cos \alpha - \tau_{\rho z}(\rho_r, z_r) \sin \alpha, \\
\tau_{r\theta}(r, \alpha) &= \tau_{\rho z}(\rho_r, z_r) \cos \alpha - \sigma_z(\rho_r, z_r) \sin \alpha,
\end{align*}
\]

(2.7)

where \( (\rho_r, z_r) \) are the cylindrical coordinates that correspond to the spherical coordinates \( (r, \theta) = (r, \alpha) \) at the junction, that is

\[
\begin{align*}
\rho_r &= \rho_0 + (r - r_0) \sin \alpha, \\
z_r &= z_0 + (r - r_0) \cos \alpha, \quad z_0 = b - (d/2) \cos \alpha.
\end{align*}
\]

(2.8)

For the Poisson ratio \( \nu = 0 \) the stress and strain tensors are related by the simple law \( \sigma = E \epsilon \). The strain–displacement relations are given as

\[
\begin{align*}
\epsilon_{\theta} &= \frac{1}{r} \frac{\partial U_{\theta}}{\partial \theta} + \frac{U_r}{r}, \quad \epsilon_{r} = \frac{\partial U_r}{\partial r}, \\
\epsilon_{\varphi} &= \frac{1}{r} (U_{\theta} \cot \theta + U_r), \\
\epsilon_{\theta \theta} &= \frac{1}{2} \left( \frac{\partial U_{\theta}}{\partial r} + \frac{1}{r} \frac{\partial U_r}{\partial \theta} - \frac{U_{\theta}}{r} \right)
\end{align*}
\]

Dome:

(2.9)

\[
\begin{align*}
\epsilon_{\rho} &= \frac{\partial U_{\rho}}{\partial \rho}, \quad \epsilon_{z} = \frac{\partial U_z}{\partial z}, \quad \epsilon_{\varphi} = \frac{U_{\rho}}{\rho}, \\
\epsilon_{\rho z} &= \frac{1}{2} \left( \frac{\partial U_{\rho}}{\partial z} + \frac{\partial U_z}{\partial \rho} \right)
\end{align*}
\]

Ring:

(2.10)
Here the displacement field components in the dome are denoted by $U_\theta, U_r$ (angular, radial) and those in the ring by $U_\rho, U_z$ (horizontal, vertical). In the problem considered there are no kinematic constraints, but as we are using different coordinate systems for the dome and for the ring, we must impose continuity conditions at the junction. The continuity is imposed by requiring that for $r_0 - d/2 < r < r_0 + d/2$

$$U_\theta(r, \alpha) = U_\rho(\rho_r, z_r) \cos \alpha - U_z(\rho_r, z_r) \sin \alpha,$$

$$U_r(r, \alpha) = U_\rho(\rho_r, z_r) \sin \alpha + U_z(\rho_r, z_r) \cos \alpha.$$  \hspace{1cm} (2.11)

where $\rho_r, z_r$ are given by Eq. (2.8).

The mathematical interpretation of the Girkmann problem as an axisymmetric 2D linear elastic problem is now completed. Following [1], we now set the more specific computational goal as: Find the stress resultants at the junction of the dome and the ring, i.e., find the reactive force and moment line densities acting on the centerline of the junction (a circular line of radius $\rho_0 = r_0 \sin \alpha$). In Fig. 2 below the total reactive force at the junction, as acting on the dome, is expressed in the traditional way (cf. [1]–[3]) as

$$\bar{F} = N \bar{e}_\theta + R \bar{e}_\rho,$$ \hspace{1cm} (2.12)

where $\bar{e}_\theta, \bar{e}_r, \bar{e}_\rho, \bar{e}_z$ are the unit vectors of the spherical and cylindrical coordinate system, respectively. At the junction these are related to each other as

$$\bar{e}_\theta = \cos \alpha \bar{e}_\rho - \sin \alpha \bar{e}_z, \quad \bar{e}_r = \sin \alpha \bar{e}_\rho + \cos \alpha \bar{e}_z.$$ \hspace{1cm} (2.13)

In Eq. (2.12), $N$ is determined by the vertical force balance as [1]

$$N = -\frac{g r_0}{1 + \cos \alpha}.$$ \hspace{1cm} (2.14)

The unknown reactions to be determined are thus the horizontal force $R$ in Eq. (2.12) and the moment $M$, the positive direction of which is taken to be $\bar{e}_\theta \times \bar{e}_r = -\bar{e}_\varphi$ when acting on the dome (see Fig. 2). When the 2D stresses $\sigma_\theta$ and $\tau_r\theta$ are known at the junction, the reactions $R$ and $M$ may be evaluated from

$$Q = -R \sin \alpha = -\frac{1}{r_0} \int_{r_0-d/2}^{r_0+d/2} \tau_r\theta(r, \alpha) r \, dr,$$

$$M = -\frac{1}{r_0} \int_{r_0-d/2}^{r_0+d/2} (r - r_0) \sigma_\theta(r, \alpha) r \, dr.$$ \hspace{1cm} (2.15)

Here $Q$ is the shear stress resultant with positive direction $-\bar{e}_r$ when acting on the dome.
3 The old mathematical model

In the engineering tradition of the modeling of a stiffened shell roof, the dome and the stiffening ring are first disconnected as corresponding to the free–body splitting of Fig. 2. This leads to a mathematical model where the two structural parts are first taken under a separate study with given reactions $R, M$ (unknown) and $N$ (known) acting at the junction. Once the dome and ring problems have been solved independently, the two unknown parameters $R, M$ are determined by enforcing the kinematic continuity of the displacements at the junction.

Below we follow the engineering tradition and focus first on the dome problem and on the ring problem separately. We start from the basic models of these structures, as presented in textbooks, and we then proceed to study possible ways of improving such traditional models without sacrificing their simplicity.

Dome models

The starting point of the traditional engineering dome model, as well as its refinements to be introduced, is the classical shell theory. In the axially symmetric case, and in case of a spherical shell of radius $r = r_0$ at the midsurface, the classical shell theory proceeds from the approximation of the 2D displacement field $(U_\theta, U_r)$ as

$$U_\theta(r, \theta) = u(\theta) + (r - r_0)\psi(\theta), \quad U_r(r, \theta) = w(\theta). \quad (3.1)$$

Here $u, w$ are the tangential and normal displacements at the midsurface and $\psi$ is the so called rotation. Using the kinematic assumptions (3.1), the strain...
expressions (2.9) come out as
\[\begin{align*}
\epsilon_\theta &= \frac{1}{r} [u' + w + (r - r_0)\psi'], \quad \epsilon_r = 0, \\
\epsilon_\varphi &= \frac{1}{r} [u \cot \theta + w + (r - r_0)\psi \cot \theta], \\
\epsilon_{r\theta} &= \frac{1}{2r^2} (r_0 \psi + u' - u).
\end{align*}\] (3.2)

The normal stresses \(\sigma_\theta\) and \(\sigma_\varphi\) may then be written as
\[\begin{align*}
\sigma_\theta(r, \theta) &= \frac{r_0}{r} \left( \frac{1}{d} n_\theta - \frac{12}{d^3} m_\theta \right), \\
\sigma_\varphi(r, \theta) &= \frac{r_0}{r} \left( \frac{1}{d} n_\varphi - \frac{12}{d^3} m_\varphi \right),
\end{align*}\] (3.3)

where
\[\begin{align*}
n_\theta &= \frac{D}{r_0} (u' + w), \\
n_\varphi &= \frac{D}{r_0} (u \cot \theta + w), \\
m_\theta &= -\frac{K}{r_0} \psi', \\
m_\varphi &= -\frac{K}{r_0} \psi \cot \theta, \\
D &= Ed, \\
K &= \frac{Ed^3}{12}.
\end{align*}\] (3.4–3.5)

In Eqs. (3.2), the Kirchhoff–Love constraint \(\epsilon_{r\theta} = 0\) is further imposed, so that the rotation is restricted to satisfy
\[r_0 \psi = u - w'.\] (3.6)

Eqs. (3.4)–(3.5) are the constitutive equations of classical shell theory that relate stress resultants \((n_\theta, n_\varphi)\) and moments \((m_\theta, m_\varphi)\) to displacements. The constitutive equations close the set of primary shell equations, the equilibrium equations for the force and momentum balance at the shell midsurface. For the assumed shell geometry and loading the equilibrium equations are [1]–[3]
\[\begin{align*}
-(n_\theta \sin \theta)' + n_\varphi \cos \theta + q \sin \theta &= gr_0 \sin^2 \theta \\
n_\theta \sin \theta + n_\varphi \sin \theta + (q \sin \theta)' &= -gr_0 \cos \theta \sin \theta \\
-(m_\theta \sin \theta)' + m_\varphi \cos \theta + r_0 q \sin \theta &= 0
\end{align*}\] (3.7–3.9)

Here \(q\) is the shear stress resultant in the direction \(-\vec{e}_r\). — Note that since \(\tau_{r\theta} = E\epsilon_{r\theta} = 0\) by Eqs. (3.2) and (3.6), \(q\) is formally set to zero by the kinematic assumptions. In the equilibrium equations, \(q = 0\) is an approximation possible under favourable edge conditions (see below); more generally a non-zero \(q\) must be allowed when solving the shell equations (3.4)–(3.9).

Eqs. (3.4)–(3.9) may be condensed to a linear system of ODE’s over the interval \(0 < \theta < \alpha\). Given the reactions \(N\) (known) and \(R, M\) (unknown) at the junction, the boundary conditions at \(\theta = \alpha\) are set as
\[n_\theta(\alpha) = N + R \cos \alpha, \quad q(\alpha) = -R \sin \alpha, \quad m_\theta(\alpha) = M.\] (3.10)

At \(\theta = 0\) the solution \((u, w, \psi, n_\theta, n_\varphi, q)\) must be continuous to be physically meaningful. The dome problem is thus formulated as a two-point boundary
value problem for a linear system of ODE’s on the interval \(0 \leq \theta \leq \alpha\). The solution is unique up to a vertical rigid displacement mode, which could be set by an extra condition such as \(w(0) = 0\).

To solve the Girkmann problem, the output needed from the boundary value problem (3.4)–(3.10) consists only of the edge values of the horizontal displacement \(u_\rho = u \cos \theta + w \sin \theta\) and rotation \(\psi\), i.e., of the numbers

\[
\Lambda_d = u_\rho(\alpha), \quad \Psi_d = \psi(\alpha).
\]  

(3.11)

Since by Eqs. (3.4)–(3.6) one has

\[
Eu_\rho = r_0 d^{-1} n_\varphi \sin \theta, \quad E\psi = d^{-1} \left[ (n_\theta - n_\varphi) \cot \theta - n'_\varphi \right],
\]  

(3.12)

it suffices to solve the boundary value problem for \(n_\theta\) and \(n_\varphi\) only. Once the expressions of \(n_\theta(\theta)\) and \(n_\varphi(\theta)\) are known in terms of parameters \(R, M\), the output (3.11) may be evaluated from Eqs. (3.12) in the form

\[
E\Lambda_d = EA_0 + k_{11} R + k_{12} M, \quad E\Psi_d = E\Psi_0 - k_{12} R - k_{22} M,
\]  

(3.13)

where the (inverse) spring coefficients \(k_{ij}\) are positive. The aim is thus to find the numerical values of \(\Lambda_0\), \(\Psi_0\) and \(k_{ij}\) in the output formulae (3.13).

Below we follow the old tradition and split the boundary value problem (3.4)–(3.10) in two subproblems: First set \(R = M = 0\) to find \(\Lambda_0\) and \(\Psi_0\) in Eqs. (3.13) (Subproblem #1), then set the gravity load to zero \((q = 0)\) and treat \(R, M\) as unknown parameters so as to find the spring coefficients \(k_{ij}\) in Eqs. (3.13) (Subproblem #2). Subproblem #1 may be solved approximately by using shell membrane theory where bending and transverse shear stresses are neglected. In Subproblem #2 the edge effect due to bending dominates, so shell bending theory needs to be applied.

**Subproblem #1: Membrane theory.** In the membrane theory one looks for an approximate solution to the shell equations such that the bending and transverse shear stresses vanish, i.e., \(m_\theta = m_\varphi = q = 0\). A particular solution to the equilibrium equations (3.7)–(3.8) when \(q = 0\) is

\[
n_\theta = -\frac{gr_0}{1 + \cos \theta}, \quad n_\varphi = gr_0 \left( \frac{1}{1 + \cos \theta} - \cos \theta \right).
\]  

(3.14)

This solution (together with \(m_\theta = m_\varphi = q = 0\)) satisfies also the edge conditions (3.10) for \(R = M = 0\) and for \(N\) given by Eq. (2.14).

We point out that, thinking of the strength of the dome as a concrete structure, the solution (3.14) is satisfactory in the assumed geometry, as it corresponds to compressive principal stresses: \(n_\theta < 0\) and also \(n_\varphi < 0\) when \(0 \leq \theta \leq \alpha\). — If instead \(\alpha\) were chosen to exceed the critical angle \(\theta_0 = \arccos \frac{1}{\sqrt{5}} = 51.8^\circ\), one would confront an undesired "primadonna" behaviour of a spherical shell: \(n_\varphi > 0\) when \(\theta_0 < \theta \leq \alpha\).
When inserted in Eq. (3.12), the membrane-theory solution (3.14) gives
\[ u_\rho = \frac{gr_0^2}{Ed} \sin \theta \left( \frac{1}{1 + \cos \theta} - \cos \theta \right), \quad (3.15) \]
\[ \psi = -\frac{2gr_0}{Ed} \sin \theta. \tag{3.16} \]
Upon evaluating these at \( \theta = \alpha \) we get the membrane-theory approximations of \( E \Lambda_0 \) and \( E \Psi_0 \) in Eq. (3.13). The displacements \( u \) and \( w \) could also be determined so that the constitutive relations (3.4) and the Kirchhoff–Love constraint (3.6) hold. Instead the membrane-theory solution fails to satisfy the constitutive relations (3.5), since \( m_\theta = m_\varphi = 0 \) but \( \psi \neq 0 \) by Eq. (3.16).

To improve the membrane-theory approximation of shell theory, a natural approach is to consider membrane theory as a first step in an iterative process for solving the boundary value problem (3.4)–(3.10). The process is started by the initial assumption \( \psi = 0 \). Eqs. (3.5) and (3.9) then lead first to the membrane-theory approximation \( m_\theta = m_\varphi = q = 0 \) and finally to the new value of \( \psi \) as given by Eq. (3.16). The latter serves as an initial condition at the next iteration step where a correction \((m_\theta^*, m_\varphi^*, q^*, n_\theta^*, n_\varphi^*, u_\rho^*, \psi^*)\) to be added to the membrane-theory solution is computed. First, Eqs. (3.5) and (3.9) together with Eq. (3.16) give
\[ m_\theta^* = m_\varphi^* = \frac{gd^2}{6} \cos \theta, \quad q^* = \frac{gd^2}{6r_0} \sin \theta. \tag{3.17} \]
Then by solving Eqs. (3.7)–(3.8) for \( g = 0 \) and \( q = q^* \) we find the corrections of \( n_\theta \) and \( n_\varphi \) to be
\[ n_\theta^* = n_\varphi^* = -\frac{gd^2}{6r_0} \cos \theta. \tag{3.18} \]
Upon inserting these in Eq. (3.12) we get finally the corrections of \( u_\rho \) and \( \psi \):
\[ u_\rho^* = -\frac{gd}{6E} \cos \theta \sin \theta, \quad \psi^* = -\frac{gd}{6Er_0} \sin \theta. \tag{3.19} \]
In comparison with Eqs. (3.15)–(3.16) these are of relative order \( d^2/r_0^2 \), thus very small. A more significant correction, however, arises from the edge conditions (3.10). These hold no more for the corrected solution, so we need to superimpose another correction by solving the homogeneous shell equations \((g = 0 \text{ in Eqs. (3.7)–(3.8)})\) with the edge conditions
\[ n_\theta(\alpha) = -n_\theta^*(\alpha), \quad q(\alpha) = -q^*(\alpha), \quad m_\theta(\alpha) = -m_\theta^*(\alpha). \tag{3.20} \]
This problem is similar to Subproblem #2 and hence solvable approximately by using the bending theory below. We refer to as bending-corrected the membrane theory where the edge correction alone is made with \((m_\theta^*, q^*, n_\theta^*)\) given by Eqs. (3.17)–(3.18).
In Table 1 the (manually computed) numerical values of $\Lambda_0$ and $\Psi_0$ in Eq. (3.13) are given as obtained with the usual membrane theory (model M) and bending-corrected membrane theory (model MB) in the Girkmann problem. The digits shown for model MB would not be affected by corrections (3.19), neither by the further corrections obtained by continuing the iteration. Thus the solution according to model MB agrees with the exact shell-theory solution to Subproblem #1 up to the digits shown in the table.

Table 1: Girkmann problem: Displacement and rotation at the dome edge according to shell membrane theory (M) and bending-corrected membrane theory (MB).

<table>
<thead>
<tr>
<th></th>
<th>$\Lambda_0$ [G/cm]</th>
<th>$\Psi_0$ [G/cm$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>$-2331$</td>
<td>$-10.000$</td>
</tr>
<tr>
<td>MB</td>
<td>$-2343$</td>
<td>$-9.538$</td>
</tr>
</tbody>
</table>

Subproblem #2: Bending theory. In the shell bending theory one makes use of the fact that when $g = 0$, the solution of the boundary value problem (3.4)–(3.10) takes the form of a boundary layer (‘edge effect’) that decays fast away from the edge. To find the solution, the classical technique is to first condense Eqs. (3.4)–(3.9) into a single differential equation for $q$. After (quite respectable, see [1, pp. 412–414]) symbolic manipulations this equation comes out in the form

$$D^2 q + \frac{12r_0^2}{2} q = 0, \quad D = \frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \cot^2 \theta. \quad (3.21)$$

From the analysis of [1] one can further extract the formulae for $n_\theta$, $m_\theta$, $u_\rho = u \cos \theta + w \sin \theta$ and $\psi$ once $q$ is given. These are

$$n_\theta = -q \cot \theta, \quad m_\theta = -\frac{d^2}{2r_0^2} (Dq)', \quad (3.22)$$
$$Eu_\rho = -r_0 d^{-1} q' \sin \theta, \quad E\psi = d^{-1} Dq. \quad (3.23)$$

Eq. (3.21) does not admit a classical solution in terms of elementary functions; however, since the solution is fast decaying, the leading term in the differential operator $D$ is dominant. By keeping only this term, i.e., using the approximation

$$D \approx \frac{d^2}{d\theta^2} \quad (3.24)$$

we are lead to the bending theory as presented in the old literature [1]–[3]. Based on this approximation the decaying solutions of Eq. (3.21) take the
form
\[ q = e^{-\kappa \phi} (A \cos \kappa \phi + B \sin \kappa \phi), \quad (3.25) \]
where the variable is \( \phi = \alpha - \theta \), the value of the decay parameter is
\[ \kappa = \sqrt{3} \sqrt{\frac{r_0}{d}} = 26.0, \quad (3.26) \]
and the coefficients \( A, B \) are determined by the edge conditions. The edge effect thus decays exponentially in the angular scale \( \sim \kappa^{-1} \text{rad} = 2.2^\circ \).

Given the characteristic angular scale \( \kappa^{-1} \ll 1 \) of the edge effect, we can use asymptotic analysis to expand the error of approximation (3.24) in terms of powers of \( \kappa^{-1} \). Starting from the expression of \( D \) in Eq. (3.21), straightforward asymptotic analysis shows that the leading error term is of relative order \( O(\kappa^{-1}) \), except in the special case \( \alpha = \pi/2 \), in which case the error is of order \( O(\kappa^{-2}) \). The analysis further indicates that the latter, higher accuracy is achieved independently of \( \alpha \) when the approximation (3.24) is improved to
\[ D \approx \left( \frac{d}{d\theta} + \gamma \right)^2, \quad \gamma = \frac{\cot \alpha}{2}, \quad (3.27) \]
Based on this approximation the decaying solutions of Eq. (3.21) are of the form
\[ q = e^{-(\kappa - \gamma) \phi} (A \cos \kappa \phi + B \sin \kappa \phi), \quad (3.28) \]
Since approximations (3.24) and (3.27) differ only when \( \alpha \neq \pi/2 \), we name the improved bending theory based on Eq. (3.27) as sloping-corrected.

For both the usual and sloping-corrected bending model, the horizontal displacement and rotation at the edge of the dome can be expressed in the form (3.13) with given coefficients \( k_{ij} \) and \( \Lambda_0 = \Psi_0 = 0 \). These expressions follow when Eq. (3.25) or (3.28) is substituted in the formulae (3.22), the edge conditions (3.10) (with \( N = 0 \)) are imposed to determine the coefficients \( A, B \), and finally the horizontal displacement and rotation are evaluated from Eqs. (3.23). In case of the usual bending model, the coefficients \( k_{ij} \) in Eq. (3.13) found in this way are [2, p. 65]
\[ k_{11} = 2r_0 d^{-1} \sin \alpha, \quad k_{12} = 2d^{-1} \kappa^2 \sin \alpha, \quad k_{22} = 4r_0^{-1} d^{-1} \kappa^3. \quad (3.29) \]
For the sloping-corrected bending model we expand \( k_{ij} \) in terms parameter \( \kappa^{-1} \), taking into account corrections of order \( O(\kappa^{-1}) \) only. In this way we find that the value of \( k_{11} \) in Eq. (3.29) remains unchanged (up to a correction of order \( O(\kappa^{-2}) \)) whereas \( k_{12} \) and \( k_{22} \) get both corrected by factor \( 1 + \gamma/\kappa \).

In Table 2 the (manually computed) numerical values of the coefficients in Eq. (3.13) are given as corresponding to the usual and sloping-corrected bending models for the Girkmann problem. For comparison we give also the values corresponding to the exact shell–theory solution of Subproblem #2. This solution was computed numerically using 1D finite elements of high order.
Table 2: Girkmann problem: Displacement and rotation at the dome edge according to the usual shell bending model (B), sloping-corrected bending model (BS) and exact shell theory (BE) for given $R$ (in G/cm) and $M$ (in G).

<table>
<thead>
<tr>
<th></th>
<th>$E\Lambda_d$ [G/cm]</th>
<th>$E\Psi_d$ [G/cm$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$8342R + 144.34M$</td>
<td>$-144.34R - 4.9950M$</td>
</tr>
<tr>
<td>BS</td>
<td>$8342R + 147.65M$</td>
<td>$-147.65R - 5.1097M$</td>
</tr>
<tr>
<td>BE</td>
<td>$8343R + 147.68M$</td>
<td>$-147.68R - 5.1115M$</td>
</tr>
</tbody>
</table>

**Ring models**

In the Girkmann problem the dimensions of the ring cross section are small compared with the radius of the ring, and the ring is symmetrically loaded. Therefore one can apply the classical ring theory, where the displacement field of the ring is allowed just two degrees of freedom. Let $\Omega_r$ be the ring cross section in cylindrical coordinates and let $P_c = (\rho_c, z_c)$ be the center of gravity of $\Omega_r$. We assume that the ring is loaded by a horizontal force $F_c \hat{e}_\rho$ and moment $M_c \hat{e}_\varphi$, both acting at the circular line of radius $\rho_c$ that passes through $P_c$ and evaluated per unit length of that line. In the classical ring theory one assumes that the shape of the ring cross section remains unchanged when the ring is deformed, so that the only degrees of freedom of the displacement field are the horizontal displacement $\Lambda_c$ of $P_c$ and the rotation $\Psi_c$ of the cross section. (The vertical rigid displacement mode can be dropped.) It is then further assumed that the resulting stress state is approximately that of a stretched bar due to $\Lambda_c$ and that of bent beam due to $\Psi_c$. Based on such assumptions, the displacement–load relations of the ring are found to be [1]

$$EA_c \Lambda_c = \rho_c^2 F_c, \quad EI_c \Psi_c = \rho_c^2 M_c,$$

where $A_c$ is the area of $\Omega_r$ and $I_c$ is the inertial moment of $\Omega_r$ with respect to the line $z = z_c$.

In the Girkmann problem the ring is loaded by the forces and moment that act at the junction of the dome and the ring (Fig. 2), by the gravity of the ring and by the assumed uniform pressure distribution at the base of the ring that balances the weight of the structure. When defining the loads $F_c$ ja $M_c$ we have to take into account that the forces and moment at the junction are evaluated per unit length of the centerline of the junction that has radius $\rho_0$. Let the centerline intersect the roof profile at point $P_0 = (\rho_0, z_0) (= \text{midpoint of line } AE \text{ in Fig. 1})$ and let $h_c = z_0 - z_c$. Then by force and momentum
balance, the loads acting on the ring at its centerline should satisfy

\[ \rho_c F_c = \rho_0(-N \cos \alpha - R), \quad \rho_c M_c = \rho_0(-Rh_c + M) + \rho_c M_0, \]  
\[ (3.31) \]
where \( M_0 \) is the moment at the centerline of the ring due to the known external loads, i.e., the moment when \( R = M = 0 \). (The formula for \( M_0 \) will be given in Eq. (3.40) below.) Upon further relating \( \Lambda_c \) and \( \Psi_c \) to the horizontal shift \( \Lambda_r \) of \( P_0 \) and to the rotation \( \Psi_r \) of the intersection line by

\[ \Lambda_r = \Lambda_c + h_c \Psi_c, \quad \Psi_r = \Psi_c, \]  
\[ (3.32) \]
and combining Eqs. (3.30)–(3.32) we come up with the formulae of classical ring theory written in analogy with Eq. (3.13) as

\[ E \Lambda_r = E \Lambda_0 - k_{11} R + k_{12} M, \quad E \Psi_r = E \Psi_0 - k_{12} R + k_{22} M, \]  
\[ (3.33) \]
where now

\[ E \Lambda_0 = -\frac{\rho_0 \rho_c}{A_c} N \cos \alpha + \frac{\rho_c^3 h_c}{I_c} M_0, \quad E \Psi_0 = \frac{\rho_c^2}{I_c} M_0, \]  
\[ (3.34) \]

\[ k_{11} = \rho_0 \rho_c \left( \frac{1}{A_c} + \frac{h_c^2}{I_c} \right), \quad k_{12} = \frac{\rho_0 \rho_c h_c}{I_c}, \quad k_{22} = \frac{\rho_0 \rho_c}{I_c}. \]  
\[ (3.35) \]

Starting from Eqs. (3.33)–(3.35) we can obtain different model variants by assuming various simplifications when evaluating the coefficients. In the simplest of these the geometric details of the junction and ring cross section are ignored, assuming simply that \( \Omega_r \) is a rectangle of width \( a \) and height \( b \) and that the dome (reduced to its midsurface) is connected to the edge of the ring. Typically one further assumes that \( \rho_c = \rho_0 \), taking into account the cross-sectional dimensions of the ring only when evaluating \( A_c \) and \( I_c \). Based on these simplifications the geometric parameters in Eqs. (3.34)–(3.35) are given by

\[ A_c = ab, \quad I_c = \frac{1}{12} ab^3, \quad \rho_c = \rho_0, \quad h_c = \frac{1}{2} b. \]  
\[ (3.36) \]

Below we refer to this model (assumed in [1]) as the basic model.

When improving the basic model, a possible first step is to correct the values of \( \rho_c \) and \( h_c \) while still assuming that \( \Omega_r \) is a rectangle. Then we have

\[ A_c = ab, \quad I_c = \frac{1}{12} ab^3, \quad \rho_c = \rho_0 + \frac{1}{2} a, \quad h_c = \frac{1}{2}(b - d \cos \alpha). \]  
\[ (3.37) \]

Concerning the evaluation of \( M_0 \), we note that \( M_0 \) arises from the known normal force \( N \) at the junction, from the volume gravity force in the ring and from the uniform pressure \( p \) assumed at the base of the ring. The formula for \( M_0 \) is then

\[ \rho_c M_0 = \rho_0(-h_c N \cos \alpha + (\rho_c - \rho_0)N \sin \alpha) \]
\[ + \int_{\Omega_r} (\rho - \rho_c) \rho \, d\rho \, dz - \int_{\Omega_r} (\rho - \rho_c) \rho \, d\rho, \]  
\[ (3.38) \]
where $\rho_1 = \rho_0 - (d/2) \sin \alpha$ is the inner radius of the ring (see Fig. 1), $\rho_2 = \rho_1 + a$ is the outer radius, $f = g/d$ is the gravity force density of the ring (in the unit G/cm$^3$), and pressure $p$ satisfies the equilibrium condition

$$\rho_0 N \sin \alpha - f \int_{\Omega_r} \rho \, d\rho\,dz + p \int_{\rho_1}^{\rho_2} \rho \, d\rho = 0. \quad (3.39)$$

Upon eliminating $p$ from Eqs. (3.38)–(3.39) we obtain

$$M_0 = \frac{\rho_0}{\rho_c} [ -h_c N \cos \alpha + (\overline{p} - \rho_0) N \sin \alpha ], \quad (3.40)$$

where $\overline{p}$ is the radial center of the supporting pressure distribution:

$$\overline{p} = \frac{\int_{\rho_1}^{\rho_2} \rho^2 \, d\rho}{\int_{\rho_1}^{\rho_2} \rho \, d\rho} = \frac{2}{3} \frac{\rho_1^2 + \rho_1 \rho_2 + \rho_2^2}{\rho_1 + \rho_2}. \quad (3.41)$$

Formulae (3.40)–(3.41) also hold for the basic model above when setting $\rho_c = \rho_0$.

Below we refer to as load-corrected the model based on Eqs. (3.33)–(3.35), (3.37) and (3.40)–(3.41). We note that both in this model and in the basic model above the effect of the ring gravity is canceled in Eq. (3.40) (and hence in the whole model), since $\Omega_r$ is assumed to be a rectangle. Thus in the models so far the ring could as well be weightless (as assumed in [1]).

We could still improve our ring model by taking into account the actual pentagonal shape of $\Omega_r$. However, such a model would still rely on the bar and beam analogies where the ring is assumed locally straight when determining its stress state. To avoid such an extra assumption we step off from the classical tradition at this point and take a completely different approach based on the energy principle.

According to the energy principle, the actual 2-dimensional displacement field of the ring, expressed as $\mathbf{U} = (U_\rho, U_z)$, minimizes the energy

$$\mathcal{F}_r(\mathbf{U}) = \frac{E}{2} \int_{\Omega_r} \left( \epsilon_\rho^2 + \epsilon_z^2 + \epsilon_\varphi^2 + 2\epsilon_\rho\epsilon_z \right) \rho \, d\rho\,dz - \mathcal{L}_r(\mathbf{U}), \quad (3.42)$$

where the leading quadratic term is the deformation energy of the ring and the last (linear) term stands for the potential energy due to the external loads. (We have assumed the value $\nu = 0$ for the Poisson ratio and scaled the energy by factor $1/(2\pi)$.) The strain-displacement relations needed in Eq. (3.42) are

$$\epsilon_\rho = \frac{\partial U_\rho}{\partial \rho}, \quad \epsilon_z = \frac{\partial U_z}{\partial z}, \quad \epsilon_\varphi = \frac{U_\rho}{\rho}, \quad \epsilon_{\rho z} = \frac{1}{2} \left( \frac{\partial U_\rho}{\partial z} + \frac{\partial U_z}{\partial \rho} \right). \quad (3.43)$$

According to the underlying kinematic assumption of classical ring theory the displacement field $\mathbf{U}$ is that of a rigid displacement of $\Omega_r$. Then

$$U_\rho(\rho, z) = \Lambda_c + \Psi_c (z - z_c), \quad U_z(\rho, z) = -\Psi_c (\rho - \rho_c), \quad (3.44)$$
where the parameters $\Lambda_c$ and $\Psi_c$ have the same meaning as in Eq. (3.32). Upon making this Ansatz in Eq. (3.42), the strains come out as

$$
\epsilon_\rho = \epsilon_z = \epsilon_{\rho z} = 0, \quad \epsilon_\phi = \frac{\Lambda_c + \Psi_c(z - z_c)}{\rho}
$$

and the load potential as

$$
L_r(U) = -\rho_c F c \Lambda_c + \rho_c (M_c - M_0) \Psi_c - \rho_c N \cos \alpha (z_0 - z_c) \Psi_c + \rho_c N \sin \alpha (\rho_c - \rho_0) \Psi_c
$$

where $F_c$ and $M_c - M_0$ are defined according to Eq. (3.31) with $h_c = z_0 - z_c$ and the last four terms sum up to $\rho_c M_0 \Psi_c$, where $M_0$ stands for the moment acting when $R = M = 0$, as before. When using the energy principle we are obviously freed of the assumptions on stresses of classical ring theory. All we need is to minimize the energy, as given by Eqs. (3.42) and (3.45)–(3.46), with respect to $\Lambda_c$ and $\Psi_c$. Using then the relations (3.32), the final output of the model can again be written in the form of Eq. (3.33). We call this the minimal-energy model of the ring.

In order to evaluate the coefficients in Eq. (3.33) for the minimal-energy model, we note that by the equilibrium condition (3.39), the potential energy expression (3.46) is independent of parameter $\rho_c$. In Eq. (3.44) we can also choose $z_c$ freely. We choose $\rho_c = \rho_0$ and set $z_c$ by the condition

$$
\int_{\Omega_r} \frac{z - z_c}{\rho} \ d\rho d\zeta = 0.
$$

With this choice of $z_c$ we find that the formulae (3.34)–(3.35) remain valid for the minimal-energy model (with $\rho_c = \rho_0$ and $h_c = z_0 - z_c$), provided the parameters $A_c$ and $I_c$ are evaluated as

$$
A_c = \int_{\Omega_r} \frac{\rho_0}{\rho} \ d\rho d\zeta, \quad I_c = \int_{\Omega_r} \frac{\rho_0}{\rho} (z - z_c)^2 \ d\rho d\zeta
$$

and the formula (3.40) for $M_0$ is corrected by an additional term $\Delta M_0$ that arises because of the ring gravity and the deviation of $\Omega_r$ from a rectangle. We may evaluate the integrals in Eqs. (3.47)–(3.48) to sufficient accuracy by using the midpoint rule to evaluate the effect of the small triangular cutoff in $\Omega_r$. Using the midpoint rule also when evaluating the mentioned correction of $M_0$ we get

$$
\Delta M_0 = \frac{1}{2} \frac{\rho_0}{\rho_0} \ f d^2 \sin \alpha \cos \alpha (\rho_0 - \bar{\rho}_0), \quad \bar{\rho}_0 = \rho_0 - (d/6) \sin \alpha.
$$
Table 3: Girkmann problem: Displacement and the rotation of the ring at the midpoint of the junction for given $R$ (in G/cm) and $M$ (in G) according to the basic ring model (R), the load-corrected model (RL) and minimal-energy model (RE).

<table>
<thead>
<tr>
<th></th>
<th>$EA_r$ [G/cm]</th>
<th>$E\Psi_r$ [G/cm$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$14569 - 3000R + 90.00M$</td>
<td>$-24.588 - 90.00R + 3.6000M$</td>
</tr>
<tr>
<td>RL</td>
<td>$13770 - 2657R + 83.36M$</td>
<td>$-75.641 - 83.36R + 3.6720M$</td>
</tr>
<tr>
<td>RE</td>
<td>$13971 - 2683R + 84.18M$</td>
<td>$-67.682 - 84.18R + 3.6964M$</td>
</tr>
</tbody>
</table>

In Table 3 the (manually computed) numerical values of the coefficients in Eq. (3.33) are given for the three ring models in the Girkmann problem. In the table, the relatively large change in the coefficients when passing from model R to model RL is mainly due to parameter $h_c$. We may interpret the change of $h_c$ to arise because in model RL the geometry of the junction is as assumed in Fig. 1, whereas in model R the intersection line is effectively shifted tangentially to the shell by the amount $\delta = (d/2) \cot \alpha = 0.60d$ so that $h_c$ achieves the assumed value $b/2$ (see Fig. 3). — Note that since the dome and the ring are made of the same material, the 'junction' is merely an imaginary line that locates the point where the reactions $R$ and $M$ are to be evaluated.

Figure 3: Change of parameter $h_c$ from model R to model RL interpreted as the shift $A'E' \rightarrow AE$ of the intersection line. $\delta = 0.60d = 3.6 \text{ cm.}$
Solution of the Girkmann problem

Having gone through the traditional engineering models, and some of their possible variations, to find the coefficients in formulae (3.13) and (3.33), we are ready to solve the Girkmann problem. In all model combinations from Tables 1–3 the solution principle is the same: After superimposing the expressions from Tables 1 and 2, the resulting $\Lambda_d$ and $\Psi_d$ should match with those given by the ring model (Table 3) at the same point, i.e., one should have

$$\Lambda_d = \Lambda_r, \quad \Psi_d = \Psi_r.$$  \hfill (3.50)

This is a $2 \times 2$ linear system for the unknowns $R, M$.

In Table 4 we give the solution of the system (3.50) for five of the possible 12 model combinations. For comparison we give also the original solution of Girkmann in the table. In Girkmann’s model it was assumed that $M_0 = 0$ in Eq. (3.34), otherwise the model was the same as our M-B-R model [1]. Concerning our 2D formulation of the Girkmann problem, we could interpret Girkmann’s assumption so that the supporting pressure at the base of the ring is not uniform but radially varying in such a way that $M_0 = 0$. Viewed in this way, Girkmann’s solution and our M-B-R solution are solutions to two different problems and hence not directly comparable. The comparison indicates anyway that the problem is quite sensitive to the assumptions made on the support, so one should be careful with such assumptions before a meaningful comparison of different models for solving the problem can be made.

<table>
<thead>
<tr>
<th>Model</th>
<th>$R$ [G/cm]</th>
<th>$M$ [G]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girkmann [1]</td>
<td>1.598</td>
<td>-11.27</td>
</tr>
<tr>
<td>M–B–R</td>
<td>1.528</td>
<td>-7.964</td>
</tr>
<tr>
<td>M–B–RL</td>
<td>1.480</td>
<td>-2.836</td>
</tr>
<tr>
<td>M–B–RE</td>
<td>1.499</td>
<td>-3.739</td>
</tr>
<tr>
<td>M–BS–RE</td>
<td>1.503</td>
<td>-4.285</td>
</tr>
<tr>
<td>MB–BS–RE</td>
<td>1.504</td>
<td>-4.238</td>
</tr>
<tr>
<td>MB–BE–RE</td>
<td>1.504</td>
<td>-4.241</td>
</tr>
<tr>
<td>2D: hp-FEM</td>
<td>1.503</td>
<td>-4.168</td>
</tr>
</tbody>
</table>

Table 4 indicates that $M$ is the more sensitive error indicator in the output $(R, M)$ of the model. Starting from the basic model M-B-R and focusing on
the percentage change in the value of $M$ at each individual step of improving the model we see that the ring-model corrections $R \rightarrow RL$ (64%) and $RL \rightarrow RE$ (32%) are the most significant ones, next comes the shell bending-model correction $B \rightarrow BS$ (15%). Compared with these the influence of the shell membrane-theory correction $M \rightarrow MB$ is by an order of magnitude smaller (1%). Still smaller (below 0.1% in $M$) is the gap between our model $MB-BS-RE$ and the (computer-based) model $MB-BE-RE$ where exact shell (bending) theory is used for the dome.

It is also of interest to see in more detail, how the different parameters of the ring model influence the value of $M$ when passing from the traditional (though corrected) model $RL$ to the modern minimal-energy model $RE$. Note that in the latter model the curvature of the ring plays a significant role in parameters $A_c$ and $I_c$, unlike in model $RL$. A closer look shows that of the observed change of $M$ from model $RL$ to model $RE$, about 60% is due to the geometric parameters (mainly due to $A_c$ and $I_c$), the remaining 40% being due to the additional ring gravity term (3.49) that contributes to the parameter $M_0$ in model $RE$. Thus the effect of the ring gravity alone is about 10% in the value of $M$ according to our model $MB-BS-RE$.

Table 4 gives finally the answer to our original question concerning the accuracy of the old manual computational model for solving the Girkmann problem. The computer-based 2D reference values here are taken from [5] (Table 4: $p$-FEM, axisymmetric solid). We see that our tuneups of the old model influence the accuracy of the model quite remarkably: For the basic $M-B-R$ model the error with respect to the 2D elastic model is seen to be about 2% in $R$ and 90% in $M$, whereas for our best manual computational model ($MB-BS-RE$) the error is reduced to below 0.1% in $R$ and to below 1.7% in $M$. — We must conclude that once we used the full potential of old manual computational methods, the resulting model turned out to be surprisingly accurate.

4 A posteriori error analysis

In this section we focus on our best manual model $MB-BS-RE$, renamed from this on as the simplified model (S). Denoting by $R_S, M_S$ the reactions at the junction according to this model and by $R, M$ their exact counterparts according to the 2D elastic model, our aim is to bound $|R - R_S|$ and $|M - M_S|$ by methods of mathematical error analysis. More specifically, we carry out an a posteriori error analysis where we need to know only the solution according to the simplified model and the numerical values of $R_S$ and $M_S$. Concerning the 2D solution we need no information beyond the problem formulation of Section 2. In the analysis we make no distinction between the simplified model and the model where exact shell theory is used for the dome. This allows us to refer directly to the known equations of shell theory, which makes
the analysis more straightforward. (In view of Table 4 the simplification is justified; see also Table 2 and the comments preceding Table 1 above.)

The a posteriori error analysis to be carried out relies on the variational formulation of the Girkmann problem as a 2D linear elastic problem. Below we first introduce the notation associated to the variational formulation.

**Girkmann problem in 2D: Variational formulation**

Let $\Omega_d$ stand for the vertical profile of the dome in 2D, and let $\Omega_r$ be the cross section of the ring as before. Let $U = (U_1, U_2)$ be a 2D displacement field that takes the value $U(P) = (U_\theta(r, \theta), U_r(\theta, r))$ when $P = (r, \theta) \in \Omega_d$ and the value $U(P) = (U_\rho(\rho, z), U_z(\rho, z))$ when $P = (\rho, z) \in \Omega_r$. Such a field $U$ is said to be **kinematically admissible**, if the associated strains according to Eqs. (2.9)–(2.10) are square integrable over $\Omega_d$ and $\Omega_r$ and the continuity conditions (2.11) hold at the junction. The **energy space** of such displacement fields is denoted by $U$. A stress field $\sigma$ is said to be kinematically admissible, if there exists $U \in U$ such that $\sigma = E\epsilon(U)$, where the strain-displacement relations are set by Eqs. (2.9)–(2.10).

The 2D solution to the Girkmann problem consists of the displacement field $U \in U$ and the associated stress field $\sigma$. The two fields are related by $\sigma = \sigma(U) = E\epsilon(U)$, so that $\sigma$ is kinematically admissible. Moreover, $\sigma$ is **statically admissible**, i.e., satisfies the equilibrium, interface and static boundary conditions (2.1)–(2.8). By the principle of virtual work, $U$ may be defined alternatively as the kinematically admissible field satisfying

$$A(U, V) = L(V) \quad \forall V \in U,$$

where the **energy product** $A$ is defined as

$$A(U, V) = \int_{\Omega_d} \sigma(U) : \epsilon(V) \, d\Omega_d + \int_{\Omega_r} \sigma(U) : \epsilon(V) \, d\Omega_r$$

$$= A_d(U, V) + A_r(U, V),$$

where further

$$d\Omega_d = r^2 \sin \theta \, dr \, d\theta, \quad d\Omega_r = \rho \, d\rho \, dz,$$

$$\sigma : \epsilon = \begin{cases} 
\sigma_{\theta \theta} + \sigma_{\rho \rho} + \sigma_{\phi \phi} + 2\sigma_{\phi \theta} \epsilon_{\theta} & \text{in } \Omega_d \\
\sigma_{\rho \rho} \epsilon_{\rho} + \sigma_{\phi \phi} \epsilon_{\phi} + 2\sigma_{\rho z} \epsilon_{\phi} & \text{in } \Omega_r
\end{cases}$$

and $L$ is the load functional defined as

$$L(U) = \int_0^\alpha g \left[ \sin \theta U_\theta(r_0, \theta) - \cos \theta U_r(r_0, \theta) \right] r_0^2 \sin \theta \, d\theta$$

$$+ \left[ - \int_{\Omega_r} f U_z \, d\Omega_r + \int_{\rho_1}^{\rho_2} p U_z(\rho, 0) \, d\rho \right] = L_d(U) + L_r(U).$$
In what follows we measure stresses in the $L_2$-norm $\| \cdot \|$ defined by

$$\| \sigma \|^2 = (\sigma, \sigma), \quad (\sigma, \tau) = \int_{\Omega_d} \sigma : \tau \, d\Omega_d + \int_{\Omega_r} \sigma : \tau \, d\Omega_r.$$  \quad (4.6)

**Error functionals**

In the error analysis we need to express the errors $R - R_S$ and $M - M_S$ in terms of functionals involving the unknown 2D stress field $\sigma$, additional stress fields to be constructed, and two auxiliary displacement fields $U^a$ and $U^b$. We begin by constructing the fields $U^a$, $U^b$.

First, $U^a$ and $U^b$ are both chosen to satisfy the kinematic assumptions of the classical shell and ring theories, so that the fields are of the form (3.1) in the dome and of the form (3.44) in the ring. Second, we assume that in the dome $U^a$ and $U^b$ are displacement fields according to the classical shell theory applied to the problem with no gravity load ($g = N = 0$ above) and the (so far unknown) reactions acting at the junction denoted by $R^a, M^a$ and $R^b, M^b$, respectively. Third, we assume that in the ring $U^a$ and $U^b$ are displacement fields according to the minimal-energy ring model applied to the problem where the reactions at the junction are opposite to those acting on the dome and the external gravity load is zero ($N = f = p = 0$).

Finally, we specify $R^a, M^a$ and $R^b, M^b$ by replacing the continuity conditions (2.11) at the junction by specific jump conditions (so that $U^a, U^b$ will not be kinematically admissible). In order to express the jump conditions, let $U_\theta, U_r$ be the components of $U = U^a$ or $U = U^b$ in the spherical coordinate system in the neighbourhood of the junction and denote by $[U_\theta], [U_r]$ the jumps at $\theta = \alpha$ when passing from $\Omega_d$ to $\Omega_r$. Within the assumed kinematic restrictions on $U$, let $\Lambda_d, Z_d, \Psi_d$ be the limit values of the horizontal displacement, vertical displacement and rotation, respectively, at the dome side of the junction and let $\Lambda_r, Z_r, \Psi_r$ be the corresponding values at the ring side. Then the jumps in $U_\theta$ and $U_r$ are expressed as

$$[U_\theta](r) = (\Lambda_r - \Lambda_d) \cos \alpha - (Z_r - Z_d) \sin \alpha + (\Psi_r - \Psi_d)(r - r_0),$$

$$[U_r](r) = (\Lambda_r - \Lambda_d) \sin \alpha + (Z_r - Z_d) \cos \alpha.$$ \quad (4.7)

Given $\Lambda_d > 0$ and $\Psi_b > 0$ we specify the right side of Eqs. (4.7) according to Table 5 below for the two fields $U^a$ and $U^b$.

By the principle of virtual work, applied separately on $\Omega_d$ and $\Omega_r$, the stress field $\sigma$ associated to the 2D solution of the Girkmann problem satisfies

$$(\sigma, \varepsilon(U)) = \mathcal{L}(U) - \mathcal{J}(U), \quad U = U^a \text{ or } U = U^b,$$ \quad (4.8)

where $\mathcal{L}(U)$ is defined by Eq. (4.5) and $\mathcal{J}(U)$ arises from the jumps at the


Table 5: Discontinuities at the junction

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\Lambda_r - \Lambda_d$</th>
<th>$Z_r - Z_d$</th>
<th>$\Psi_r - \Psi_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U^a$</td>
<td>$-\Lambda_a$</td>
<td>$-\Lambda_a \cot \alpha$</td>
<td>0</td>
</tr>
<tr>
<td>$U^b$</td>
<td>0</td>
<td>0</td>
<td>$\Psi_b$</td>
</tr>
</tbody>
</table>

junction:

$$J(U) = \sin \alpha \int_{r_0 - d/2}^{r_0 + d/2} \sigma_\theta(r, \alpha)[U_\theta](r) r dr$$

$$+ \sin \alpha \int_{r_0 - d/2}^{r_0 + d/2} \tau_\theta(r, \alpha)[U_r](r) r dr.$$  \hspace{1cm} (4.9)

Recall that in the dome, the reactions $R$ and $M$ are related to $\sigma$ by Eqs. (2.15). In view Eq. (4.9), Table 5, Eq. (4.7) and Eq. (2.15) we have then

$$J(U^a) = -\Lambda_a \int_{r_0 - d/2}^{r_0 + d/2} \tau_\theta(r, \alpha) r dr = -\rho_0 \Lambda_a R,$$

$$J(U^b) = \Psi_b \sin \alpha \int_{r_0 - d/2}^{r_0 + d/2} (r - r_0) \sigma_\theta(r, \alpha) r dr = -\rho_0 \Psi_b M.$$  \hspace{1cm} (4.10)

Therefore, writing

$$\epsilon^a = \epsilon(U^a), \quad \epsilon^b = \epsilon(U^b)$$  \hspace{1cm} (4.11)

and using Eqs. (4.10) in Eq. (4.8) we get

$$\rho_0 \Lambda_a R = (\sigma, \epsilon^a) - L(U^a), \quad \rho_0 \Psi_b M = (\sigma, \epsilon^b) - L(U^b).$$  \hspace{1cm} (4.12)

In Eqs. (4.12) the unknown field $\sigma$ is both statically and kinematically admissible. However, only the static admissibility is actually required for the identity (4.8), and hence also Eqs. (4.12), to hold. Therefore if $\sigma^s$ is a stress field that is statically admissible only (i.e., satisfies Eqs. (2.1)–(2.8)) and if $R^s$ and $M^s$ are the reactions associated to $\sigma = \sigma^s$ by Eqs. (2.15), then Eqs. (4.12) remain valid when $\sigma, R, M$ are replaced by $\sigma^s, R^s, M^s$. By subtracting Eqs. (4.12) from their mentioned counterparts we then obtain

$$\rho_0 \Lambda_a (R - R^s) = (\sigma - \sigma^s, \epsilon^a),$$

$$\rho_0 \Psi_b (M - M^s) = (\sigma - \sigma^s, \epsilon^b).$$  \hspace{1cm} (4.13)

We introduce still another stress field, a kinematically admissible field $\sigma^k$, and rearrange Eqs. (4.13) as

$$\rho_0 \Lambda_a (R - R_S) = \varepsilon_a + \delta_a + \gamma_a,$$

$$\rho_0 \Psi_b (M - M_S) = \varepsilon_b + \delta_b + \gamma_b,$$  \hspace{1cm} (4.14)
where

\[
\begin{align*}
\varepsilon_a &= \left(\sigma - \frac{1}{2}(\sigma^s + \sigma^k), \varepsilon^a\right), & \varepsilon_b &= \left(\sigma - \frac{1}{2}(\sigma^s + \sigma^k), \varepsilon^b\right) \\
\delta_a &= \frac{1}{2}(\sigma^k - \sigma^s), & \delta_b &= \frac{1}{2}(\sigma^k - \sigma^s, \varepsilon^b) \\
\gamma_a &= \rho_0 \Lambda_a (R^s - R_S), & \gamma_b &= \rho_0 \Psi_b (M^s - M_S)
\end{align*}
\] (4.15)

The error analysis that follows will be based on Eqs. (4.14)–(4.15). In order to bound the terms \(\varepsilon_a\) and \(\varepsilon_b\) that contain the unknown field \(\sigma\) we apply the following famous theorem [12]

**Theorem 4.1** (Hypercircle theorem) For any statically admissible \(\sigma^s\) and kinematically admissible \(\sigma^k\) it holds that

\[
\|\sigma - \frac{1}{2}(\sigma^s + \sigma^k)\| = \frac{1}{2}\|\sigma^s - \sigma^k\|.
\] (4.16)

By applying in Eqs. (4.15) the Cauchy–Schwarz inequality

\[
|\langle \sigma^s, \varepsilon^a \rangle| \leq \|\sigma^s\| \|\varepsilon^a\|
\] (4.17)

together with Eq. (4.16) we obtain the bounds

\[
|\varepsilon_a| + |\delta_a| \leq \|\sigma^s - \sigma^k\|\|\varepsilon^a\|, \quad |\varepsilon_b| + |\delta_b| \leq \|\sigma^s - \sigma^k\|\|\varepsilon^b\|.
\] (4.18)

These are computable bounds in so far as the fields \(\sigma^s\) and \(\sigma^k\) (or at least the difference \(\sigma^s - \sigma^k\)) are known. In what follows we will construct \(\sigma^s\) and \(\sigma^k\) in terms of three stress fields \(\sigma^{ss}, \sigma^{kk}\) and \(\sigma^X\) in such a way that

\[
\sigma^s = \sigma^{ss} + \sigma^X, \quad \sigma^k = \sigma^{kk} + \sigma^X.
\] (4.19)

Here the fields \(\sigma^{ss}\) and \(\sigma^{kk}\) will be constructed explicitly, whereas \(\sigma^X\) remains an unknown field to be defined as the kinematically admissible field such that \(\sigma^{ss} + \sigma^X\) is statically admissible. Thus \(\sigma^X\) is defined as the solution of an auxiliary (2D elastic) problem in the Girkmann geometry with no kinematic constraints and with the load determined by \(\sigma^{ss}\).

The known reactions \(R_S, M_S\) associated to the simplified model will be connected to the definition of \(\sigma^{ss}\) in the dome: Denoting by \(R^{ss}, M^{ss}\) the associated reactions acting on the dome at the junction according to Eqs. (2.15), the field \(\sigma^{ss}\) will be defined so that

\[
R^{ss} = R_S, \quad M^{ss} = M_S.
\] (4.20)

In addition, \(\sigma^{ss}\) will be statically admissible in \(\Omega_d\), so that Eqs. (2.1)–(2.3) hold for \(\sigma = \sigma^{ss}\). Instead, \(\sigma^{ss}\) satisfies neither the interface continuity conditions (2.7) nor the equilibrium conditions (2.4)–(2.6) in the ring, so we need to superimpose the auxiliary field \(\sigma^X\) so as to achieve the static
admissibility of $\sigma^s$. Upon denoting the (unknown) reactions associated to $\sigma^X$ by $R^X, M^X$ (limit values from $\Omega_d$ defined by Eqs. (2.15)), we have

$$R^s - R_S = R^X, \quad M^s - M_S = M^X$$

(4.21)

by Eqs. (4.19) and (4.20). In view Eqs. (4.14)–(4.15), (4.18)–(4.19) and (4.21) we have then the error bounds

$$|R - R_S| \leq (\rho_0\Lambda_a)^{-1}|\sigma^{ss} - \sigma^{kk}||e^a| + |R^X|,$$

$$|M - M_S| \leq (\rho_0\Psi_b)^{-1}|\sigma^{ss} - \sigma^{kk}||e^b| + |M^X|.$$

(4.22)

In what follows our aim is to construct the fields $\sigma^{ss}$ and $\sigma^{kk}$ with the assumed properties in such a way that the computable first terms in estimates (4.22) become as small as possible. For the unknown terms $R^X, M^X$ we have no computable absolute bounds, but we will give order of magnitude estimates indicating that in the final numerical bounds the unknown terms are small, or at least not dominant.

Below we start the construction of $\sigma^{ss}$ and $\sigma^{kk}$ by first defining $\sigma^{ss}$ in $\Omega_d$. This initial step is based on the resultants $(n_\theta, n_\varphi, q)$ and moments $(m_\theta, m_\varphi)$ taken from the known solution of the Girkmann problem according to the simplified model. We extend then $\sigma^{ss}$ to $\Omega_r$ as

$$\sigma^{ss} = \sigma^{akk} \text{ in } \Omega_r,$$

(4.23)

where $\sigma^{akk}$ is the kinematically admissible field satisfying $\sigma^{akk} = E\epsilon(U)$, where $U$ is the known displacement field according to the simplified model, i.e., the field given by Eqs. (3.1) in the dome and by Eqs. (3.44) in the ring, where $u, w, \psi$ and $\Lambda_c, \Psi_c$ are defined according to the simplified model. Finally we define the field $\sigma^{kk}$ in terms of $\sigma^{akk}$ and $\sigma^{ss}$ by setting

$$\sigma^{kk} = \sigma^{akk} + \sigma^{bkk},$$

(4.24)

where $\sigma^{bkk}$ is another kinematically admissible field to be constructed in such a way that $|\sigma^{ss} - \sigma^{kk}| = |\sigma^{ss} - \sigma^{akk} - \sigma^{bkk}|$ is approximately minimized.

**The field $\sigma^{ss}$ in the dome**

Let us combine the equilibrium equations (2.1) and the load conditions (2.2) into the equations

$$-\frac{\partial \sigma_\theta}{\partial \theta} - \frac{1}{r} \frac{\partial (r^2 \tau_{r\theta})}{\partial y} + \sigma_\varphi \cot \theta = gr_0^2 \delta(y) \sin^2 \theta,$$

(4.25)

$$-\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\partial (r \tau_{r\theta})}{\partial y} + \sigma_\theta + \sigma_\varphi = -gr_0^2 \delta(y) \cos \theta \sin \theta,$$

(4.26)
where we have introduced the scaled pseudostress tensor

\[ \underline{\sigma} = r \sin \theta \sigma \]  

(4.27)

and written \( y = r - r_0 \), with \( \delta(y) \) standing for the delta distribution at \( y = 0 \). We assume that the solution of the Girkmann problem according to the simplified model is available, so that the stress resultants \( n_\theta, n_\varphi, q \), moments \( m_\theta, m_\varphi \), displacements \( u, w \) and rotation \( \psi \) are known. In analogy with Eq. (4.27), let us introduce the pseudoresultants

\[ (\overline{m}_\theta, \overline{m}_\varphi, \overline{m}_\theta, \overline{m}_\varphi, \overline{q}) = r_0 \sin \theta \ (n_\theta, n_\varphi, m_\theta, m_\varphi, q) \]  

(4.28)

so as to rewrite the shell equations (3.7)–(3.9) as

\[ \begin{align*}
-\overline{m}_\theta' + \overline{m}_\varphi \cot \theta + \overline{q} & = gr_0^2 \sin^2 \theta \\
\overline{m}_\theta + \overline{m}_\varphi + \overline{q}' & = -gr_0^2 \cos \theta \sin \theta \\
-\overline{m}_\theta' + \overline{m}_\varphi \cot \theta + r_0 \overline{q} & = 0
\end{align*} \]  

(4.29)–(4.31)

Given the pseudoresultants satisfying Eqs. (4.29)–(4.31) we now choose the components \( \overline{\sigma}_\theta^s \) and \( \overline{\sigma}_\varphi^s \) of \( \underline{\sigma}^s \) so as to satisfy the relations (3.3):

\[ \begin{align*}
\overline{\sigma}_\theta^s & = d^{-1} \overline{m}_\theta - 12d^{-3}y \overline{m}_\varphi, \\
\overline{\sigma}_\varphi^s & = d^{-1} \overline{m}_\varphi - 12d^{-3}y \overline{m}_\varphi.
\end{align*} \]  

(4.32)

With this choice the pseudoresultants associated to \( \overline{\sigma}_\theta^s \) and \( \overline{\sigma}_\varphi^s \) agree with the corresponding resultants according to the simplified model, viz.

\[ \int_{-d/2}^{d/2} \overline{\sigma}_\theta^s \ dy = \overline{m}_\theta, \quad -\int_{-d/2}^{d/2} y\overline{\sigma}_\theta^s \ dy = \overline{m}_\theta, \]  

\[ \int_{-d/2}^{d/2} \overline{\sigma}_\varphi^s \ dy = \overline{m}_\varphi, \quad -\int_{-d/2}^{d/2} y\overline{\sigma}_\varphi^s \ dy = \overline{m}_\varphi. \]  

(4.33)

With \( \overline{\sigma}_\theta^s \) and \( \overline{\sigma}_\varphi^s \) given by Eq. (4.32), our aim is now to define \( \overline{\tau}_\theta^s \) and \( \overline{\tau}_\varphi^s \) in \( \Omega_d \) so as to satisfy both the equilibrium Eqs. (4.25)–(4.26) and the boundary conditions \( \overline{\tau}_{\theta\theta}^s = \overline{\tau}_{\varphi\varphi}^s = 0 \) at \( y = \pm d/2 \). To this end, insert expressions (4.32) in Eq. (4.25), solve for \( \tau_{\theta\theta}^s = \tau_{\varphi\varphi}^s \) and use Eqs. (4.29) and (4.31) to obtain

\[ \frac{1}{r} \partial \left( r^2 \overline{\tau}_{\theta\theta}^s \right) \frac{\partial}{\partial y} = -\frac{\partial \overline{\sigma}_\theta^s}{\partial \theta} + \overline{\sigma}_\varphi^s \cot \theta - gr_0^2 \delta(y) \sin^2 \theta \\
= - [d^{-1} + 12r_0d^{-3}y] \overline{\tau}(\theta) + [d^{-1} - \delta(y)] gr_0^2 \sin^2 \theta. \]  

(4.34)

By Eq. (4.34) and by the boundary condition \( \overline{\tau}_{\theta\theta}^s(\theta, -d/2) = 0 \) one must set

\[ \overline{\tau}_{\theta\theta}^s(\theta, y) = r^{-2} A_1(y) \overline{\tau}(\theta) + r^{-2} A_2(y) gr_0^2 \sin^2 \theta, \]  

(4.35)

where \( A_1(y) \) and \( A_2(y) \) are solutions to the initial value problems

\[ \begin{align*}
\{ A_1' & = r \left( d^{-1} + 12r_0d^{-3}y \right) \\
A_1(-d/2) & = 0 \\
\{ A_2' & = r \left( d^{-1} - \delta(y) \right) \\
A_2(-d/2) & = 0
\end{align*} \]  

(4.36)
Eqs. (4.36) imply that $A_1(d/2) = A_2(d/2) = 0$ and
\begin{equation}
\int_{-d/2}^{d/2} r^{-2} A_1(y) \, dy = -1, \quad \int_{-d/2}^{d/2} r^{-2} A_2(y) \, dy = 0,
\end{equation}
hence $\overline{\sigma}_{r\theta}^{ss}(\theta, y)$, as defined by Eqs. (4.35)–(4.36), satisfies $\overline{\sigma}_{r\theta}^{ss}(\theta, \pm d/2) = 0$ and
\begin{equation}
-\int_{-d/2}^{d/2} \overline{\sigma}_{r\theta}^{ss}(\theta, y) \, dy = \overline{\tau}(\theta).
\end{equation}

So far we have found $\overline{\sigma}_\theta = \overline{\sigma}_\theta^{ss}$, $\overline{\sigma}_\varphi = \overline{\sigma}_\varphi^{ss}$ and $\overline{\sigma}_{r\theta} = \overline{\sigma}_{r\theta}^{ss}$ such that Eq. (4.25) holds, together with the boundary conditions $\overline{\tau}_{r\theta} = 0$ at $y = \pm d/2$. To satisfy the remaining conditions of static admissibility, we must solve Eq. (4.26) for $\overline{\sigma}_r = \overline{\sigma}_r^{ss}$. To this end, use Eqs. (4.35) and (4.30) in Eq. (4.26) to obtain
\begin{equation}
\frac{\partial (r \overline{\sigma}_r^{ss})}{\partial y} = -\frac{\partial \overline{\sigma}_\theta^{ss}}{\partial \theta} + \overline{\sigma}_\varphi^{ss} + \overline{\sigma}_{r\varphi}^{ss} + 2gr_0^2 \delta(y) \cos \theta \sin \theta
\end{equation}
\begin{equation}
= [r^{-2} A_1(y) + d^{-1} (\overline{\tau}_\theta + \overline{\tau}_\varphi)(\theta) - 12d^{-3} y (\overline{\tau}_\theta + \overline{\tau}_\varphi)(\theta)
+ [r^{-2} A_1(y) - 2r^{-2} A_2(y) + \delta(y)] gr_0^2 \cos \theta \sin \theta.
\end{equation}

By Eq. (4.39) and by the boundary condition $\overline{\sigma}_r^{ss}(\theta, -d/2) = 0$ one must set
\begin{equation}
\overline{\sigma}_r^{ss}(\theta, y) = r^{-1} B_1(y)(\overline{\tau}_\theta + \overline{\tau}_\varphi)(\theta) + r^{-1} B_2(y)(\overline{\tau}_\theta + \overline{\tau}_\varphi)(\theta)
+ r^{-1} B_3(y) gr_0^2 \cos \theta \sin \theta,
\end{equation}
where $B_1(y)$, $B_2(y)$ and $B_3(y)$ satisfy
\begin{align}
B_1' &= r^{-2} A_1(y) + d^{-1} \\
B_2' &= -12d^{-3} y \\
B_3' &= r^{-2} A_1(y) - 2r^{-2} A_2(y) + \delta(y) \\
B_1(-d/2) &= B_2(-d/2) = B_3(-d/2) = 0
\end{align}

By Eqs. (4.41) and (4.37) one has $B_1(d/2) = B_2(d/2) = B_3(d/2) = 0$, thus $\overline{\sigma}_r^{ss} = 0$ at $y = \pm d/2$. We conclude that when $\sigma = \sigma^{ss}$ is defined by Eqs. (4.27), (4.32), (4.35)–(4.36) and (4.40)–(4.41), all the conditions of static admissibility are fulfilled in $\Omega_4$. Moreover, since the stress resultants of the field $\sigma^{ss}$ agree with those defined according to the simplified model by Eqs. (4.33) and (4.38), there is agreement in particular at $\theta = \alpha$, hence the static edge conditions (4.26) are fulfilled as well.

The field $\sigma^{kk}$

We construct first the field $\sigma^{akk}$ in Eq. (4.24). This is the kinematically admissible field consistent with the simplified model, i.e. $\sigma^{akk} = E\varepsilon(U)$, where
\( \mathbf{U} \) satisfies Eqs. (3.1) and (3.44) in the dome and in the ring, respectively, with \( u, w, \psi \) and \( \Lambda_c, \Psi_c \) defined according to the simplified model. Comparing \( \sigma^s_{\vartheta} \) with \( \sigma^{akk}_{\vartheta} \) and \( \sigma^s_{\varphi} \) with \( \sigma^{akk}_{\varphi} \) in the dome we conclude from Eqs. (4.32), (4.27), (4.28) and (3.2)–(3.5) that

\[
\sigma^s_{\vartheta} = \frac{r_0}{r} \left( d^{-1} n_{\vartheta} - 12 d^{-3} y m_{\vartheta} \right) = \frac{E}{r} \left( u' + w + y \psi' \right) = \sigma^{akk}_{\vartheta} \quad \text{in } \Omega_d \tag{4.42}
\]

and similarly,

\[
\sigma^s_{\varphi} = \frac{r_0}{r} \left( d^{-1} n_{\varphi} - 12 d^{-3} y m_{\varphi} \right) = \frac{E}{r} \left( u \cot \theta + w + y \psi \cot \theta \right) = \sigma^{akk}_{\varphi} \quad \text{in } \Omega_d. \tag{4.43}
\]

From Eqs. (3.2) and (3.6) we also see that

\[
\sigma^{akk}_{\rho} = \tau^{akk}_{r\vartheta} = 0 \quad \text{in } \Omega_d. \tag{4.44}
\]

In \( \Omega_r \) we define \( \sigma^{akk} = E \varepsilon(\mathbf{U}) \) according to Eq. (3.45), so that

\[
\sigma^{akk}_{\rho} = \sigma^{akk}_{\varphi} = \tau^{akk}_{r\varphi} = 0 \quad \text{in } \Omega_r. \tag{4.45}
\]

To finish the construction of \( \sigma^{kk} \), the remaining task is to specify the field \( \sigma^{bkk} \) in Eq. (4.24) so as to approximately minimize \( \| \sigma^s - \sigma^{kk} \| \). The field \( \sigma^{bkk} \) is kinematically admissible, so it is related to a 2D displacement field \( \mathbf{U} = \mathbf{U}^{bkk} \) via the constitutive relations (2.9)–(2.10). In what follows we restrict the field \( \mathbf{U}^{bkk} \) to be constant on \( \Omega_r \), so that

\[
\sigma^{bkk}_{\rho} = \sigma^{bkk}_{\varphi} = \tau^{bkk}_{r\varphi} = 0 \quad \text{in } \Omega_r. \tag{4.46}
\]

In view of Eqs. (4.6), (4.4), (4.23)–(4.24) and (4.42)–(4.45) the expression to be minimized under constraint (4.46) is then

\[
\| \sigma^s - \sigma^{kk} \|^2 = \int_{\Omega_d} \left[ (\sigma^{bkk}_{\vartheta})^2 + (\sigma^s_{\vartheta} - \sigma^{bkk}_{\vartheta})^2 + (\sigma^{bkk}_{\varphi})^2 + 2(\tau^{s}_{r\vartheta} - \tau^{bkk}_{r\vartheta})^2 \right] d\Omega_d
\]

\[
+ \int_{\Omega_r} (\sigma^{bkk}_{\varphi})^2 d\Omega_r, \tag{4.47}
\]

where \( \sigma^{bkk} = E \varepsilon(\mathbf{U}^{bkk}) \) according to Eqs. (2.9)–(2.10).

When constructing \( \mathbf{U}^{bkk} \) in \( \Omega_d \) we drop the superindices, denoting the components of \( \mathbf{U}^{bkk} \) simply by \( U_{\vartheta}(\theta, y) \) and \( U_{r}(\theta, y) \), where \( y = r - r_0 \). We first split \( \tau^{s}_{r\vartheta} \) in two parts as

\[
\tau^{s}_{r\vartheta} = \tau_0 + \tau_1, \tag{4.48}
\]
where $\tau_0$ is constant in $y$ and chosen so that the remainder $\tau_1$ satisfies

$$\int_{-d/2}^{d/2} \left( \frac{d^2}{4} - y^2 \right) \tau_1(\theta, y) \, dy = 0, \quad 0 \leq \theta \leq \alpha. \quad (4.49)$$

We also split the field components $U_\theta$ and $U_r$ to be constructed as

$$U_\theta = U_0 + U_1, \quad U_r = W_0 + W_1, \quad (4.50)$$

where $U_0(\theta)$ and $W_0(\theta)$ are the averages of $U_\theta$ and $U_r$ in $y$, so that the remainders satisfy

$$\int_{-d/2}^{d/2} U_1(\theta, y) \, dy = \int_{-d/2}^{d/2} W_1(\theta, y) \, dy = 0, \quad 0 \leq \theta \leq \alpha. \quad (4.51)$$

Below we further replace $\tau_1$ in Eq. (4.48) and $\sigma_{ss}^{sr}$ by the modified stresses

$$\tilde{\tau}_1(\theta, y) = \tau_1(\theta, y) - e_1^{c_1(\theta - \alpha)} \tau_1(\alpha, y),$$

$$\tilde{\sigma}_{ss}^{sr}(\theta, y) = \sigma_{ss}^{sr}(\theta, y) - e_2^{c_2(\theta - \alpha)} \sigma_{ss}^{sr}(\alpha, y), \quad (4.52)$$

which vanish at $\theta = \alpha$. Here $c_1$ and $c_2$ are so far free positive parameters.

With the above preparations we now set $U_1, W_1$ on $\Omega_d$ and $U_0, W_0$ on the interval $0 < \theta < \alpha$ so as to satisfy

$$E \frac{\partial U_1}{\partial y} = \tilde{\tau}_1, \quad (4.53)$$

$$E \frac{\partial W_1}{\partial y} = \tilde{\sigma}_{ss}^{sr}, \quad (4.54)$$

$$-\frac{E}{2r_0} \frac{dW_0}{d\theta} = \tau_0, \quad W_0(0) = 0 \quad (4.55)$$

$$-\frac{dU_0}{d\theta} = W_0, \quad U_0(0) = 0. \quad (4.56)$$

These equations arise as an attempt to minimize the right side of Eq. (4.47) under the constitutive relations (2.9) for given $\tau_{ss}^{sr}$ and $\sigma_{ss}^{sr}$. First, in view of Eqs. (2.9) and (4.48), we have chosen Eqs. (4.53)–(4.55) in such a way that the terms $\tau_{ss}^{sr} - \tau_{bkk}^{sr}$ and $\sigma_{ss}^{sr} - \sigma_{bkk}^{sr}$ in Eq. (4.47) vanish when $\tilde{\tau}_1 = \tau_1$ and $\tilde{\sigma}_{ss}^{sr} = \sigma_{ss}^{sr}$. By using the modified stresses (4.52) in Eqs. (4.53)–(4.54) we enforce the necessary edge condition that $U_1$ and $W_1$ both vanish at $\theta = \alpha$.

Finally we define $U_0$ according to (4.56) so as to cancel the contribution of $W_0$ to the term $\sigma_{bkk}^{sr}$ in Eq. (4.47).

Eqs. (4.53)–(4.56) together with Eqs. (4.50)–(4.51) define $U_\theta$ and $U_r$ uniquely for given $\tau_0, \tilde{\tau}_1$ and $\tilde{\sigma}_{ss}^{sr}$. Since $\tau_{ss}^{sr}$ vanishes at $\theta = 0$, it follows from Eqs. (4.48)–(4.56) that $U_\theta$ likewise vanishes at $\theta = 0$, as required for
kinematic admissibility. Finally note that if $0 < \theta < \alpha$, Eqs. (4.53), (4.49) and (4.52) imply that

$$
\int_{-d/2}^{d/2} y U_1(\theta, y) dy = \frac{1}{2} \int_{-d/2}^{d/2} \left( \frac{d^2}{4} - y^2 \right) \frac{\partial U_1(\theta, y)}{\partial y} dy = 0.
$$

We want to impose this constraint to minimize the contribution of $U_1$ to the term $\sigma_{y}^{kk}$ in Eq. (4.47), so this explains our splitting (4.48)–(4.49) of $\tau_{s\theta}^{ss}$.

By the construction so far, we have defined the displacement field $U_{bkk} = (U_0 + U_1, W_0 + W_1)$ in $\Omega_d$ by Eqs. (4.48)–(4.56), with $c_1$ and $c_2$ in Eq. (4.52) as free parameters. Since $U_0$ and $W_0$ are constant along the junction line at $\theta = \alpha$ and since $U_1$ and $W_1$ both vanish at the junction, we obtain a continuous, hence admissible field on the entire domain by extending $U_{bkk}$ as a constant field to $\Omega_r$. For the field $U_{bkk}$ so defined we finally set the parameters $c_1$ and $c_2$ in Eqs. (4.52) to values that minimize the right side of Eq. (4.47).

The field $\sigma^X$

Since $\sigma_{p}^{ss} = \sigma_{z}^{ss} = \tau_{pz}^{ss} = 0$ in $\Omega_r$, the field $\sigma^{ss}$ fails to satisfy the conditions (2.4)–(2.8) of static admissibility in the ring and at the junction. We therefore supplement $\sigma^{ss}$ with an auxiliary field $\sigma^X$ such that $\sigma^X = \sigma^{ss} + \sigma^X$ is statically admissible. In addition, $\sigma^X$ is required to be kinematically admissible, so there is a displacement field $U^X \in \mathcal{U}$ such that $\sigma^X = E \varepsilon(U^X)$. Our aim is to approximate the reactions $R^X$ and $M^X$ as defined by

$$
\rho_0 R^X = \int_{r_0-d/2}^{r_0+d/2} \tau_{r\theta}^X(r, \alpha) r dr,
$$

$$
\rho_0 M^X = -\sin \alpha \int_{r_0-d/2}^{r_0+d/2} (r - r_0) \sigma_{\theta}^X(r, \alpha) r dr,
$$

where $\tau_{r\theta}^X, \sigma_{\theta}^X$ are defined at the junction as their limit values from $\Omega_d$, i.e., as the limits when $\theta \to \alpha^-$. The fields $\sigma^X, U^X$ are the solution to an auxiliary linear elastic problem, the variational formulation of which is stated as: Find $U^X \in \mathcal{U}$ such that the associated stress field $\sigma^X = E \varepsilon(U^X)$ satisfies

$$
\int_{\Omega_d} (\sigma^{ss} + \sigma^X) : \varepsilon(V) d\Omega_d + \int_{\Omega_r} (\sigma^{ss} + \sigma^X) : \varepsilon(V) d\Omega_r = \mathcal{L}_d(V) + \mathcal{L}_r(V), \quad V \in \mathcal{U},
$$

where $\mathcal{L}_d$ and $\mathcal{L}_r$ are the external load potentials of the original Girkmann problem in the dome and in the ring, respectively. When approximately solving problem (4.59) we follow the same procedure as when solving the
original problem. First we split the problem in two parts by reformulating it as: Find \( \mathbf{U}^X \in \mathcal{U} \) and \( \mathbf{\sigma}^X = E \mathbf{\varepsilon}(\mathbf{U}^X) \) such that

\[
\int_{\Omega_d} (\mathbf{\sigma}^{ss} + \mathbf{\sigma}^X) : \mathbf{\varepsilon}(\mathbf{V}) \, d\Omega = \mathcal{L}_d(\mathbf{V}) + \mathcal{L}_{d1}(\mathbf{V}), \quad \mathbf{V} \in \mathcal{U}_d, \tag{4.60}
\]

\[
\int_{\Omega_t} (\mathbf{\sigma}^{ss} + \mathbf{\sigma}^X) : \mathbf{\varepsilon}(\mathbf{V}) \, d\Omega = \mathcal{L}_t(\mathbf{V}) - \mathcal{L}_{dt}(\mathbf{V}), \quad \mathbf{V} \in \mathcal{U}_t, \tag{4.61}
\]

where \( \mathcal{U}_d \) and \( \mathcal{U}_t \) stand for the restrictions of the energy space \( \mathcal{U} \) on \( \Omega_d \) and \( \Omega_t \), respectively, and \( \mathcal{L}_{d1} \) is the load potential due to the surface tractions acting at the junction of the dome and the ring. For \( \mathbf{V} = (u, w) \) expressed in spherical coordinates one has

\[
\mathcal{L}_{d1}(\mathbf{V}) = \sin \alpha \int_{r_0-d/2}^{r_0+d/2} \left[ \mathbf{\sigma}^{ss}_\theta(r,\alpha) + \mathbf{\sigma}^X_\theta(r,\alpha) \right] u(r,\alpha) \, r \, dr \\
+ \sin \alpha \int_{r_0-d/2}^{r_0+d/2} \left[ \mathbf{\tau}^{ss}_r(r,\alpha) + \mathbf{\tau}^X_r(r,\alpha) \right] w(r,\alpha) \, r \, dr, \tag{4.62}
\]

where the stresses are again defined as their limit values from \( \Omega_d \). In Eqs. (4.60)–(4.62) we have taken into account that, due to the static admissibility of \( \mathbf{\sigma} = \mathbf{\sigma}^{ss} + \mathbf{\sigma}^X \), the continuity conditions (2.7) hold for this field.

Still following the solution procedure of the original Girkmann problem, we break temporarily the continuity of the displacement field \( \mathbf{U}^X \) at the junction by introducing the splitting

\[
\mathbf{\sigma}^X = \mathbf{\sigma}^X_0 + \mathbf{\sigma}^X_1, \quad \mathbf{U}^X = \mathbf{U}^X_0 + \mathbf{U}^X_1, \tag{4.63}
\]

where \( (\mathbf{\sigma}^X_0, \mathbf{U}^X_0) \) is determined first as the solution to problems (4.60) and (4.61) when \( \mathbf{\sigma}^X_0 = \mathbf{\tau}^X_0 = 0 \) in Eq. (4.62). Then \( (\mathbf{\sigma}^X_1, \mathbf{U}^X_1) \) is found by setting \( \mathcal{L}_d(\mathbf{V}) = \mathcal{L}_t(\mathbf{V}) = 0 \) and \( \mathbf{\sigma}^{ss} = 0 \) in Eqs. (4.60)–(4.62) and defining the unknown stress distributions \( \sigma^X_\theta(r,\alpha) \) and \( \tau^X_r(r,\alpha) \) in Eq. (4.62) in such a way that the discontinuity of \( \mathbf{U}^X_1 \) at the junction cancels that of \( \mathbf{U}^X_0 \) in Eq. (4.63).

We note that when \( \sigma^X_\theta = \tau^X_\theta = 0 \) in Eq. (4.62), problem (4.60) has the simple solution \( \mathbf{U}^X = \mathbf{\sigma}^X = 0 \) in \( \Omega_d \), since \( \mathbf{\sigma}^{ss} \) is statically admissible in the dome. Hence \( \mathbf{\sigma}^X_0 = 0 \) in \( \Omega_d \) and thus we may set \( \mathbf{\sigma}^X = \mathbf{\sigma}^X_1 \) in the formulae (4.58) for \( R^X \) and \( M^X \). From this starting point we approximate \( R^X \) and \( M^X \) in two steps as follows.

The first step is to find an approximation \( \mathbf{U}^X_0 = (u^X_0, u^X_0) \) of the field \( \mathbf{U}^X_0 = (u^X_0, u^X_0) \) in the ring. With such an approximation of the displacement field available, we can approximate the average horizontal shift and average
rotation at the junction due to the field \( U_0^X \) as (recall formulae (2.8))

\[
\Lambda_0^* = \frac{1}{d} \int_{r_0-d/2}^{r_0+d/2} u^*_\rho(\rho_r, z_r) \, dr, \\
\Psi_0^* = \frac{12}{d^3} \int_{r_0-d/2}^{r_0+d/2} (\cos \alpha u^*_\rho - \sin \alpha u^*_z)(\rho_r, z_r)(r - r_0) \, dr.
\] (4.64)

The second step in our approximation is to relax the full continuity requirement of \( U^X \) in Eq. (4.63) and enforce instead the continuity of the average horizontal shift and rotation only. Assuming the approximate values as given by Eq. (4.64) on the ring side, we can then approximate \((\sigma_1^X, u_1^X)\) by using the simplified model. In this way we find the desired approximation \((R^*, M^*)\) of \((R^X, M^X)\) simply by solving the linear system

\[
\Lambda_r - \Lambda_d = -\Lambda_0^*, \quad \Psi_r - \Psi_d = -\Psi_0^*,
\] (4.65)

where the left sides are related to \( R = R^* \) and \( M = M^* \) according to Eqs. (3.13) and (3.33) (with \( \Lambda_0 = \Psi_0 = 0 \) and \( k_{ij} \) given by Tables 2 [BS] and 3 [RE]).

The remaining task is thus to construct the approximation \( U_0^X = (u^*_\rho, u^*_z) \) of the field \( U_0^X \in \mathcal{U}_0 \) that is needed in Eq. (4.64). According to the definition of \( U_0^X \) as stated, \( \sigma_0^X = E\epsilon(U_0^X) \) satisfies

\[
\int_{\Omega_t} (\sigma^{ss} + \sigma_0^X) : \epsilon(V) \, d\Omega_t = \mathcal{L}_r(V) - \mathcal{L}_{dr}^0(V), \quad V \in \mathcal{U}_0,
\] (4.66)

where \( \mathcal{L}_{dr}^0 \) denotes the load potential of Eq. (4.62) when \( \sigma_0^X(r, \alpha) \) and \( \tau_0^X(r, \alpha) \) are set to zero. Recall that the field \( \sigma^{ss} \) was defined in the ring as being kinematically consistent with the solution of the Girkmann problem according to the simplified model. Therefore \( \sigma^{ss} \) satisfies

\[
\int_{\Omega_t} \sigma^{ss} : \epsilon(V) \, d\Omega_t = \mathcal{L}_r(V) - \rho_0(N \cos \alpha + R_S)\Lambda(V) + \rho_0 M_S \Psi(V), \quad V \in \mathcal{U}_0,
\] (4.67)

where \( \mathcal{U}_0 \) is the two-dimensional space of rigid displacements that replaces \( \mathcal{U}_t \) in the ring model RE, and \( \Lambda(V), \Psi(V) \) are the horizontal displacement and rotation at the midpoint of the junction line as corresponding to the field \( V \).

Now let us compare Eqs. (4.66) and (4.67) when choosing \( V \in \mathcal{U}_0 \) also in Eq. (4.66). When expressing \( V = (u_\rho, u_z) \in \mathcal{U}_0 \) in the rotated basis \( \{\vec{e}_\theta, \vec{e}_r\} \) at \( \theta = \alpha \) we get

\[
\mathcal{L}_{dr}^0(V) = \sin \alpha \int_{r_0-d/2}^{r_0+d/2} \left[ \sigma^{ss}_\theta(r, \alpha)u(r) + \tau^{ss}_\theta(r, \alpha)w(r) \right] \, r \, dr,
\] (4.68)

36
where

\[ u(r) = \Lambda(V) \cos \alpha + \Psi(V)(r - r_0), \quad w(r) = \Lambda(V) \sin \alpha. \quad (4.69) \]

By the construction of the field \( \sigma^{ss} \) in the dome we have (recall Eqs. (4.27), (4.28), (4.33), (4.38) and (3.10))

\[
\int_{r_0-d/2}^{r_0+d/2} \sigma_{\theta}^{ss}(r, \alpha) r dr = r_0 n_\theta(\alpha) = r_0 (N + R_S \cos \alpha),
\]

\[
\int_{r_0-d/2}^{r_0+d/2} \tau_{r\theta}^{ss}(r, \alpha) r dr = -r_0 q(\alpha) = r_0 R_S \sin \alpha,
\]

\[
\int_{r_0-d/2}^{r_0+d/2} (r - r_0) \sigma_{\theta}^{ss}(r, \alpha) r dr = -r_0 m_\theta(\alpha) = -r_0 M_S,
\]

so by Eqs. (4.68)–(4.70)

\[
\mathcal{L}_{tr}^0(V) = \rho_0(N \cos \alpha + R_S)\Lambda(V) - \rho_0 M_S \Psi(V), \quad V \in \mathcal{U}_0. \quad (4.71)
\]

From Eqs. (4.66), (4.67) and (4.71) we conclude then that

\[
\int_{\Omega_t} \sigma^X_0 : \epsilon(V) d\Omega_t = 0, \quad V \in \mathcal{U}_0. \quad (4.72)
\]

Since all strains except \( \epsilon_\phi(V) \) vanish for \( V \in \mathcal{U}_0 \), Eq. (4.72) is equivalent to the weighted orthogonality condition

\[
\int_{\Omega_t} \rho^{-1} u^X_\rho \epsilon_\rho d\rho dz = 0, \quad (\epsilon_\mu, \epsilon_z) \in \mathcal{U}_0. \quad (4.73)
\]

In the approximation of \( \mathcal{U}_0^X \) that follows we preserve constraint (4.73). The approximation is based on three simplifications as follows.

First we postulate that the displacements along the junction line are mainly caused by the load functional \( \mathcal{L}_{tr}^0(V) \) in Eq. (4.66), i.e., by the surface tractions that act at the junction. Thus we drop the external load functional \( \mathcal{L}_t(V) \) when approximating \( \mathcal{U}_0^X \) at the junction.

The second, major simplification is to replace \( \Omega_t \) by a cone (sector), the end of which is at point \( O \) where \( \rho = \rho_0 - (d/2) \sin \alpha \) and \( z = b \), and secondly, to replace the surface tractions at the junction by idealized point forces \( F \hat{e}_\rho \) and \( -G \hat{e}_z \) and moment \( M \hat{e}_\phi \) acting at \( O \) (see Fig. 4 below). The amplitudes here are chosen so as to match with the resultants at the junction according to the simplified model, so we set \( F = -N \cos \alpha - R_S, \) \( G = -N \sin \alpha \) and \( M = M_S \). Free boundary conditions are assumed along the boundary lines of the cone, consistently with the original problem formulation.

After the simplifications so far we are looking for the displacement field such that the associated stress field according to Eqs. (2.10) satisfies the free
boundary conditions, the required three resultant conditions at \( O \), and the homogeneous equilibrium equations (Eqs. (2.4) with \( f = 0 \)) inside the cone. Our final simplification is to set \( \sigma_\phi = \epsilon_\phi = 0 \) and \( \rho = \text{const.} \) in Eqs. (2.4) and (2.10), i.e., we replace these equations by the corresponding plane elasticity equations.

After all the simplifications we are facing a plane elastic problem, the equilibrium solution of which is known from classical theory of elasticity. In polar coordinates \((\varrho, \phi)\) where \( \varrho \) is the distance from \( O \) and \( \phi \) is measured clockwise from the direction of \( e_\rho \) (see Fig. 4), this solution is [11, pp. 97–113]

\[
\begin{align*}
E u^*_\rho &= -\frac{4F}{\pi} \left( \log \varrho + \frac{1}{2} \sin^2 \phi \right) + \frac{2G}{\pi} \left( \phi - \frac{\pi}{2} + \sin \phi \cos \phi \right) \\
&\quad + \frac{2M}{\varrho} (3 \cos^3 \phi - 2 \cos \phi) + A - B \varrho \sin \phi, \\
E u^*_z &= \frac{2F}{\pi} (\phi - \sin \phi \cos \phi) + \frac{4G}{\pi} \left( \log \varrho + \frac{1}{2} \cos^2 \phi \right) \\
&\quad + \frac{2M}{\varrho} (3 \sin^3 \phi - 2 \sin \phi) - B \varrho \cos \phi,
\end{align*}
\]

where the terms with coefficients \( A, B \) represent undetermined rigid body modes (the vertical mode has been removed). Our approximation process is now finished by first eliminating the two rigid body modes by enforcing constraint (4.73), then evaluating the integrals in Eq. (4.64) numerically, and finally solving the system (4.65) for \((R^*, M^*)\).
Numerical a posteriori bounds

We are now ready to convert estimates (4.22) into numerical bounds. For $R_S, M_S$ we assume the values $R_S = 1.504 \text{ G/cm}$, $M_S = -4.238 \text{ G}$ as given by the simplified model. Since the value of $E$, neither the values of $\Lambda_a$ and $\Psi_b$ in Table 5 (as far as non-zero) will not matter, we set $E = 1 \text{ G/cm}^2$, $\Lambda_a = 1 \text{ cm}$ and $\Psi_b = 1$ below. We drop the units in the numerical calculation, assuming Girkmann’s units throughout.

To evaluate the known leading terms in estimates (4.22) we recall that here $|\sigma_{ss} - \sigma_{kk}|$ is given by Eq. (4.47), where $\tau_{ss}^a$ and $\sigma_{ss}^b$ are further defined in $\Omega_d$ by Eqs. (4.35)–(4.36), (4.40)–(4.41) and (4.27)–(4.28) and $\sigma_{bkk} = E\epsilon(U)$ is a kinematically admissible field where $U = (U_\theta, U_r)$ is defined in $\Omega_d$ by Eqs. (4.48)–(4.56) and extended continuously to a constant field in $\Omega_r$. From Eqs. (4.35)–(4.36) we can derive the approximation

$$
\tau_{ss}^a(y, \theta) \approx \frac{6}{d^3} \left(y^2 - \frac{d^2}{4}\right) q(\theta) + g \left(\frac{y}{d} - \frac{\text{sgn}(y)}{2}\right) \sin \theta,
$$

which is sufficient for our purposes. For the shear stress resultant $q$ in Eq. (4.75) we may assume (again to sufficient accuracy) the expression given by the basic engineering bending theory. In the variable $\phi = \alpha - \theta$ the expression is [1, pp. 418–419]

$$
q(\phi) = e^{-\kappa \phi} \left[(-R_S \sin \alpha) \cos \kappa \phi + \left(\frac{2\kappa M_S}{r_0} + R_S \sin \alpha\right) \sin \kappa \phi\right],
$$

where $\kappa = 26.0$ in our case.

When defining the displacement field $U_{bkk} = (U_\theta, U_r)$ by Eqs. (4.48)–(4.56) we assume the approximation (4.75) and interpret the initial conditions in Eqs. (4.55)–(4.56) as conditions set at $\phi = \infty$. Then we end up defining $U_1$ by

$$
U_1(\phi, y) = \frac{12}{d^3} \left(\frac{y^3}{3} - \frac{d^2 y}{20}\right) \left[q(\phi) - q(0)e^{-c_1 r_0 \phi}\right]
+ g \left(\frac{y^2}{2d} - \frac{|y|}{2} + \frac{d}{12}\right) \left[\sin(\alpha - \phi) - \sin \alpha e^{-c_1 r_0 \phi}\right]
$$

and $W_0$ and $U_0$ so as to satisfy

$$
W_0'(\phi) = \frac{12 r_0}{5d} q(\phi), \quad W_0(\infty) = 0,
$$

$$
U_0'(\phi) = W_0(\phi), \quad U_0(\infty) = 0.
$$

When numerically evaluating the right side of Eq. (4.47) it turns out that the second terms in Eqs. (4.75) and (4.77) do not contribute significantly to the
numerical value, neither the terms involving $\sigma_{ss}$ or $W_1$, so we can drop these terms, and we can also set $d\Omega_d = r_0^2 \sin \alpha \, d\phi \, dy$ in the numerical evaluation. With these simplifications we obtain

$$W_0(\phi) = e^{-\kappa \phi}(1.695 \cos \kappa \phi - 36.464 \sin \kappa \phi),$$
$$U_0(\phi) = e^{-\kappa \phi}(0.6045 \cos \kappa \phi + 0.6698 \sin \kappa \phi),$$

(4.80)

and we find that the minimum of the right side of Eq. (4.47) is obtained when choosing $c_1 = 9.17$ in Eq. (4.52). At the minimum the five terms of the right side of Eq. (4.47) are evaluated as

$$|\sigma_{ss} - \sigma_{kk}|^2 = 35.4 + 0.0 + 19.5 + 36.7 + 4.7 = 96.4.$$  

(4.81)

Here the dominant first and fourth terms both arise from the local 'hot spot' near the junction where the very fast–decaying exponential terms of $\tau^1$ and $U_1$ in Eqs. (4.52) and (4.77) are active. For comparison, let us note that by Eqs. (4.42)–(4.45), (4.23) and (4.75)–(4.76)

$$|\sigma_{ss} - \sigma_{kk}^{akk}| \approx \int_{\Omega} (\tau_{rs})^2 d\Omega_d = 12600.$$  

(4.82)

Thus defining $\sigma_{kk}$ by Eq. (4.24) instead of choosing $\sigma_{kk} = \sigma_{kk}^{akk}$ reduces $|\sigma_{ss} - \sigma_{kk}|$ by an order of magnitude.

In order to find the numerical values of $|\epsilon^a|$ and $|\epsilon^b|$ in Eq. (4.22) we express the displacements and rotations at the junction, as corresponding to the fields $U^a$ and $U^b$, in terms of the unknown reactions $R_a, M_a$ and $R_b, M_b$ and use the simplified model to determine the reactions from the given jump conditions at the junction. In view of Table 5 we find the reactions by solving the linear systems

$$R_a, M_a : \quad \Lambda_r - \Lambda_d = \Lambda_a, \quad \Psi_r - \Psi_d = 0$$
$$R_b, M_b : \quad \Lambda_r - \Lambda_d = 0, \quad \Psi_r - \Psi_d = \Psi_b.$$  

(4.83)

The energy principle then states that

$$E|\epsilon^a|^2 = \rho_0 |R_a \Lambda_a|, \quad E|\epsilon^b|^2 = \rho_0 |M_b \Psi_b|.$$  

(4.84)

The numerical values found in this way are

$$|\epsilon^a| = 0.377, \quad |\epsilon^b| = 13.3.$$  

(4.85)

Using now Eqs. (4.81) and (4.85) in (4.22) we get the error bounds

$$|R - R_S| \leq 0.0025 \, \text{G/cm} + |R^X|,$$
$$|M - M_S| \leq 0.087 \, \text{G} + |M^X|.$$  

(4.86)
Finally to estimate $|R^X|$ and $|M^X|$ we follow the approximation process as described above. Upon inserting the numerical values of $F, G, M$ in Eq. (4.74), enforcing the constraint (4.73), and evaluating the integrals (4.64) numerically over the junction line $AE$, we get

$$\Lambda_0^* = +49.90, \quad \Psi_0^* = -0.4164. \quad (4.87)$$

Upon solving the system (4.65) we then obtain

$$R^* = +0.00444, \quad M^* = +0.0153. \quad (4.88)$$

We cannot guarantee the accuracy of these values as the approximations of $R^X, M^X$, but we may postulate that at least the orders of magnitudes are correct. We predict then that

$$|R^X| \sim 10^{-3} \text{ G/cm}, \quad |M^X| \sim 10^{-2} \text{ G}. \quad (4.89)$$

Combining estimates (4.86) and (4.89) our error analysis thus predicts that

$$|R - R_S| \sim 10^{-3}..10^{-2} \text{ G/cm},$$

$$|M - M_S| \sim 10^{-2}..10^{-1} \text{ G}. \quad (4.90)$$

This agrees reasonably well with the observed errors in Table 4.

5 Conclusions and historical remarks

We have approached the Girkmann problem, as originally introduced in [1], in two quite different ways. We first carried out a historical expedition to study in detail the classical simplified mathematical model for solving the problem. A comparison to recent finite element solutions based on the 2D linear elastic formulation of the problem indicates that the classical model, as found in textbooks, is rather inaccurate when evaluating the reactions, especially the moment, acting at the junction of the dome and the ring. This observation gave us the initial impulse to the search of possible improvements of the classical model. We wanted to obey here the historical limitations, so we did all computations manually and limited the search to relatively simple modifications of the classical model, knowing the 2D reference solution as our target. For the dome model we found simple improvements that practically close the gap between classical shell theory and the simplified membrane and bending models used as part of the classical model. In the ring model the straightforward energy method was found to be the most efficient alternative of the classical ring theory as presented in textbooks. In the end our improvements of the classical model turned out to be very significant, reducing the error with respect to the 2D reference solution by almost two orders of magnitude.
From the historical point of view it appears quite surprising that our relatively simple corrections of the traditional shell membrane and bending models are not found (or at least seem hard to find) in the old literature. Perhaps the most likely explanation is that the basic models as presented were considered sufficiently accurate in actual applications where other error sources, such as errors arising from the assumed material behavior in the underlying linear elastic model, most likely dominate. Of course, the motivation to improve the basic models was missing also because reference solutions based on more complex models were not available in the pre-computer era.

Another source of surprise is that the rather natural minimal-energy ring model that we presented was neither found in the old literature. Here a possible explanation is that, although the energy principle and variational methods have a long history in the linear theory of elasticity, the connection of this approach to numerical approximations was apparently not understood before the era of computers. For example, the early paper by Courant [14], often (disputably) cited as to pioneer the finite element method, discusses both traditional variational methods and finite difference methods but leaves the connection between the two approaches obscure.

As the second approach to the Girkmann problem we performed an a posteriori error analysis of our improved classical model. Our aim here was to both certify and explain the observed accuracy of the model, which indeed exceeded our expectations. The key tool of our error analysis was the well known Hypercircle theorem of the linear theory of elasticity. Originally the theorem is due to Prager and Synge [15]. To be able to use the theorem we needed to construct two stress fields, one statically admissible and one kinematically admissible for the problem according to its 2D linear elastic formulation. The starting point in both constructions was the known solution of the problem according to the simplified model. To obtain sharp error bounds, the idea was to bring the two fields as close to each other as possible. The main obstacle in our constructions was the fact that the two fields, when desired to be very close to each other, necessarily contained an unknown part (denoted by \( \sigma^X \) above), which caused our a posteriori error bounds to contain unknown terms as well. These unknown terms we were only able to bound approximately.

The guidelines of the a posteriori error analysis of classical shell theory that we have followed were presented originally by Koiter [16] as an extension of related earlier work on thin plates and membranes by Morgenstern and Szabó [17]. As usual, our construction of the statically admissible stress field in the dome is based on the static equilibrium, interface and boundary conditions of the 2D formulation of the problem and on the equilibrium equations of classical shell theory that the solution according to our simplified model was assumed to satisfy. Only the technical details in this construction, as associated to the assumed specific surface load, appear new. Instead, our
construction of the kinematically admissible stress field (the field $\sigma^{kk}$ above) is non-standard. Here we first attempted the standard construction of [16], but the resulting field (denoted by $\sigma^{akk}$ above) turned out to lead to quite pessimistic bounds predicting errors of an order of magnitude larger than those observed. Therefore we ended up in improving the construction by adding a non-standard correction term that we denoted by $\sigma^{bkk}$. This finally closed the gap between our error bounds and the actual errors observed. — We point out that adding the field $\sigma^{bkk}$ may be viewed as postprocessing of the field $\sigma^{akk}$ given by classical shell theory. That such an approach is possible seems to indicate that classical shell theory (at least in case of a spherical shell) is to some extent "superconvergent", i.e., more accurate than the underlying kinematic assumptions of the theory would directly imply.

Our a posteriori error analysis thus confirms that the observed accuracy of our simplified model has a mathematical reasoning behind. The analysis also shows that the error of the simplified model has two main sources in comparison with the 2D linear elastic model. The first error source is the shell (bending) theory used in the simplified model to approximate the bending edge effect of the shell, and the second main error source arises because the genuinely two-dimensional deformations of the structure (mainly in the vicinity of the junction) cannot be captured by the simplified model. The first of the mentioned two error terms we were able to bound numerically, the second only approximately. Under the slight uncertainty involved in our approximation we came to the conclusion that the two main error components both match in order of magnitude with the error actually observed.

Our approximation process for the field $\sigma^X$ seems also to explain, why the unknown reactions associated to this field are much smaller than straightforward energy estimates would predict. — We note that, by an argument of dimension analysis, the order of magnitude of $|\sigma^X|^2$ is expected to be $|\sigma^X|^2 \sim \rho_0 N^2 \sim 10^6$ in the assumed Girkmann units. Energy estimates involving $|\sigma^X|$ would then predict values of $R^X$ and $M^X$ that are orders of magnitude larger than those given by our approximation.

Our error analysis finally gives an answer to the most obvious question: How can a simple model be so accurate despite ignoring the actual complex behavior of the 2D stress field in the problem? We note that in our error analysis all the genuinely two-dimensional features of the stress field, such as the corner singularities at the two re-entrant corners of the roof profile (points $A$ and $E$ in Fig. 1), are captured by the unknown field component $\sigma^X$. In the simplified model $\sigma^X$ is effectively set to zero, but as our analysis indicates, this causes only an error that is of the same order of magnitude as the error actually observed, i.e., about 1% in $M$ and 0.1% in $R$. This obviously answers the final question, so we are at the end of our story.
References