

# ICES REPORT 10-34

---

August 2010

## A multipoint flux mixed finite element method on distorted quadrilaterals and hexahedra

by

Mary F. Wheeler, Guangri Xue, and Ivan Yotov



**The Institute for Computational Engineering and Sciences**  
The University of Texas at Austin  
Austin, Texas 78712

*Reference: Mary F. Wheeler, Guangri Xue, and Ivan Yotov, "A multipoint flux mixed finite element method on distorted quadrilaterals and hexahedra", ICES REPORT 10-34, The Institute for Computational Engineering and Sciences, The University of Texas at Austin, August 2010.*

# A multipoint flux mixed finite element method on distorted quadrilaterals and hexahedra

Mary Wheeler <sup>\*</sup>      Guangri Xue <sup>†</sup>      Ivan Yotov <sup>‡</sup>

## Abstract

In this paper, we develop a new mixed finite element method for elliptic problems on general quadrilateral and hexahedral grids that reduces to a cell-centered finite difference scheme. A special non-symmetric quadrature rule is employed that yields a positive definite cell-centered system for the pressure by eliminating local velocities. The method is shown to be accurate on highly distorted rough quadrilateral and hexahedral grids, including hexahedra with non-planar faces. Theoretical and numerical results indicate first-order convergence for the pressure and face fluxes.

**Keywords:** mixed finite element, multipoint flux approximation, cell-centered finite difference, mimetic finite difference, full tensor coefficient, quadrilaterals, hexahedra.

## 1 Introduction

Single phase incompressible flow in porous media is governed by coupled Darcy's law and continuity equation:

$$\mathbf{u} = -K \nabla p, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = f, \quad (1.2)$$

where  $p$  is the pressure,  $\mathbf{u}$  is the velocity,  $K$  represents the rock permeability divided by fluid kinematic viscosity, and  $f$  is the source term. In [44, 28], a special mixed finite element (MFE) method called the multipoint flux mixed finite element (MFMFE) has been developed for elliptic problems (1.1)–(1.2). This method reduces to a cell-centered finite difference scheme for the scalar variable and is accurate for discontinuous full tensor coefficients on  $h^2$ -perturbed parallelograms, simplicial grids, and  $h^2$ -perturbed parallelepipeds. The goal of this paper is to develop a new MFMFE method that is accurate on general quadrilaterals and hexahedra such as shown in Fig. 1.

---

<sup>\*</sup>Center for Subsurface Modeling (CSM), Institute for Computational Engineering and Sciences (ICES), The University of Texas at Austin, Austin, TX 78712; [mfw@ices.utexas.edu](mailto:mfw@ices.utexas.edu); partially supported by the NSF-CDI under contract number DMS 0835745, the DOE grant DE-FG02-04ER25617, and the Center for Frontiers of Subsurface Energy Security under Contract No. DE-SC0001114.

<sup>†</sup>(CSM), (ICES), The University of Texas at Austin, Austin, TX 78712; [gxue@ices.utexas.edu](mailto:gxue@ices.utexas.edu); supported by Award No. KUS-F1-032-04, made by King Abdullah University of Science and Technology (KAUST).

<sup>‡</sup>Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260; [yotov@math.pitt.edu](mailto:yotov@math.pitt.edu); partially supported by the DOE grant DE-FG02-04ER25618, the NSF grant DMS 0813901, and the J. Tinsley Oden Faculty Fellowship, ICES, The University of Texas at Austin.

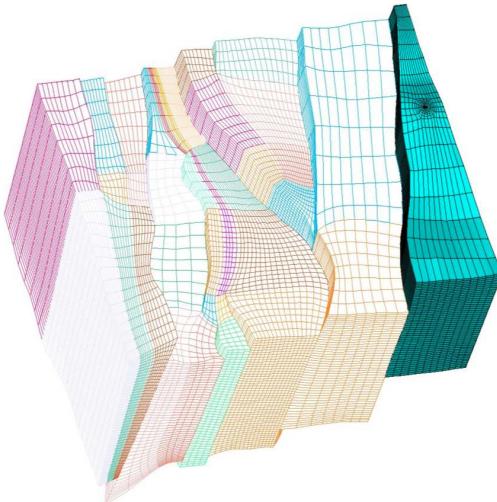


Figure 1: General hexahedral mesh for a geological computational domain with layers and faults

MFE methods [36, 40, 35, 12, 11, 17, 19] are widely used to discretize this coupled system, as they provide accurate and locally mass conservative velocities and can accurately treat discontinuous full tensor coefficients. A standard formulation of MFE yields a saddle point system and requires solving pressure and velocity simultaneously. Another approach, the hybrid MFE formulation [9, 13], reduces the discrete problem to a symmetric positive definite system for the pressure Lagrange multipliers on element faces. In addition, efficient formulations that reduce to cell-centered pressure schemes have also been developed. These apply appropriate MFE spaces and numerical quadrature rules for the velocity mass term. Examples include [37, 42] for diagonal permeability on rectangular grids based on the lowest order Raviart-Thomas MFE method [36] and the expanded mixed finite element (EMFE) method for smooth full tensor permeability on rectangular and cuboid grids [5] and smooth curvilinear grids [4]. In the case of discontinuous permeability, EMFE loses accuracy unless the pressure Lagrange multipliers are introduced along discontinuous interfaces [4]. The MFMFE method [44, 28] has been designed to be a cell-centered pressure scheme that is accurate for both smooth and discontinuous full tensor permeability, but unfortunately it is not accurate on general hexahedral grids.

The velocity finite element spaces in the above mentioned MFE methods on quadrilateral and hexahedral meshes are defined via a mapping from the reference rectangle or cube using the Piola transformation [40, 33]. This transformation preserves normal components of vectors and leads to continuous normal flux approximations on the physical grid. Optimal order velocity convergence in the  $L^2$ -norm on quadrilaterals has been established for the Raviart-Thomas (RT) spaces [36, 40] in [40, 41, 8]. However, approximation accuracy for the divergence of the velocity as well as for the  $L^2$ -norm of the velocity in other MFE spaces may be reduced due to failure of the classical scaling arguments [8]. On hexahedra, the deterioration of accuracy is even more severe due the fact that the constant vector is not contained in the velocity space [34, 38]. Optimal approximation and superconvergence properties in some MFE methods can be obtained under a grid restrictions to  $h^2$ -perturbed

parallelograms or parallelepipeds [22, 44, 28]. Efforts have been made to define accurate MFE methods on highly distorted elements; proposed developments include [39, 7, 8, 24] where new families of quadrilateral finite element spaces and lowest order hexahedral finite element spaces are obtained by enriching the classical Raviart-Thomas finite element spaces. Other approaches employ composite-element techniques, see [31, 38]. All of these methods require solving saddle point problems in their standard forms.

The multipoint flux approximation (MPFA) method [1, 2, 21, 20] has gained significant popularity since it combines the advantages of the above mentioned methods: it is accurate for rough grids and rough full tensor coefficients, and reduces to a cell-centered pressure scheme. The method was first developed as a non-variational finite volume method. However, the convergence analysis was done by formulating it as a MFE method with appropriate finite element spaces and numerical quadrature rules. Motivated by the MPFA method, the MFMFE method was defined in [44] using the lowest order Brezzi-Douglas-Marini ( $BDM_1$ ) finite element spaces and applying a special symmetric quadrature rule for the local velocity elimination. In [28], this scheme was extended to hexahedral elements by introduction of finite element spaces based on enhanced  $BDDF_1$  spaces. These methods were shown to be accurate on  $h^2$ -perturbed parallelograms and  $h^2$ -perturbed parallelepipeds. Similar approaches have been developed on simplicial elements in [14] and on quadrilaterals in [29, 30], using a broken Raviart-Thomas space. The method in [30] uses a non-symmetric quadrature rule and it is first order accurate on general quadrilaterals. It is shown to be equivalent to the physical space MPFA method [1].

In this paper we develop a new non-symmetric MFMFE method that is first order accurate on general quadrilaterals and hexahedra. As in the symmetric MFMFE method [44, 28, 43], we employ the lowest order  $BDM_1$  or enhanced  $BDDF_1$  spaces. A non-symmetric quadrature rule is introduced for the velocity mass matrix that allows for local velocity elimination and reduces the method to a cell-centered pressure system. Unlike its symmetric variant, the quadrature rule satisfies certain critical properties on the physical elements, which are key ingredients in the analysis. The method is equivalent to the symmetric MFMFE method on cuboids and rectangles.

The quadrature rule is similar to the one proposed in [30] for quadrilateral grids. However, the analysis in [30] does not extend to three dimensions, as it relies on the approximation property of the  $RT_0$  velocity space, which is lost on general hexahedral elements. Motivated by the fact that the Piola transformation provides accurate interpolation of normal velocities on faces of physical elements, we employ techniques from the analysis of mimetic finite difference (MFD) methods [15, 32]. We prove first order convergence for the pressure and the normal velocity on the element faces. Our analysis applies to hexahedra with non-planar faces and it is confirmed by the numerical experiments. A key property is that the Piola image of a constant vector on a physical element has a linear normal component in the reference element, which belongs to the  $BDDF_1$  space. This allows us to establish first order approximation of the  $BDDF_1$  interpolant in a face-based  $L^2$ -norm. We note that such property does not hold for the  $RT_0$  interpolant, which have a constant normal component on the reference element. We also note that existing methods with pressure and velocity degrees of freedom similar to our method, such as MFD methods [15], composite  $RT_0$  methods [31, 38], or finite volume methods [23], are limited to planar faces or require extra tangential velocity degrees of freedom to handle non-planar faces [16].

The rest of the paper is organized as follows. In Section 2, the non-symmetric MFMFE method is developed. In Sections 3 and 4, the error analysis for the velocity and pressure, respectively, is presented. In Section 5, numerical examples demonstrate the superior convergence properties of the non-symmetric MFMFE method compared to the symmetric MFMFE method [44, 28]. Conclusions are given in Section 6.

Throughout the paper we use the notation  $X \lesssim (\gtrsim) Y$  to denote that there exists a constant  $C$ , independent of the mesh size  $h$ , such that  $X \leq (\geq) CY$ . The notation  $X \approx Y$  means that both  $X \lesssim Y$  and  $X \gtrsim Y$  hold.

For a domain  $G \subset \mathbb{R}^d$ , the  $L^2(G)$  inner product and norm for scalar and vector valued functions are denoted  $(\cdot, \cdot)_G$  and  $\|\cdot\|_G$ , respectively. The norms and seminorms of the Sobolev spaces  $W^{k,p}(G)$ ,  $k \in \mathbb{R}$ ,  $p > 0$  are denoted by  $\|\cdot\|_{k,p,G}$  and  $|\cdot|_{k,p,G}$ , respectively. The norms and seminorms of the Hilbert spaces  $H^k(G)$  are denoted by  $\|\cdot\|_{k,G}$  and  $|\cdot|_{k,G}$ , respectively. We omit  $G$  in the subscript if  $G = \Omega$ . For a section of the domain, subdomain, or element boundary  $S \subset \mathbb{R}^{d-1}$  we write  $\langle \cdot, \cdot \rangle_S$  and  $\|\cdot\|_S$  for the  $L^2(S)$  inner product (or duality pairing) and norm, respectively. For a tensor-valued function  $M$ , let  $\|M\|_{k,\infty} = \max_{i,j} \|M_{ij}\|_{k,\infty}$ . Furthermore, let

$$H(\text{div}; G) = \left\{ \mathbf{v} \in (L^2(G))^d : \nabla \cdot \mathbf{v} \in L^2(G) \right\},$$

$$\|\mathbf{v}\|_{\text{div}; G} = (\|\mathbf{v}\|_G^2 + \|\nabla \cdot \mathbf{v}\|_G^2)^{1/2}.$$

## 2 Definition of the method

We consider the system (1.1)–(1.2) in a domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  with Lipschitz continuous boundary. For simplicity we assume homogeneous Dirichlet boundary conditions

$$p = 0 \quad \text{on } \partial\Omega,$$

although more general boundary conditions can also be treated. We assume that  $K$  is a symmetric and uniformly positive definite tensor with  $L^\infty(\Omega)$  components satisfying, for some  $0 < k_0 \leq k_1 < \infty$ ,

$$k_0 \xi^T \xi \leq \xi^T K(\mathbf{x}) \xi \leq k_1 \xi^T \xi, \quad \forall \mathbf{x} \in \Omega, \text{ and } \forall \xi \in \mathbb{R}^d. \quad (2.1)$$

### 2.1 Finite element mappings

Let  $\mathcal{T}_h$  be a conforming, shape-regular, quasi-uniform partition of  $\Omega$  [18]. The elements considered are quadrilaterals and hexahedra in two and three dimensions, respectively. We assume that for any  $E \in \mathcal{T}_h$  there exists a bilinear (trilinear in 3D) bijection mapping  $F_E : E \rightarrow \hat{E}$ , where  $\hat{E}$  is the reference square (cube). Denote the Jacobian matrix by  $DF_E$  and let  $J_E = |\det(DF_E)|$ . Denote the inverse mapping by  $F_E^{-1}$ , its Jacobian matrix by  $DF_E^{-1}$ , and let  $J_{F_E^{-1}} = |\det(DF_E^{-1})|$ . We have that

$$DF_E^{-1}(\mathbf{x}) = (DF_E)^{-1}(\hat{\mathbf{x}}), \quad J_{F_E^{-1}}(\mathbf{x}) = \frac{1}{J_E(\hat{\mathbf{x}})}.$$

In the case of hexahedra,  $\hat{E}$  is the unit cube with vertices  $\hat{\mathbf{r}}_1 = (0, 0, 0)^T$ ,  $\hat{\mathbf{r}}_2 = (1, 0, 0)^T$ ,  $\hat{\mathbf{r}}_3 = (1, 1, 0)^T$ ,  $\hat{\mathbf{r}}_4 = (0, 1, 0)^T$ ,  $\hat{\mathbf{r}}_5 = (0, 0, 1)^T$ ,  $\hat{\mathbf{r}}_6 = (1, 0, 1)^T$ ,  $\hat{\mathbf{r}}_7 = (1, 1, 1)^T$ ,  $\hat{\mathbf{r}}_8 = (0, 1, 1)^T$ .

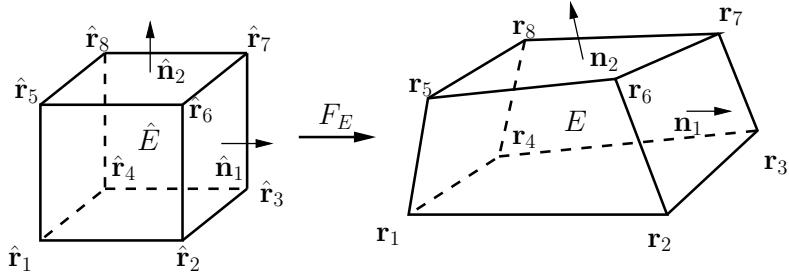


Figure 2: Trilinear hexahedral mapping.

Denote by  $\mathbf{r}_i = (x_i, y_i, z_i)^T, i = 1, \dots, 8$ , the eight corresponding vertices of element  $E$  as shown in Fig. 2. We note that the element can have non-planar faces. The outward unit normal vectors to the faces of  $E$  and  $\hat{E}$  are denoted by  $\mathbf{n}_i$  and  $\hat{\mathbf{n}}_i, i = 1, \dots, 6$ , respectively. In this case  $F_E$  is a trilinear mapping given for  $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z}) \in \hat{E}$  by

$$\begin{aligned}
F_E(\hat{\mathbf{x}}) = & \mathbf{r}_1(1 - \hat{x})(1 - \hat{y})(1 - \hat{z}) + \mathbf{r}_2\hat{x}(1 - \hat{y})(1 - \hat{z}) + \mathbf{r}_3\hat{x}\hat{y}(1 - \hat{z}) \\
& + \mathbf{r}_4(1 - \hat{x})\hat{y}(1 - \hat{z}) + \mathbf{r}_5(1 - \hat{x})(1 - \hat{y})\hat{z} + \mathbf{r}_6\hat{x}(1 - \hat{y})\hat{z} \\
& + \mathbf{r}_7\hat{x}\hat{y}\hat{z} + \mathbf{r}_8(1 - \hat{x})\hat{y}\hat{z} \\
= & \mathbf{r}_1 + \mathbf{r}_{21}\hat{x} + \mathbf{r}_{41}\hat{y} + \mathbf{r}_{51}\hat{z} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}\hat{y} + (\mathbf{r}_{65} - \mathbf{r}_{21})\hat{x}\hat{z} \\
& + (\mathbf{r}_{85} - \mathbf{r}_{41})\hat{y}\hat{z} + (\mathbf{r}_{21} - \mathbf{r}_{34} - \mathbf{r}_{65} + \mathbf{r}_{78})\hat{x}\hat{y}\hat{z},
\end{aligned} \tag{2.2}$$

where  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ . It is easy to see that each component of  $DF_E$  is a bilinear function of two space variables:

$$\begin{aligned}
& DF_E(\hat{\mathbf{x}}) \\
= & [\mathbf{r}_{21} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y} + (\mathbf{r}_{65} - \mathbf{r}_{21})\hat{z} + (\mathbf{r}_{21} - \mathbf{r}_{34} - \mathbf{r}_{65} + \mathbf{r}_{78})\hat{y}\hat{z}, \\
& \mathbf{r}_{41} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x} + (\mathbf{r}_{85} - \mathbf{r}_{41})\hat{z} + (\mathbf{r}_{21} - \mathbf{r}_{34} - \mathbf{r}_{65} + \mathbf{r}_{78})\hat{x}\hat{z}, \\
& \mathbf{r}_{51} + (\mathbf{r}_{65} - \mathbf{r}_{21})\hat{x} + (\mathbf{r}_{85} - \mathbf{r}_{41})\hat{y} + (\mathbf{r}_{21} - \mathbf{r}_{34} - \mathbf{r}_{65} + \mathbf{r}_{78})\hat{x}\hat{y}].
\end{aligned} \tag{2.3}$$

In the case of quadrilaterals,  $\hat{E}$  is the unit square with vertices  $\hat{\mathbf{r}}_1 = (0, 0)^T, \hat{\mathbf{r}}_2 = (1, 0)^T, \hat{\mathbf{r}}_3 = (1, 1)^T, \hat{\mathbf{r}}_4 = (0, 1)^T$ . Let  $\mathbf{r}_i, i = 1, \dots, 4$ , be the corresponding vertices of  $E$ . The bilinear mapping of quadrilaterals has the form, for  $\hat{\mathbf{x}}(\hat{x}, \hat{y}) \in \hat{E}$ ,

$$\begin{aligned}
F_E(\hat{\mathbf{x}}) = & \mathbf{r}_1(1 - \hat{x})(1 - \hat{y}) + \mathbf{r}_2\hat{x}(1 - \hat{y}) + \mathbf{r}_3\hat{x}\hat{y} + \mathbf{r}_4(1 - \hat{x})\hat{y} \\
= & \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)\hat{x} + (\mathbf{r}_4 - \mathbf{r}_1)\hat{y} + (\mathbf{r}_3 + \mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_4)\hat{x}\hat{y}.
\end{aligned} \tag{2.4}$$

The Jacobian matrix and its determinant are

$$DF_E(\hat{\mathbf{x}}) = [\mathbf{r}_2 - \mathbf{r}_1 + (\mathbf{r}_3 + \mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_4)\hat{y}, \mathbf{r}_4 - \mathbf{r}_1 + (\mathbf{r}_3 + \mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_4)\hat{x}], \tag{2.5}$$

$$J_E = 2|T_1| + 2(|T_2| - |T_1|)\hat{x} + 2(|T_4| - |T_1|)\hat{y}, \tag{2.6}$$

where  $|T_i|$  is the area of the triangle formed by the two edges sharing  $\mathbf{r}_i$ .

Using the above mapping definitions and the classical formula, for scalar  $\phi(\mathbf{x}) = \hat{\phi}(\hat{\mathbf{x}})$ ,  $\nabla\phi = (DF_E^{-1})^T \hat{\nabla} \hat{\phi}$ , it is easy to see that, for any face or edge  $e_i \subset E$ ,

$$\mathbf{n}_i = \frac{1}{J_{e_i}} J_E (DF_E^{-1})^T \hat{\mathbf{n}}_i, \quad J_{e_i} = |J_E (DF_E^{-1})^T \hat{\mathbf{n}}_i|_{\mathbb{R}^d}, \quad (2.7)$$

where  $|\cdot|_{\mathbb{R}^d}$  is the Euclidean norm in  $\mathbb{R}^d$ . Also, the shape regularity and quasiuniformity of the grids imply that, for all  $E \in \mathcal{T}_h$ ,

$$\begin{aligned} \|DF_E\|_{0,\infty,\hat{E}} &\lesssim h, & \|DF_E^{-1}\|_{0,\infty,E} &\lesssim h^{-1}, \\ \|J_E\|_{0,\infty,\hat{E}} &\asymp h^d, & \|J_{F_E^{-1}}\|_{0,\infty,E} &\asymp h^{-d}. \end{aligned} \quad (2.8)$$

## 2.2 Mixed finite element spaces

Let  $\hat{\mathbf{V}}(\hat{E})$  and  $\hat{W}(\hat{E})$  be the finite element spaces on the reference element  $\hat{E}$ . For convenience, we use the same notation in two and three dimensions.

On the reference square, let  $\hat{\mathbf{V}}(\hat{E}) \times \hat{W}(\hat{E})$  be the lowest order BDM<sub>1</sub> finite element space [12]:

$$\begin{aligned} \hat{\mathbf{V}}(\hat{E}) &= (P_1(\hat{E}))^2 + r \operatorname{curl}(\hat{x}^2 \hat{y}) + s \operatorname{curl}(\hat{x} \hat{y}^2), \\ \hat{W}(\hat{E}) &= P_0(\hat{E}), \end{aligned} \quad (2.9)$$

where  $r$  and  $s$  are real constants and  $(P_k)^d$  denotes the space of  $d$ -dimensional polynomials of degree  $\leq k$ . In the case of the unit cube, the space is defined by the enhanced BDDF<sub>1</sub> space [28]:

$$\begin{aligned} \hat{\mathbf{V}}(\hat{E}) &= \text{BDDF}_1(\hat{E}) + r_2 \operatorname{curl}(0, 0, \hat{x}^2 \hat{z})^T + r_3 \operatorname{curl}(0, 0, \hat{x}^2 \hat{y} \hat{z})^T \\ &\quad + s_2 \operatorname{curl}(\hat{x} \hat{y}^2, 0, 0)^T + s_3 \operatorname{curl}(\hat{x} \hat{y}^2 \hat{z}, 0, 0)^T \\ &\quad + t_2 \operatorname{curl}(0, \hat{y} \hat{z}^2, 0)^T + t_3 \operatorname{curl}(0, \hat{x} \hat{y} \hat{z}^2, 0)^T, \\ \hat{W}(\hat{E}) &= P_0(\hat{E}), \end{aligned} \quad (2.10)$$

where the BDDF<sub>1</sub> velocity spaces on the unit cube [11] is defined as

$$\begin{aligned} \text{BDDF}_1(\hat{E}) &= (P_1(\hat{E}))^3 + r_0 \operatorname{curl}(0, 0, \hat{x} \hat{y} \hat{z})^T + r_1 \operatorname{curl}(0, 0, \hat{x} \hat{y}^2)^T \\ &\quad + s_0 \operatorname{curl}(\hat{x} \hat{y} \hat{z}, 0, 0)^T + s_1 \operatorname{curl}(\hat{y} \hat{z}^2, 0, 0)^T \\ &\quad + t_0 \operatorname{curl}(0, \hat{x} \hat{y} \hat{z}, 0)^T + t_1 \operatorname{curl}(0, \hat{x}^2 \hat{z}, 0)^T. \end{aligned} \quad (2.11)$$

Here  $r_i, s_i, t_i, (i = 0, \dots, 3)$  are real constants. Note that in both cases

$$\hat{\nabla} \cdot \hat{\mathbf{V}}(\hat{E}) = \hat{W}(\hat{E}),$$

and that for all  $\hat{\mathbf{v}} \in \hat{V}(\hat{E})$  and for any face (or edge)  $\hat{e}$  of  $\hat{E}$ ,

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}} \in P_1(\hat{e}) \text{ on the unit square, } \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}} \in Q_1(\hat{e}) \text{ on the unit cube,}$$

where  $Q_1$  is the space of bilinear functions. The degrees of freedom of  $\hat{\mathbf{V}}(\hat{E})$  are chosen to be the values of  $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}}$  at the corners of  $\hat{e}$  for all faces (or edges) of  $\hat{E}$ .

The spaces  $\mathbf{V}(E)$  and  $W(E)$  on any physical element  $E \in \mathcal{T}_h$  are defined via the Piola transformation:

$$\mathbf{v} \leftrightarrow \hat{\mathbf{v}} : \hat{\mathbf{v}} = \frac{1}{J_E} DF_E \hat{\mathbf{v}} \circ F_E^{-1}, \quad (2.12)$$

and the standard scalar transformation

$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1}. \quad (2.13)$$

Under these transformations, the normal components of the velocity vectors on the faces (edges) are preserved [13]:

$$(\nabla \cdot \mathbf{v}, w)_E = (\hat{\nabla} \cdot \hat{\mathbf{v}}, \hat{w})_{\hat{E}} \quad \text{and} \quad \langle \mathbf{v} \cdot \mathbf{n}_e, w \rangle_e = \langle \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}}, \hat{w} \rangle_{\hat{e}}. \quad (2.14)$$

In addition, (2.7) implies that

$$\mathbf{v} \cdot \mathbf{n}_e = \frac{1}{J_e} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}} \circ F_E^{-1}(\mathbf{x}), \quad (2.15)$$

and (2.14) implies that

$$\nabla \cdot \mathbf{v} = \left( \frac{1}{J_E} \hat{\nabla} \cdot \hat{\mathbf{v}} \right) \circ F_E^{-1}(\mathbf{x}). \quad (2.16)$$

Clearly, for  $\mathbf{v} \in \mathbf{V}(E)$ ,  $\nabla \cdot \mathbf{v} \neq \text{constant}$  since  $J_E$  is not constant on quadrilaterals or hexahedra. Furthermore, while  $J_e = |e| = \text{constant}$  on quadrilaterals, this is not true on hexahedra. As a result  $\mathbf{v} \cdot \mathbf{n}_e \in P_1(e)$  on quadrilaterals, but  $\mathbf{v} \cdot \mathbf{n}_e \notin Q_1(e)$  on hexahedra.

The finite element spaces  $\mathbf{V}_h$  and  $W_h$  are given by

$$\begin{aligned} \mathbf{V}_h &= \left\{ \mathbf{v} \in \mathbf{V} : \mathbf{v}|_E \leftrightarrow \hat{\mathbf{v}}, \hat{\mathbf{v}} \in \hat{\mathbf{V}}(\hat{E}), \forall E \in \mathcal{T}_h \right\}, \\ W_h &= \left\{ w \in W : w|_E \leftrightarrow \hat{w}, \hat{w} \in \hat{W}(\hat{E}), \forall E \in \mathcal{T}_h \right\}, \end{aligned} \quad (2.17)$$

where

$$\mathbf{V} = H(\text{div}; \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}, \quad W = L^2(\Omega).$$

Recall the projection operator in the space  $\mathbf{V}_h$ . The operator  $\hat{\Pi} : (H^1(\hat{E}))^d \rightarrow \hat{\mathbf{V}}(\hat{E})$  is defined locally on each element by

$$\langle (\hat{\Pi} \hat{\mathbf{q}} - \hat{\mathbf{q}}) \cdot \hat{\mathbf{n}}_{\hat{e}}, \hat{q}_1 \rangle_{\hat{e}} = 0, \quad \forall \hat{e} \subset \partial \hat{E}, \quad (2.18)$$

where  $\hat{q}_1 \in P_1(\hat{e})$  for the case of the unit square  $\hat{E}$ , and  $\hat{q}_1 \in Q_1(\hat{e})$  for the case of the unit cube  $\hat{E}$ . The global operator  $\Pi : \mathbf{V} \cap (H^1(\Omega))^d \rightarrow \mathbf{V}_h$  on each element  $E$  is defined by the Piola transformation:

$$\Pi \mathbf{q} \leftrightarrow \widehat{\Pi \mathbf{q}}, \quad \widehat{\Pi \mathbf{q}} = \hat{\Pi} \hat{\mathbf{q}}. \quad (2.19)$$

Furthermore, (2.15) and (2.18) imply that  $\Pi \mathbf{q} \cdot \mathbf{n}$  is continuous across element interfaces, which gives  $\Pi \mathbf{q} \in \mathbf{V}_h$ , and (2.16) implies that

$$(\nabla \cdot (\Pi \mathbf{q} - \mathbf{q}), w) = 0, \quad \forall w \in W_h. \quad (2.20)$$

In the analysis, we will require a similar projection operator onto the lowest order Raviart-Thomas velocity space [36, 35]. The  $RT_0$  spaces are defined on the unit cube as

$$\hat{\mathbf{V}}^R(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1 \hat{x} \\ \alpha_2 + \beta_2 \hat{y} \\ \alpha_3 + \beta_3 \hat{z} \end{pmatrix}, \quad \hat{W}^R(\hat{E}) = P_0(\hat{E}), \quad (2.21)$$

and on the unit square as

$$\hat{\mathbf{V}}^R(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1 \hat{x} \\ \alpha_2 + \beta_2 \hat{y} \end{pmatrix}, \quad \hat{W}^R(\hat{E}) = P_0(\hat{E}). \quad (2.22)$$

Here  $\alpha_i$  and  $\beta_i$  ( $i = 1, 2, 3$ ) are real constants. In both cases,  $\hat{\nabla} \cdot \hat{\mathbf{V}}^R(\hat{E}) = \hat{W}^R(\hat{E})$  and  $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}} \in P_0(\hat{e})$ . The degrees of freedom of  $\hat{\mathbf{V}}^R(\hat{E})$  are chosen to be the values of  $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_{\hat{e}}$  at the midpoints of all faces (or edges) of  $\hat{E}$ . The projection operator  $\hat{\Pi}_R : (H^1(\hat{E}))^d \rightarrow \hat{\mathbf{V}}_R(\hat{E})$  satisfies

$$\langle (\hat{\Pi}_R \hat{\mathbf{q}} - \hat{\mathbf{q}}) \cdot \hat{\mathbf{n}}_{\hat{e}}, \hat{q}_0 \rangle_{\hat{e}} = 0, \quad \forall \hat{e} \subset \partial \hat{E}, \quad \forall \hat{q}_0 \in P_0(\hat{E}). \quad (2.23)$$

The spaces  $\mathbf{V}_h^R$  and  $W_h^R$  on  $\mathcal{T}_h$  and the projection operator  $\Pi_R : (H^1(\Omega))^d \rightarrow \mathbf{V}_h^R$  are defined similarly to the case of  $\mathbf{V}_h$  and  $W_h$ . By definition, we have

$$\mathbf{V}_h^R \subset \mathbf{V}_h, \quad W_h^R = W_h. \quad (2.24)$$

The projection operator  $\Pi_R$  satisfies

$$(\nabla \cdot (\Pi_R \mathbf{q} - \mathbf{q}), w) = 0, \quad \forall w \in W_h^R, \quad (2.25)$$

and

$$\nabla \cdot \mathbf{v} = \nabla \cdot \Pi_R \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (2.26)$$

It has been shown in [41, 8, 28] that on general quadrilaterals and  $h^2$ -perturbed parallelepipeds,

$$\|\mathbf{q} - \Pi \mathbf{q}\| + \|\mathbf{q} - \Pi_R \mathbf{q}\| = O(h).$$

However, on general hexahedra, it only holds that [34, 38, 24]

$$\|\mathbf{q} - \Pi \mathbf{q}\| + \|\mathbf{q} - \Pi_R \mathbf{q}\| = O(1).$$

Due to the above property, the analysis in [30] of the non-symmetric MPFA method on quadrilaterals does not extend to hexahedra.

Let  $\hat{Q}$  be the  $L^2(\hat{E})$ -orthogonal projection onto  $\hat{W}(\hat{E})$ , satisfying for any  $\hat{\varphi} \in L^2(\hat{E})$ ,

$$(\hat{\varphi} - \hat{Q} \hat{\varphi}, \hat{w})_{\hat{E}} = 0, \quad \forall \hat{w} \in \hat{W}(\hat{E}).$$

Let  $\mathcal{Q}_h : L^2(\Omega) \rightarrow W_h$  be the projection operator satisfying for any  $\varphi \in L^2(\Omega)$ ,

$$\mathcal{Q}_h \varphi = \hat{Q} \hat{\varphi} \circ F_E^{-1} \quad \text{on all } E.$$

It is easy to see that, due to (2.14),

$$(\varphi - \mathcal{Q}_h \varphi, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (2.27)$$

Using a scaling argument and the Bramble-Hilbert lemma [18], it can be shown that

$$\|\varphi - \mathcal{Q}_h \varphi\| \lesssim h |\varphi|_1. \quad (2.28)$$

### 2.3 Quadrature rule

The integration for the velocity mass matrix on any element  $E$  is performed by mapping to the reference element  $\hat{E}$  and applying quadrature rule defined on  $\hat{E}$ . Using (2.17) and (2.12), we map the physical integral to the reference one; namely for all  $\mathbf{q}, \mathbf{v} \in \mathbf{V}_h$ , we have

$$(K^{-1}\mathbf{q}, \mathbf{v})_E = \left( \frac{1}{J_E} DF_E^T K^{-1}(F_E(\hat{\mathbf{x}})) DF_E \hat{\mathbf{q}}, \hat{\mathbf{v}} \right)_{\hat{E}} \equiv (\mathcal{M}_E \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}},$$

where

$$\mathcal{M}_E = \frac{1}{J_E} DF_E^T K^{-1}(F_E(\hat{\mathbf{x}})) DF_E. \quad (2.29)$$

Define a constant matrix  $\bar{K}_E$  such that  $\bar{K}_E^{ij}$  is the mean value of  $K^{ij}$  on  $E$ , where  $\bar{K}_E^{ij}$  and  $K^{ij}$  denote the elements on the  $i$ -th row and  $j$ -th column of matrix  $\bar{K}_E$  and  $K$  respectively. Let  $\hat{\mathbf{r}}_{c,\hat{E}}$  denote the center of mass of  $\hat{E}$ . Replacing  $DF_E^T$  and  $K$  by the constant matrices  $DF_E^T(\hat{\mathbf{r}}_{c,\hat{E}})$  and  $\bar{K}_E$  respectively, we define

$$\tilde{\mathcal{M}}_E = \frac{1}{J_E} DF_E^T(\hat{\mathbf{r}}_{c,\hat{E}}) \bar{K}_E^{-1} DF_E. \quad (2.30)$$

In addition, we use  $(\cdot, \cdot)_{\hat{Q},\hat{E}}$  to denote the trapezoidal rule on  $\hat{E}$ :

$$(\hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{Q},\hat{E}} \equiv \frac{|\hat{E}|}{2^d} \sum_{i=1}^{2^d} \hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i), \quad (2.31)$$

where  $\hat{\mathbf{r}}_i$  are the vertices of element  $\hat{E}$  defined in Section 2.1.

The quadrature rule on an element  $E$  is defined as

$$(K^{-1}\mathbf{q}, \mathbf{v})_{Q,E} \equiv (\tilde{\mathcal{M}}_E \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{Q},\hat{E}} = \frac{|\hat{E}|}{2^d} \sum_{i=1}^{2^d} \tilde{\mathcal{M}}_E(\hat{\mathbf{r}}_i) \hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i). \quad (2.32)$$

Mapping back to the physical element  $E$ , we have the quadrature rule on  $E$ :

$$(K^{-1}\mathbf{q}, \mathbf{v})_{Q,E} = \frac{1}{2^d} \sum_{i=1}^{2^d} J_E(\hat{\mathbf{r}}_i) (DF_E^{-1})^T(\mathbf{r}_i) DF_E^T(\hat{\mathbf{r}}_{c,\hat{E}}) \bar{K}_E^{-1} \mathbf{q}(\mathbf{r}_i) \cdot \mathbf{v}(\mathbf{r}_i). \quad (2.33)$$

This is related to a non-symmetric inner product used in mimetic finite difference methods [27, 32]. Note that this trapezoidal rule for the modified integrand induces a non-symmetric quadrature rule unless the Jacobian matrix  $DF_E$  is constant. A similar quadrature rule on quadrilateral elements was introduced in [30], where the mean value of  $K^{-1}$  was used.

The global quadrature rule on  $\Omega$  is defined as

$$(K^{-1}\mathbf{q}, \mathbf{v})_Q \equiv \sum_{E \in \mathcal{T}_h} (K^{-1}\mathbf{q}, \mathbf{v})_{Q,E}.$$

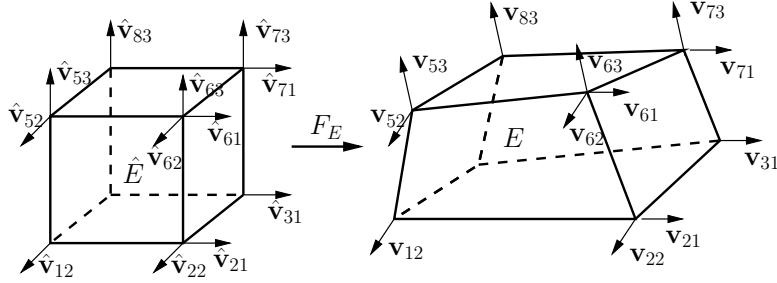


Figure 3: Degrees of freedom and basis functions for the enhanced BDDF1 velocity space on hexahedra.

**Remark 2.1.** *The quadrature rule (2.33) becomes symmetric on parallelograms and parallelepipeds. In such cases the method is equivalent to the symmetric MFMFE method introduced in [44, 28] when the permeability is an element by element piecewise constant tensor.*

Next we discuss a property of the quadrature rule that leads to a cell-centered pressure scheme. The corner vector  $\hat{\mathbf{q}}(\hat{\mathbf{r}}_i)$  is uniquely determined by its normal components to the faces that share the vertex. Since we chose the velocity degrees of freedom as the normal components to the face (edge) at each corner  $\hat{\mathbf{r}}_i$ , the corner vector  $\hat{\mathbf{q}}(\hat{\mathbf{r}}_i)$  can be uniquely expressed in terms of the degrees of freedom that share the vertex  $\hat{\mathbf{r}}_i$ . More precisely,

$$\hat{\mathbf{q}}(\hat{\mathbf{r}}_i) = \sum_{j=1}^d \hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_{ij}(\hat{\mathbf{r}}_i) \hat{\mathbf{n}}_{ij},$$

where  $\hat{\mathbf{n}}_{ij}$ ,  $j = 1, \dots, d$ , are the outward unit normal vectors to the faces (or edges) sharing  $\hat{\mathbf{r}}_i$ , and  $\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_{ij}(\hat{\mathbf{r}}_i)$  are the velocity degrees of freedom associated with this corner. More precisely, the first index  $i$  denotes the node and the second index  $j$  denotes the direction. Denote the basis functions associated with  $\hat{\mathbf{r}}_i$  by  $\hat{\mathbf{v}}_{ij}$ ,  $j = 1, \dots, d$ , see Fig. 3:

$$\hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{ij}(\hat{\mathbf{r}}_i) = 1, \quad \hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{ik}(\hat{\mathbf{r}}_i) = 0, \quad k \neq j, \quad \hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{lk}(\hat{\mathbf{r}}_l) = 0, \quad l \neq i, \quad k = 1, \dots, d.$$

The quadrature rule (2.32) couples only the  $d$  basis functions associated with a corner. For example, on the unit cube,

$$\begin{aligned} (\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{11})_{\hat{Q}, \hat{E}} &= \frac{\widetilde{\mathcal{M}}_E^{11}(\hat{\mathbf{r}}_1)}{8}, & (\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{12})_{\hat{Q}, \hat{E}} &= \frac{\widetilde{\mathcal{M}}_E^{21}(\hat{\mathbf{r}}_1)}{8}, \\ (\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{13})_{\hat{Q}, \hat{E}} &= \frac{\widetilde{\mathcal{M}}_E^{31}(\hat{\mathbf{r}}_1)}{8}, & \text{and } (\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{ij})_{\hat{Q}, \hat{E}} &= 0, \end{aligned} \quad (2.34)$$

where  $i \neq 1, j = 1, 2, 3$ .

The following are some properties of the quadrature rule that are key ingredients in the velocity error analysis.

**Lemma 2.1** ([44, 28]). *For any  $\hat{\mathbf{v}} \in \hat{V}(\hat{E})$  and constant vector  $\hat{\mathbf{q}}_0$  on  $\hat{E}$ , the numerical quadrature rule satisfies*

$$(\hat{\mathbf{v}} - \hat{\Pi}_R \hat{\mathbf{v}}, \hat{\mathbf{q}}_0)_{\hat{Q}, \hat{E}} = 0. \quad (2.35)$$

**Lemma 2.2.** For any constant vector  $\mathbf{q}_0$  on  $E$  and  $\mathbf{v} \in \mathbf{V}_h(E)$ ,

$$(K^{-1}\mathbf{q}_0, \mathbf{v} - \Pi_R \mathbf{v})_{Q,E} = 0. \quad (2.36)$$

*Proof.* By the definition (2.32) of the quadrature rule and the Piola transformation (2.12),

$$\begin{aligned} (K^{-1}\mathbf{q}_0, \mathbf{v} - \Pi_R \mathbf{v})_{Q,E} &= (\widetilde{\mathcal{M}}_E \hat{\mathbf{q}}_0, \hat{\mathbf{v}} - \hat{\Pi}_R \hat{\mathbf{v}})_{\hat{Q}, \hat{E}} \\ &= \left( \frac{1}{J_E} DF_E^T(\hat{\mathbf{r}}_{c, \hat{E}}) \bar{K}_E^{-1} DF_E J_E DF_E^{-1} \mathbf{q}_0, \hat{\mathbf{v}} - \hat{\Pi}_R \hat{\mathbf{v}} \right)_{\hat{Q}, \hat{E}} \\ &= \left( DF_E^T(\hat{\mathbf{r}}_{c, \hat{E}}) \bar{K}_E^{-1} \mathbf{q}_0, \hat{\mathbf{v}} - \hat{\Pi}_R \hat{\mathbf{v}} \right)_{\hat{Q}, \hat{E}} = 0, \end{aligned} \quad (2.37)$$

where we have used (2.35) in the last equality.  $\square$

**Lemma 2.3.** For any constant vector  $\mathbf{q}_0$  on  $\hat{E}$  and  $\hat{\mathbf{v}} \in RT_0(\hat{E})$ ,

$$(DF_E(\hat{\mathbf{r}}_{c, \hat{E}}) \hat{\mathbf{v}}, \mathbf{q}_0)_{\hat{E}} = (DF_E \hat{\mathbf{v}}, \mathbf{q}_0)_{\hat{E}}. \quad (2.38)$$

*Proof.* Since the proofs in two and three dimensions are very similar, we give only the proof for the 3D case. Let  $\hat{\mathbf{v}} = [\hat{v}_1(\hat{x}), \hat{v}_2(\hat{y}), \hat{v}_3(\hat{z})]^T$ ,  $\hat{\mathbf{r}}_{c, \hat{E}} = (\hat{x}_c, \hat{y}_c, \hat{z}_c)$ , and  $DF_E = [\xi_1(\hat{y}, \hat{z}), \xi_2(\hat{x}, \hat{z}), \xi_3(\hat{x}, \hat{y})]$ . We have

$$\begin{aligned} (DF_E \hat{\mathbf{v}}, \mathbf{q}_0)_{\hat{E}} &= \int_{\hat{E}} \hat{v}_1(\hat{x}) \xi_1(\hat{y}, \hat{z}) \cdot \mathbf{q}_0 d\hat{x} d\hat{y} d\hat{z} \\ &\quad + \int_{\hat{E}} \hat{v}_2(\hat{y}) \xi_2(\hat{x}, \hat{z}) \cdot \mathbf{q}_0 d\hat{x} d\hat{y} d\hat{z} + \int_{\hat{E}} \hat{v}_3(\hat{z}) \xi_3(\hat{x}, \hat{y}) \cdot \mathbf{q}_0 d\hat{x} d\hat{y} d\hat{z}. \end{aligned} \quad (2.39)$$

Since the midpoint quadrature rule is exact for bilinear functions on a square, the first term in the right in (2.39) can be written as

$$\begin{aligned} \int_{\hat{E}} \hat{v}_1(\hat{x}) \xi_1(\hat{y}, \hat{z}) \cdot \mathbf{q}_0 d\hat{x} d\hat{y} d\hat{z} &= \int_0^1 \hat{v}_1(\hat{x}) d\hat{x} \int_0^1 \int_0^1 \xi_1(\hat{y}, \hat{z}) \cdot \mathbf{q}_0 d\hat{y} d\hat{z} \\ &= \int_0^1 \hat{v}_1(\hat{x}) d\hat{x} \int_0^1 \int_0^1 \xi_1(\hat{y}_c, \hat{z}_c) \cdot \mathbf{q}_0 d\hat{y} d\hat{z} = \int_{\hat{E}} \hat{v}_1(\hat{x}) \xi_1(\hat{y}_c, \hat{z}_c) \cdot \mathbf{q}_0 d\hat{x} d\hat{y} d\hat{z}. \end{aligned}$$

Similar identities hold for the other terms on the right in (2.39).  $\square$

**Lemma 2.4.** For any constant vector  $\mathbf{q}_0$  on  $E$  and  $\mathbf{v} \in RT_0(E)$ ,

$$(K^{-1}\mathbf{q}_0, \mathbf{v})_{Q,E} = (\bar{K}_E^{-1}\mathbf{q}_0, \mathbf{v})_E. \quad (2.40)$$

*Proof.* The proof is similar to the proof of Lemma 3.1 in [30]. By the definition (2.32) of the quadrature rule and the Piola transformation (2.12),

$$\begin{aligned} (K^{-1}\mathbf{q}_0, \mathbf{v})_{Q,E} &= \left( \frac{1}{J_E} DF_E^T(\hat{\mathbf{r}}_{c, \hat{E}}) \bar{K}_E^{-1} DF_E \hat{\mathbf{q}}_0, \hat{\mathbf{v}} \right)_{\hat{Q}, \hat{E}} \\ &= \left( DF_E^T(\hat{\mathbf{r}}_{c, \hat{E}}) \bar{K}_E^{-1} \mathbf{q}_0, \hat{\mathbf{v}} \right)_{\hat{Q}, \hat{E}} = \left( DF_E^T(\hat{\mathbf{r}}_{c, \hat{E}}) \bar{K}_E^{-1} \mathbf{q}_0, \hat{\mathbf{v}} \right)_{\hat{E}} \\ &= \left( \bar{K}_E^{-1} \mathbf{q}_0, DF_E(\hat{\mathbf{r}}_{c, \hat{E}}) \hat{\mathbf{v}} \right)_{\hat{E}} = \left( \bar{K}_E^{-1} \mathbf{q}_0, DF_E \hat{\mathbf{v}} \right)_{\hat{E}} = (\bar{K}_E^{-1} \mathbf{q}_0, \mathbf{v})_E, \end{aligned} \quad (2.41)$$

where we have used the fact that trapezoidal rule is exact for bilinear (trilinear) functions on  $\hat{E}$ , and (2.38).  $\square$

## 2.4 The multipoint flux mixed finite element method

The method is defined as follows: find  $\mathbf{u}_h \in \mathbf{V}_h$  and  $p_h \in W_h$  such that

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_Q - (p_h, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h \quad (2.42)$$

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w), \quad \forall w \in W_h \quad (2.43)$$

Following the terminology from [44, 28], we call the method (2.42)–(2.43) a non-symmetric multipoint flux mixed finite element (MFMFE) method, due to its relation to the MPFA method and the non-symmetric quadrature rule (2.33).

For the solvability of this system, we require the bilinear form  $(\cdot, \cdot)_Q$  to be coercive. This holds if the matrix  $\mathbf{M}_c$  associated with  $(\cdot, \cdot)_Q$  is positive definite, see (3.27) in Section 3.2.

**Lemma 2.5.** *If (3.27) holds, then the method (2.42)–(2.43) has a unique solution.*

*Proof.* It is sufficient to show the uniqueness since (2.42)–(2.43) is a finite-dimensional square linear system. We let  $f = 0$  and choose  $\mathbf{v} = \mathbf{u}_h$  and  $w = p_h$  to conclude that  $(K^{-1}\mathbf{u}_h, \mathbf{u}_h)_{Q, \Omega_i} = 0$ . Since (3.27) holds, the norm equivalence (3.28) implies that  $\mathbf{u}_h = 0$ . For  $p_h \in W_h \subset L^2(\Omega)$ , there exists  $\mathbf{q} \in (H^1(\Omega))^d$  such that  $\nabla \cdot \mathbf{q} = p_h$  [26, 25]. Taking  $\mathbf{v} = \Pi \mathbf{q}$  in (2.42) and using (2.20) implies that  $p_h = 0$ .  $\square$

## 2.5 Reduction to a cell-centered stencil

In this section we describe how the multipoint flux mixed finite element method reduces to a system for the pressure at the cell centers. Consider any interior vertex  $\mathbf{r}_c$  that is shared by  $k$  elements  $E_1, \dots, E_k$ ; see Fig. 4 for a two-dimensional example of five quadrilaterals sharing a vertex. Denote the faces that share the vertex by  $e_1, \dots, e_k$ , and the corresponding velocity basis functions associated with the vertex by  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . Let the normal components of  $\mathbf{u}_h$  on the edges be denoted by  $u_1, \dots, u_k$ . Note that for clarity the normal velocities in Fig. 4 are drawn at a distance from the vertex.

Recall that the quadrature rule  $(K^{-1}, \cdot)_Q$  decouples the basis functions that do not share a vertex (see (2.34)). Setting  $\mathbf{v}_1$  in (2.42) leads to coupling of  $u_1$  only with  $u_2$  and  $u_5$ . Similarly,  $u_2$  is coupled only with  $u_1$  and  $u_3$ , etc. Thus, the five equations obtained by taking  $\mathbf{v} = \mathbf{v}_1, \dots, \mathbf{v}_5$  form a linear system for  $u_1, \dots, u_5$ .

We derive the local linear system for the example in Fig. 4. Taking  $\mathbf{v} = \mathbf{v}_3$  in (2.42) gives

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_3)_Q = (K^{-1}\mathbf{u}_h, \mathbf{v}_3)_{Q, E_3} + (K^{-1}\mathbf{u}_h, \mathbf{v}_3)_{Q, E_4}. \quad (2.44)$$

By applying the quadrature rule (2.32), we have

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_3)_{Q, E_3} &= (\widetilde{\mathcal{M}}_E \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_3)_{\hat{Q}, \hat{E}} \\ &= u_2(\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3)_{\hat{Q}, \hat{E}} + u_3(\widetilde{\mathcal{M}}_E \hat{\mathbf{v}}_3, \hat{\mathbf{v}}_3)_{\hat{Q}, \hat{E}} \\ &= \frac{1}{4} \left( \widetilde{\mathcal{M}}_E^{12} |e_2| u_2 + \widetilde{\mathcal{M}}_E^{11} |e_3| u_3 \right) |e_3|, \end{aligned} \quad (2.45)$$

where we have used (2.15) in the last equality. Similarly, the second term on the right of (2.45) can be written as

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_3)_{Q, E_4} = \frac{1}{4} \left( \widetilde{\mathcal{M}}_E^{11} |e_3| u_3 + \widetilde{\mathcal{M}}_E^{12} |e_4| u_4 \right) |e_3|. \quad (2.46)$$

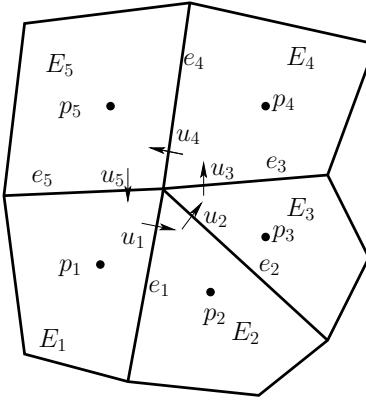


Figure 4: Five quadrilaterals sharing a vertex and associated degrees of freedom.

For the second term on the left of (2.42), we write

$$\begin{aligned}
 (p_h, \nabla \cdot \mathbf{v}_3) &= (p_h, \nabla \cdot \mathbf{v}_3)_{E_3} + (p_h, \nabla \cdot \mathbf{v}_3)_{E_4} \\
 &= \langle p_h, \mathbf{v}_3 \cdot \mathbf{n}_{E_3} \rangle_{e_3} + \langle p_h, \mathbf{v}_3 \cdot \mathbf{n}_{E_4} \rangle_{e_3} \\
 &= \langle \hat{p}_h, \hat{\mathbf{v}}_3 \cdot \hat{\mathbf{n}}_{E_3} \rangle_{\hat{e}_3} + \langle \hat{p}_h, \hat{\mathbf{v}}_3 \cdot \hat{\mathbf{n}}_{E_4} \rangle_{\hat{e}_3} = \frac{1}{2}(p_3 - p_4)|e_3|.
 \end{aligned} \tag{2.47}$$

In the last equality we used the fact that trapezoidal rule is exact for the integrals on  $\hat{e}_3$  since  $\hat{p}_h$  is constant and  $\hat{\mathbf{v}}_3 \cdot \hat{\mathbf{n}}$  is linear (or bilinear for 3D elements). The above results gives the equation associated with test function  $\mathbf{v}_3$  as:

$$\frac{1}{2}\widetilde{\mathcal{M}}_{E_3}^{12}|e_2|u_2 + \left(\frac{1}{2}\widetilde{\mathcal{M}}_{E_3}^{11} + \frac{1}{2}\widetilde{\mathcal{M}}_{E_4}^{11}\right)|e_3|u_3 + \frac{1}{2}\widetilde{\mathcal{M}}_{E_4}^{12}|e_4|u_4 = p_3 - p_4.$$

Similarly we derive the other four equations of the local system and write the system as

$$\mathbf{A}\widetilde{\mathbf{M}}_c\mathbf{A} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \mathbf{A} \begin{pmatrix} 2p_1 - 2p_2 \\ 2p_2 - 2p_3 \\ 2p_3 - 2p_4 \\ 2p_4 - 2p_5 \\ 2p_5 - 2p_1 \end{pmatrix}, \tag{2.48}$$

where  $\mathbf{A} = \text{diag}(|e_1|, |e_2|, |e_3|, |e_4|, |e_5|)$  and

$$\begin{aligned}
 \widetilde{\mathbf{M}}_c &= \begin{pmatrix} \widetilde{\mathcal{M}}_{E_1}^{11} + \widetilde{\mathcal{M}}_{E_2}^{11} & \widetilde{\mathcal{M}}_{E_2}^{12} & & & \widetilde{\mathcal{M}}_{E_1}^{12} \\ \widetilde{\mathcal{M}}_{E_2}^{21} & \widetilde{\mathcal{M}}_{E_2}^{22} + \widetilde{\mathcal{M}}_{E_3}^{22} & \widetilde{\mathcal{M}}_{E_3}^{21} & & \\ & \widetilde{\mathcal{M}}_{E_3}^{12} & \widetilde{\mathcal{M}}_{E_3}^{11} + \widetilde{\mathcal{M}}_{E_4}^{11} & \widetilde{\mathcal{M}}_{E_4}^{12} & \\ & & \widetilde{\mathcal{M}}_{E_4}^{21} & \widetilde{\mathcal{M}}_{E_4}^{22} + \widetilde{\mathcal{M}}_{E_5}^{22} & \widetilde{\mathcal{M}}_{E_5}^{21} \\ \widetilde{\mathcal{M}}_{E_1}^{12} & & & \widetilde{\mathcal{M}}_{E_5}^{12} & \widetilde{\mathcal{M}}_{E_5}^{11} + \widetilde{\mathcal{M}}_{E_1}^{11} \end{pmatrix}. \tag{2.49}
 \end{aligned}$$

Let  $\mathbf{M}_c = \widetilde{\mathbf{A}}\mathbf{M}_c\mathbf{A}$ . On general quadrilateral and hexahedral elements,  $\mathbf{M}_c$  is not symmetric unless the meshes are parallelograms and parallelepipeds. The stability condition (3.27) ensures the solvability of this local system. Let  $\mathbf{M}$  be the matrix induced by the bilinear form  $(K^{-1}\cdot, \cdot)_Q$ . Clearly  $\mathbf{M}$  is block-diagonal with each block  $\mathbf{M}_c$  associated with a corner point  $c$ . The size of  $\mathbf{M}_c$  is  $n_c \times n_c$ , where  $n_c$  is the number of faces (or edges in 2D) that share the vertex point  $c$ . The solution of the local  $n_c \times n_c$  linear system allows for the velocities  $u_i, i = 1, \dots, n_c$  to be expressed in term of the pressure degrees of freedom  $p_i, i = 1, \dots, N_c$  at the cell centers, where  $N_c$  is the number of elements that share the vertex  $c$ . Then, substituting these expressions into the discrete mass conservation equation (2.43) gives a cell-centered stencil for the pressures. The pressure in each element  $E$  is coupled only with the pressures in elements that share a vertex with  $E$ . This gives a 9-point or a 27-point stencil on logically rectangular or cuboid grids, respectively.

### 3 Convergence of the velocity

In the error analysis, we will make use of the following auxiliary lemmas on general quadrilateral and hexahedral elements.

#### 3.1 Auxiliary results

**Lemma 3.1** ([28]). *For any element  $E \in \mathcal{T}_h$ ,*

$$\|\mathbf{q}\|_{j,E} \lesssim h^{-1} \|\mathbf{q}\|_{j-1,E}, \quad j = 1, 2, \quad \forall \mathbf{q} \in \mathbf{V}_h(E). \quad (3.1)$$

The following are bounds on the Jacobian matrix terms that appear in the Piola transformation (2.12).

**Lemma 3.2.** *For a general quadrilateral and hexahedral element  $E \in \mathcal{T}_h$ ,*

$$\left| \frac{1}{J_E} DFE \right|_{0,\infty,E} \lesssim h^{1-d}, \quad (3.2)$$

$$|J_E DFE|_{j,\infty,\hat{E}} \lesssim h^{d-1}, \quad j = 0, 1. \quad (3.3)$$

*Proof.* The proof for general hexahedra can be found in [28]. The proof for general quadrilaterals is similar.  $\square$

**Remark 3.1.** *Better bounds can be obtained under a certain restriction on the element geometry. A quadrilateral in 2D or a hexahedral face in 3D with vertices  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ , and  $\mathbf{r}_4$  (numbered counter clockwise) is called an  $h^2$ -parallelogram if*

$$|\mathbf{r}_{34} - \mathbf{r}_{21}|_{\mathbb{R}^d} \lesssim h^2.$$

*A hexahedron is called an  $h^2$ -parallelepiped when all of its faces are  $h^2$ -parallelograms. For  $h^2$ -parallelograms and  $h^2$ -parallelepipeds, one order higher estimate holds for the one-seminorm; namely,  $|J_E DFE|_{1,\infty,\hat{E}} \lesssim h^d$  (see [44, 28]), as compared to (3.3) for general quadrilateral and hexahedral elements.*

**Lemma 3.3.** For any element  $E \in \mathcal{T}_h$ ,

$$\|\hat{\mathbf{q}}\|_{\hat{E}} \lesssim h^{-1+d/2} \|\mathbf{q}\|_E, \quad \forall \mathbf{q} \in (L^2(E))^d, \quad (3.4)$$

$$|\hat{\mathbf{q}}|_{1,\hat{E}} \lesssim (h^{-1+d/2} \|\mathbf{q}\|_E + h^{d/2} |\mathbf{q}|_{1,E}), \quad \forall \mathbf{q} \in (H^1(E))^d. \quad (3.5)$$

*Proof.* Let  $\tilde{\mathbf{q}}(\hat{\mathbf{x}}) = \mathbf{q} \circ F(\hat{\mathbf{x}})$ . By the standard scaling argument [18] and the shape regularity of mesh,

$$|\tilde{\mathbf{q}}|_{j,\hat{E}} \lesssim h^{j-d/2} |\mathbf{q}|_{j,E}, \quad j = 0, 1. \quad (3.6)$$

The definition of the Piola transformation (2.12) implies,

$$\hat{\mathbf{q}}(\hat{\mathbf{x}}) = J_E D F_E^{-1} \tilde{\mathbf{q}}(\hat{\mathbf{x}}). \quad (3.7)$$

Then, by (3.3) and (3.6),

$$\|\hat{\mathbf{q}}\|_{\hat{E}} \leq |J_E D F_E^{-1}|_{0,\infty,\hat{E}} \|\tilde{\mathbf{q}}\|_{\hat{E}} \lesssim h^{d-1} h^{-d/2} \|\mathbf{q}\|_E = h^{-1+d/2} \|\mathbf{q}\|_E. \quad (3.8)$$

Similarly, using (3.2), one can show

$$\|\mathbf{q}\|_E \lesssim h^{1-d/2} \|\hat{\mathbf{q}}\|_{\hat{E}}. \quad (3.9)$$

Next, using (3.7), (3.3), and (3.6), we get

$$\begin{aligned} |\hat{\mathbf{q}}|_{1,\hat{E}} &\lesssim |J_E D F_E^{-1}|_{1,\infty,\hat{E}} \|\tilde{\mathbf{q}}\|_{\hat{E}} + |J_E D F_E^{-1}|_{0,\infty,\hat{E}} |\tilde{\mathbf{q}}|_{1,\hat{E}} \\ &\lesssim h^{d-1} h^{-d/2} \|\mathbf{q}\|_E + h^{d-1} h^{1-d/2} |\mathbf{q}|_{1,E} \\ &= h^{-1+d/2} \|\mathbf{q}\|_E + h^{d/2} |\mathbf{q}|_{1,E}. \end{aligned} \quad (3.10)$$

□

**Lemma 3.4.** The following trace inequality holds:

$$\|\mathbf{v} \cdot \mathbf{n}\|_{\partial E} \lesssim h^{-1/2} \|\mathbf{v}\|_E \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad (3.11)$$

*Proof.* First, note that (2.8) implies that  $J_e \lesssim h^{d-1}$ . Now, (3.11) follows from (2.15), the norm equivalence on the reference element  $\hat{E}$ ,  $\|\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\|_{\partial \hat{E}} \lesssim \|\hat{\mathbf{v}}\|_{\hat{E}}$ , and the scaling estimate (3.4). □

**Lemma 3.5.** The following hold for all  $\mathbf{q} \in (H^1(E))^d$ :

$$\|\Pi_R \mathbf{q}\|_E + \|\Pi \mathbf{q}\|_E \lesssim \|\mathbf{q}\|_E + h |\mathbf{q}|_{1,E}, \quad (3.12)$$

$$\|\nabla \cdot \Pi_R \mathbf{q}\|_E + \|\nabla \cdot \Pi \mathbf{q}\|_E \lesssim \|\nabla \cdot \mathbf{q}\|_E. \quad (3.13)$$

*Proof.* We present the proof for  $\Pi_R$ . The proof for  $\Pi$  is similar. First, note that

$$\|\hat{\Pi}_R \hat{\mathbf{q}}\|_{\hat{E}} \lesssim \|\hat{\mathbf{q}}\|_{\hat{E}} + \|\hat{\Pi}_R \hat{\mathbf{q}} - \hat{\mathbf{q}}\|_{\hat{E}} \lesssim \|\hat{\mathbf{q}}\|_{\hat{E}} + |\hat{\mathbf{q}}|_{1,\hat{E}} \lesssim \|\hat{\mathbf{q}}\|_{1,\hat{E}}, \quad (3.14)$$

where we have used the Bramble-Hilbert lemma in the second inequality. Now, using (3.4), we have

$$\begin{aligned}\|\Pi_R \mathbf{q}\|_E &\lesssim h^{1-d/2} \|\hat{\Pi}_R \hat{\mathbf{q}}\|_{\hat{E}} \lesssim h^{1-d/2} \|\hat{\mathbf{q}}\|_{1,E} \\ &\lesssim h^{1-d/2} (h^{-1+d/2} \|\mathbf{q}\|_E + h^{d/2} |\mathbf{q}|_{1,E}) = \|\mathbf{q}\|_E + h |\mathbf{q}|_{1,E},\end{aligned}\quad (3.15)$$

using (3.5) in the last inequality. This gives (3.12). To show (3.13) we write, using (2.16) and (2.8),

$$\|\nabla \cdot \Pi_R \mathbf{q}\|_E \lesssim h_E^{-d/2} \|\hat{\nabla} \cdot \hat{\Pi}_R \hat{\mathbf{q}}\|_{\hat{E}} \leq h_E^{-d/2} \|\hat{\nabla} \cdot \hat{\mathbf{q}}\|_{\hat{E}} \lesssim \|\nabla \cdot \mathbf{q}\|_E,$$

where we have used that  $\hat{\nabla} \cdot \hat{\Pi}_R \hat{\mathbf{q}}$  is the  $L^2(\hat{E})$ -projection of  $\hat{\nabla} \cdot \hat{\mathbf{q}}$ . This completes the proof of (3.13).  $\square$

The next lemmas establish important properties of the Piola transformation acting on constant (in 3D) and linear (in 2D) vectors in a physical element.

**Lemma 3.6.** *In the case of hexahedra, for any constant vector  $\mathbf{q}_0$  on  $E$ ,*

$$\forall e \subset \partial E, \quad \Pi \mathbf{q}_0 \cdot \mathbf{n}_e = \mathbf{q}_0 \cdot \mathbf{n}_e. \quad (3.16)$$

*Proof.* The Piola transformation gives  $\hat{\mathbf{q}}_0 = J_E D F_E^{-1} \mathbf{q}_0$  with  $J_E D F_E^{-1} = C_E^T$ , where  $C_E$  is the cofactor matrix of  $D F_E$ . Then

$$\hat{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_{\hat{e}} = C_E^T \mathbf{q}_0 \cdot \hat{\mathbf{n}}_{\hat{e}} = C_E \hat{\mathbf{n}}_{\hat{e}} \cdot \mathbf{q}_0.$$

It is easy to check by direct computation that

$$C_E = [\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3], \quad \mathbf{c}_1 \in (P_{2,1,1}(E))^3, \mathbf{c}_2 \in (P_{1,2,1}(E))^3, \mathbf{c}_3 \in (P_{1,1,2}(E))^3,$$

where  $P_{\alpha_1, \alpha_2, \alpha_3}$  is the space of tensor-product polynomials of degree at most  $\alpha_i$  in variable  $x_i$ ,  $i = 1, 2, 3$ . A closer inspection shows that  $C_E \hat{\mathbf{n}}_{\hat{e}} \in P_1(\hat{e})^3$ . For example on the face ( $\hat{x} = 0$ ) of the reference element  $\hat{E}$ , it is easy to calculate using (2.3)

$$C_E \hat{\mathbf{n}}_{\hat{e}} = - \begin{bmatrix} (c_2 + g_2 \hat{z})(d_3 + g_3 \hat{y}) - (d_2 + g_2 \hat{y})(c_3 + g_3 \hat{z}) \\ (d_1 + g_1 \hat{y})(c_3 + g_3 \hat{z}) - (c_1 + g_1 \hat{z})(d_3 + g_3 \hat{y}) \\ (c_1 + g_1 \hat{z})(d_2 + g_2 \hat{y}) - (d_1 + g_1 \hat{y})(c_2 + g_2 \hat{z}) \end{bmatrix},$$

where the constant vectors are  $[c_1 \ c_2 \ c_3]^T = \mathbf{r}_{41}$ ,  $[d_1 \ d_2 \ d_3]^T = \mathbf{r}_{51}$ , and  $[g_1 \ g_2 \ g_3]^T = \mathbf{r}_{85} - \mathbf{r}_{21}$ . This gives  $C_E \hat{\mathbf{n}}_{\hat{e}} \cdot \mathbf{q}_0 \in P_1(\hat{e})$ , implying that  $\hat{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_{\hat{e}} \in P_1(\hat{e}) \subset \hat{\mathbf{V}}(\hat{E}) \cdot \hat{\mathbf{n}}_{\hat{e}}$ .

Now, (2.18) implies that  $\hat{\Pi} \hat{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_{\hat{e}} = \hat{\mathbf{q}}_0 \cdot \hat{\mathbf{n}}_{\hat{e}}$  and the proof is completed by using (2.15).  $\square$

**Lemma 3.7.** *In the case of quadrilaterals, for any linear vector  $\mathbf{q}_1$  on  $E$ ,*

$$\forall e \subset \partial E, \quad \Pi \mathbf{q}_1 \cdot \mathbf{n}_e = \mathbf{q}_1 \cdot \mathbf{n}_e. \quad (3.17)$$

*Proof.* Let  $\hat{\mathbf{q}}_1 = J_E D F_E^{-1} \mathbf{q}_1$ . Due to (2.15), it is enough to show on any edge  $\hat{e}$ ,

$$\hat{\Pi} \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{n}}_{\hat{e}} = \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{n}}_{\hat{e}}. \quad (3.18)$$

Let  $\mathbf{q}_1 = [a_1 + b_1 x + c_1 y \ a_2 + b_2 x + c_2 y]^T$  and the constant normal vector  $\mathbf{n}_e = [d_1 \ d_2]^T$ . By (2.15)

$$\hat{\mathbf{q}}_1 \cdot \hat{\mathbf{n}}_{\hat{e}} = |e| \mathbf{q}_1 \cdot \mathbf{n}_e = |e| \sum_{i=1}^2 (a_i + b_i x + c_i y) d_i. \quad (3.19)$$

Then by (2.4), we have  $\hat{\mathbf{q}}_1 \cdot \hat{\mathbf{n}}_{\hat{e}} \in P_1(\hat{e})$ . Now, (2.18) implies (3.18).  $\square$

**Remark 3.2.** *The above result in 3D relies on the key property that the Piola image of a constant vector on a physical element has a linear normal component in the reference element, which belongs to the BDDF<sub>1</sub> space and the enhanced BDDF<sub>1</sub> space. This property does not hold for the RT<sub>0</sub> interpolant, which has a constant normal component on the reference element.*

In the analysis, we will use the following well-known estimates [10]. There exists  $q^1 \in P_1(E)$  such that

$$\|p - q^1\|_{j,E} \lesssim h^{2-j} \|p\|_{2,E}, \quad j = 0, 1, \quad (3.20)$$

and

$$\|p - q^1\|_E \lesssim h \|p\|_{1,E}. \quad (3.21)$$

Using the trace inequality for Lipschitz domains [3, 6],

$$\forall e \in \partial E, \quad \|\phi\|_e \lesssim h^{-1/2} \|\phi\|_E + h^{1/2} |\phi|_{1,E}, \quad \forall \phi \in H^1(E), \quad (3.22)$$

we have the interpolation estimate on  $e$ :

$$\|p - q^1\|_e \lesssim h^{3/2} |p|_{2,E}. \quad (3.23)$$

We also have [18]

$$\|K - \bar{K}_E\|_E \lesssim h \|K\|_{1,E}. \quad (3.24)$$

### 3.2 Coercivity of the velocity bilinear form

For the analysis of non-symmetric MFMFE method we require some properties of the bilinear form  $(K^{-1} \cdot, \cdot)_Q$  defined on the space  $\mathbf{V}_h$ . Note that

$$(K^{-1} \mathbf{q}, \mathbf{v})_Q = \sum_{E \in \mathcal{T}_h} (K^{-1} \mathbf{q}, \mathbf{v})_{Q,E} = \sum_{c \in \mathcal{C}_h} \mathbf{v}_c^T \mathbf{M}_c \mathbf{q}_c = \mathbf{v}_\Omega^T \mathbf{M} \mathbf{q}_\Omega, \quad (3.25)$$

where  $\mathcal{C}_h$  denotes the set of corner or vertex points in  $\mathcal{T}_h$ ,  $\mathbf{v}_c := \{(\mathbf{v} \cdot \mathbf{n}_e)(\mathbf{x}_c)\}_{e=1}^{n_c}$ ,  $\mathbf{x}_c$  is the coordinate vector of point  $c$ , and  $n_c$  is the number of faces (or edges in 2D) that share the vertex point  $c$ . An example of the  $n_c \times n_c$  matrix  $\mathbf{M}_c$  is given in (2.49). In the above,  $\mathbf{v}_\Omega$  is the global vector of values of the normal trace degrees of freedom of  $\mathbf{v}$  in  $\mathbf{V}_h$  and  $\mathbf{M}$  is the block-diagonal matrix with blocks  $\mathbf{M}_c$ ,  $c \in \mathcal{C}_h$ . From (2.30) and (2.8) one can see that

$$\|\mathbf{M}_c\|_{0,\infty,\hat{E}} \lesssim h^d \|K^{-1}\|_{0,\infty,E}. \quad (3.26)$$

**Lemma 3.8.** Assume that  $\mathbf{M}_c$  is uniformly positive definite for all  $c \in \mathcal{C}_h$ :

$$h^d \boldsymbol{\xi}^T \boldsymbol{\xi} \lesssim \boldsymbol{\xi}^T \mathbf{M}_c \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{n_c}. \quad (3.27)$$

Then the bilinear form  $(K^{-1} \cdot, \cdot)_Q$  is coercive in  $\mathbf{V}_h$  and induces a norm in  $\mathbf{V}_h$  equivalent to the  $L^2$ -norm:

$$(K^{-1} \mathbf{v}, \mathbf{v})_Q \asymp \|\mathbf{v}\|^2, \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (3.28)$$

If in addition

$$\boldsymbol{\xi}^T \mathbf{M}_c^T \mathbf{M}_c \boldsymbol{\xi} \lesssim h^{2d} \boldsymbol{\xi}^T \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{n_c}, \quad (3.29)$$

then the following Cauchy-Schwarz type inequality holds:

$$(K^{-1} \mathbf{q}, \mathbf{v})_Q \lesssim \|\mathbf{q}\| \|\mathbf{v}\| \quad \forall \mathbf{q}, \mathbf{v} \in \mathbf{V}_h, \quad (3.30)$$

*Proof.* Let  $\mathbf{v} \in \mathbf{V}_h$ . Using the shape regularity of  $\mathcal{T}_h$ , it is easy to check by direct calculation that for all  $E \in \mathcal{T}_h$

$$\|\mathbf{v}\|_E^2 \lesssim h^d \mathbf{v}_E^T \mathbf{v}_E,$$

where  $\mathbf{v}_E$  is the vector of values of the degrees of freedom of  $\mathbf{v}$  in  $E$ . We then have

$$\|\mathbf{v}\|^2 \lesssim \sum_{E \in \mathcal{T}_h} h^d \mathbf{v}_E^T \mathbf{v}_E \lesssim \sum_{c \in \mathcal{C}_h} h^d \mathbf{v}_c^T \mathbf{v}_c \lesssim \sum_{c \in \mathcal{C}_h} \mathbf{v}_c^T \mathbf{M}_c \mathbf{v}_c = (K^{-1} \mathbf{v}, \mathbf{v})_Q,$$

where we have used (3.27) and (3.25). This implies that  $\mathbf{M}^s = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$  is symmetric and positive definite and that  $(K^{-1} \mathbf{v}, \mathbf{v})_Q = \mathbf{v}_\Omega^T \mathbf{M} \mathbf{v}_\Omega = \mathbf{v}_\Omega^T \mathbf{M}^s \mathbf{v}_\Omega$  is a norm. The equivalence to  $\|\mathbf{v}\|$  follows from (3.26). If (3.29) holds, then (3.30) follows from the Cauchy-Schwarz inequality for the norm induced by the matrix  $\mathbf{M}^T \mathbf{M}$ .  $\square$

In practice, we can verify the assumptions (3.27) and (3.29) by considering only the local  $d \times d$  matrix associated with a vertex of an element  $E$ . In particular,  $\mathbf{M} = \sum_{E \in \mathcal{T}_h} \mathbf{M}_E$ , where  $\mathbf{M}_E$  is a block diagonal matrix with  $2^d$  blocks of size  $d \times d$  corresponding to the vertices of  $E$ . For any vertex  $c$  of  $E$ , denote the associated  $d \times d$  matrix by  $\mathbf{M}_E^c$ . It is easy to check that

$$h^d \boldsymbol{\eta}^T \boldsymbol{\eta} \lesssim \boldsymbol{\eta}^T \mathbf{M}_E^c \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in \mathbb{R}^d \quad (3.31)$$

and

$$\boldsymbol{\eta}^T (\mathbf{M}_E^c)^T \mathbf{M}_E^c \boldsymbol{\eta} \lesssim h^{2d} \boldsymbol{\eta}^T \boldsymbol{\eta}, \quad \forall \boldsymbol{\eta} \in \mathbb{R}^d \quad (3.32)$$

imply (3.27) and (3.29), respectively.

Conditions (3.31) and (3.32) impose restrictions on the element geometry and the anisotropy of the permeability tensor  $K$ . We refer readers to [30, 32] for discussions on these conditions.

For the permeability tensor  $K$ , we will use the following notation. Let  $W_{\mathcal{T}_h}^{\alpha, \infty}$  consist of functions  $\phi$  such that  $\phi|_E \in W^{\alpha, \infty}(E)$  for all  $E \in \mathcal{T}_h$ . Here  $\alpha$  is an integer. Let  $\|\phi\|_{\alpha, \infty} = \max_{E \in \mathcal{T}_h} \|\phi\|_{\alpha, \infty, E}$ .

### 3.3 Convergence of the velocity to the interpolant of the solution

We are now ready to establish first order convergence of the computed velocity to the MFE interpolant of the true solution.

**Theorem 3.1.** *Let  $K \in W_{\mathcal{T}_h}^{1,\infty}(\Omega)$  and  $K^{-1} \in W^{0,\infty}(\Omega)$ . If (3.27) and (3.29) hold, then the velocity  $\mathbf{u}_h$  of the non-symmetric MFMFE method (2.42)–(2.43) satisfies*

$$\|\Pi\mathbf{u} - \mathbf{u}_h\| \lesssim h(|\mathbf{u}|_1 + \|p\|_2), \quad (3.33)$$

*Proof.* Let  $\mathbf{v} = \Pi\mathbf{u} - \mathbf{u}_h$ , and note that (1.2), (2.43), and (2.20) imply that

$$(\nabla \cdot \mathbf{v}, w) = 0 \quad \forall w \in W_h.$$

Taking  $w \in W_h$  such that  $w|_E = J_E \nabla \cdot \mathbf{v}$  for all  $E \in \mathcal{T}_h$  implies that  $\nabla \cdot \mathbf{v} = 0$ . Then from (2.42) we have

$$\begin{aligned} & (K^{-1}(\Pi\mathbf{u} - \mathbf{u}_h), \Pi\mathbf{u} - \mathbf{u}_h)_Q \\ &= (K^{-1}\Pi\mathbf{u}, \mathbf{v})_Q - (p_h, \nabla \cdot \mathbf{v}) = (K^{-1}\Pi\mathbf{u}, \mathbf{v})_Q \\ &= \sum_{E \in \mathcal{T}_h} (K^{-1}\Pi(\mathbf{u} + \bar{K}_E \nabla q^1), \mathbf{v})_{Q,E} - \sum_{E \in \mathcal{T}_h} (K^{-1}\Pi(\bar{K}_E \nabla q^1), \mathbf{v})_{Q,E} \\ &\equiv I_1 + I_2, \end{aligned} \quad (3.34)$$

where  $q^1$  is defined in (3.20). The term  $I_2$  in (3.34) can be manipulated as

$$\begin{aligned} (K^{-1}\Pi(\bar{K}_E \nabla q^1), \mathbf{v})_{Q,E} &= (K^{-1}\bar{K}_E \nabla q^1, \mathbf{v})_{Q,E} = (K^{-1}\bar{K}_E \nabla q^1, \Pi_R \mathbf{v})_{Q,E} \\ &= (\nabla q^1, \Pi_R \mathbf{v})_E, \end{aligned}$$

using (3.16), (2.36) and (2.40). Note that

$$\sum_{E \in \mathcal{T}_h} (\nabla p, \Pi_R \mathbf{v})_E = (\nabla p, \Pi_R \mathbf{v}) = (p, \nabla \cdot \Pi_R \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) = 0,$$

using (2.25) and the fact that  $p = 0$  on  $\partial\Omega$ . Then  $I_2$  can be further written as

$$\begin{aligned} I_2 &= \sum_{E \in \mathcal{T}_h} (\nabla(p - q^1), \Pi_R \mathbf{v})_E \lesssim \sum_{E \in \mathcal{T}_h} \|\nabla(p - q^1)\|_E \|\Pi_R \mathbf{v}\|_E \\ &\lesssim \sum_{E \in \mathcal{T}_h} h \|p\|_{2,E} (\|\mathbf{v}\|_E + h|\mathbf{v}|_{1,E}) \lesssim h \|p\|_2 \|\mathbf{v}\|, \end{aligned} \quad (3.35)$$

where we have used (3.20), (3.12), and (3.1). To bound  $I_1$ , we write

$$\begin{aligned} (K^{-1}\Pi(\mathbf{u} + \bar{K}_E \nabla q^1), \mathbf{v})_{Q,E} &\lesssim \|\Pi(\mathbf{u} + \bar{K}_E \nabla q^1)\|_E \|\mathbf{v}\|_E \\ &\lesssim (\|\mathbf{u} + \bar{K}_E \nabla q^1\|_E + h|\mathbf{u} + \bar{K}_E \nabla q^1|_{1,E}) \|\mathbf{v}\|_E \\ &= (\|\mathbf{u} + \bar{K}_E \nabla q^1\|_E + h|\mathbf{u}|_{1,E}) \|\mathbf{v}\|_E, \end{aligned} \quad (3.36)$$

using (3.30), (3.12), and the fact that  $\bar{K}_E \nabla q^1$  is a constant vector. It remains to estimate  $\|\mathbf{u} + \bar{K}_E \nabla q^1\|_E$ . From (3.20) and (3.24), we have

$$\begin{aligned} \|\mathbf{u} + \bar{K}_E \nabla q^1\|_E &\leq \|(K - \bar{K}_E) \nabla p\|_E + \|\bar{K}_E \nabla(p - q^1)\|_E \\ &\lesssim h(\|\nabla p\|_E + \|p\|_{2,E}) \lesssim h\|p\|_{2,E}. \end{aligned} \quad (3.37)$$

A combination of (3.36) and (3.37) gives

$$|I_1| \lesssim h(\|p\|_2 + |\mathbf{u}|_1)\|\mathbf{v}\|. \quad (3.38)$$

The proof is completed by combining (3.34), (3.35), (3.38), and (3.28).  $\square$

### 3.4 Convergence of the velocity on the element faces

In this section we show that the result from the previous section implies convergence of the computed normal velocity to the true normal velocity on the element faces. We introduce a norm for vectors in  $\Omega$  based on the normal components on the faces of  $\mathcal{T}_h$ :

$$\|\mathbf{v}\|_{\mathcal{F}_h}^2 := \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \frac{|E|}{|e|} \|\mathbf{v} \cdot \mathbf{n}_e\|_e^2, \quad (3.39)$$

where  $|E|$  is the volume of  $E$  and  $|e|$  is the area of  $e$ . This norm gives an appropriate scaling of size of  $|\Omega|$  for a unit vector. The shape regularity assumption on the mesh implies that  $|E|/|e| \asymp h$ , which gives

$$\|\mathbf{v}\|_{\mathcal{F}_h} \asymp h^{1/2} \left( \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \|\mathbf{v} \cdot \mathbf{n}_e\|_e^2 \right)^{1/2}. \quad (3.40)$$

The next result establishes approximation for the enhanced BDDF<sub>1</sub> interpolant in the norm  $\|\cdot\|_{\mathcal{F}_h}$ .

**Lemma 3.9.** *On hexahedra,*

$$\|(\mathbf{u} - \Pi \mathbf{u}) \cdot \mathbf{n}_e\|_e \lesssim h^{1/2} |\mathbf{u}|_{1,E}, \quad (3.41)$$

and

$$\|\mathbf{u} - \Pi \mathbf{u}\|_{\mathcal{F}_h} \lesssim h |\mathbf{u}|_1. \quad (3.42)$$

*On quadrilaterals,*

$$\|(\mathbf{u} - \Pi \mathbf{u}) \cdot \mathbf{n}_e\|_e \lesssim h^{j+1/2} |\mathbf{u}|_{j+1,E}, \quad j = 0, 1 \quad (3.43)$$

and

$$\|\mathbf{u} - \Pi \mathbf{u}\|_{\mathcal{F}_h} \lesssim h^{j+1} |\mathbf{u}|_{j+1}, \quad j = 0, 1. \quad (3.44)$$

*Proof.* Let  $\mathbf{q}$  be any constant vector on  $E$  in the case of hexahedra or a linear vector on  $E$  in the case of quadrilaterals. Lemma 3.6 and Lemma 3.7 imply that

$$\begin{aligned}\|(\mathbf{u} - \Pi\mathbf{u}) \cdot \mathbf{n}_e\|_e &= \|(\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_e - (\Pi\mathbf{u} - \Pi\mathbf{q}) \cdot \mathbf{n}_e\|_e \\ &\leq \|(\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_e\|_e + \|\Pi(\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_e\|_e.\end{aligned}\quad (3.45)$$

By the trace inequality (3.22), we have

$$\|(\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_e\|_e \lesssim h^{-1/2}\|\mathbf{u} - \mathbf{q}\|_E + h^{1/2}|\mathbf{u} - \mathbf{q}|_{1,E}. \quad (3.46)$$

The trace inequality (3.11) implies that

$$\|\Pi(\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_e\|_e \lesssim h^{-1/2}\|\Pi(\mathbf{u} - \mathbf{q})\|_E \lesssim h^{-1/2}\|\mathbf{u} - \mathbf{q}\|_E + h^{1/2}|\mathbf{u} - \mathbf{q}|_{1,E}, \quad (3.47)$$

where we have also used (3.12) in the last inequality. Taking  $\mathbf{q}$  to be the  $L^2$ -projection of  $\mathbf{u}$  into the space of constant vectors on  $E$  in the case of hexahedra or into the space of linear vectors in the case quadrilaterals in (3.45)–(3.47) yields (3.41) and (3.43). Inequalities (3.42) and (3.44) follow from (3.41) and (3.43), respectively, and (3.40).  $\square$

**Remark 3.3.** *The above result for the enhanced BDDF<sub>1</sub> interpolant relies on Lemma 3.6. Therefore, due to Remark 3.2, similar approximation result also holds for the standard BDDF<sub>1</sub> interpolant, but it does not hold for the RT<sub>0</sub> interpolant.*

**Theorem 3.2.** *Let  $K \in W_{T_h}^{1,\infty}(\Omega)$  and  $K^{-1} \in W_{T_h}^{0,\infty}(\Omega)$ . If (3.27) and (3.29) hold, then the velocity  $\mathbf{u}_h$  of the non-symmetric MFMFE method (2.42)–(2.43) satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{F}_h} \lesssim h(\|\mathbf{u}\|_1 + \|p\|_2). \quad (3.48)$$

*Proof.* The triangle inequality gives

$$\begin{aligned}\|(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}_e\|_e &\leq \|(\mathbf{u} - \Pi\mathbf{u}) \cdot \mathbf{n}_e\|_e + \|(\Pi\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}_e\|_e \\ &\lesssim h^{1/2}|\mathbf{u}|_{1,E} + h^{-1/2}\|\Pi\mathbf{u} - \mathbf{u}_h\|_E,\end{aligned}$$

where we have used (3.41) and (3.11) in the second inequality. The assertion of the theorem follows by combining the above inequality with (3.40) and (3.33).  $\square$

**Remark 3.4.** *It is easy to see that the analysis presented in this section is valid also for the method (2.42)–(2.43) using the BDDF<sub>1</sub> space instead of the enhanced BDDF<sub>1</sub> space. Recall that the standard BDDF<sub>1</sub> MFE method does not converge on general hexahedra. Our results indicate that, while the quadrature rule  $(\cdot, \cdot)_Q$  does not lead to a cell-centered finite difference system for the pressure in the case of the BDDF<sub>1</sub> space, it does lead to a method that gives a first order accurate fluxes on the hexahedral faces.*

## 4 Convergence of the pressure

**Lemma 4.1.** *The spaces  $\mathbf{V}_h^R \times W_h$  satisfy the inf-sup condition:*

$$\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h^R} \frac{(\nabla \cdot \mathbf{v}, w)}{\|\mathbf{v}\|_{\text{div}}} \gtrsim \|w\|, \quad \forall w \in W_h. \quad (4.1)$$

*Proof.* It is enough to show that  $\forall w \in W_h$  there exists  $\mathbf{v} \in \mathbf{V}_h^R$  such that

$$(\nabla \cdot \mathbf{v}, w) = \|w\|^2 \quad \text{and} \quad \|\mathbf{v}\|_{\text{div}} \lesssim \|w\|. \quad (4.2)$$

As noted in the proof of Lemma 2.5, for any  $w \in W_h \subset L^2(\Omega)$ , there exists  $\mathbf{q} \in H^1(\Omega)$  such that [26, 25]

$$\nabla \cdot \mathbf{q} = w \quad \text{and} \quad \|\mathbf{q}\|_1 \lesssim \|w\|, \quad (4.3)$$

Using (2.25), we have

$$(\nabla \cdot \Pi_R \mathbf{q}, w) = (\nabla \cdot \mathbf{q}, w) = \|w\|^2. \quad (4.4)$$

Furthermore, (3.12), (3.13), and (4.3) imply that

$$\|\Pi_R \mathbf{q}\|_{\text{div}} \lesssim \|\mathbf{q}\|_1 \lesssim \|w\|. \quad (4.5)$$

Now, (4.4) and (4.5) imply that  $\mathbf{v} = \Pi_R \mathbf{q}$  satisfies (4.2), which completes the proof.  $\square$

**Theorem 4.1.** *Assume that  $K \in W_{T_h}^{1,\infty}(\Omega)$ ,  $K^{-1} \in W_{T_h}^{0,\infty}(\Omega)$ , and that (3.27) and (3.29) hold. Then the pressure  $p_h$  of the non-symmetric MFMFE method (2.42)–(2.43) satisfies*

$$\|p - p_h\| \lesssim h(|\mathbf{u}|_1 + \|p\|_2). \quad (4.6)$$

*Proof.* By Lemma 4.1, we have

$$\|Q_h p - p_h\| \lesssim \sup_{\mathbf{v} \in \mathbf{V}_h^R} \frac{(\nabla \cdot \mathbf{v}, Q_h p - p_h)}{\|\mathbf{v}\|_{\text{div}}}. \quad (4.7)$$

Using (2.27) and (2.42), the numerator can be written as

$$\begin{aligned} (\nabla \cdot \mathbf{v}, Q_h p - p_h) &= (\nabla \cdot \mathbf{v}, p - p_h) = (\nabla \cdot \mathbf{v}, p - q^1) + (\nabla \cdot \mathbf{v}, q^1) - (\nabla \cdot \mathbf{v}, p_h) \\ &= (\nabla \cdot \mathbf{v}, p - q^1) + \sum_{E \in T_h} \sum_{e \in \partial E} \langle q_1, \mathbf{v} \cdot \mathbf{n} \rangle_e + (K^{-1}(\Pi \mathbf{u} - \mathbf{u}_h), \mathbf{v})_Q \\ &\quad - \sum_{E \in T_h} \left( (K^{-1} \Pi \mathbf{u}, \mathbf{v})_{Q,E} + (\nabla q_1, \mathbf{v}) \right)_E \\ &\equiv I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where  $q_1$  is defined by (3.21). Term  $I_1$  is estimated using (3.21):

$$|I_1| \lesssim h\|p\|_1\|\nabla \cdot \mathbf{v}\|. \quad (4.8)$$

Using the continuity of  $p$ , (3.23), and (3.11), term  $I_2$  can be written as

$$\begin{aligned} |I_2| &= \left| \sum_{E \in T_h} \sum_{e \in \partial E} \langle q^1 - p, \mathbf{v} \cdot \mathbf{n} \rangle_e \right| \leq \sum_{E \in T_h} \sum_{e \in \partial E} \|q^1 - p\|_e \|\mathbf{v} \cdot \mathbf{n}\|_e \\ &\lesssim h^{3/2} |p|_2 h^{-1/2} \|\mathbf{v}\| = h|p|_2 \|\mathbf{v}\|. \end{aligned} \quad (4.9)$$

Using (3.30), term  $I_3$  is estimated from Theorem 3.1 as:

$$|I_3| \lesssim h(|\mathbf{u}|_1 + \|p\|_2) \|\mathbf{v}\|. \quad (4.10)$$

Term  $I_4$  can be written as

$$\begin{aligned}
I_4 &= \sum_{E \in \mathcal{T}_h} ((K^{-1} \Pi K \nabla p, \mathbf{v})_{Q,E} - (\nabla q_1, \mathbf{v})_E) \\
&= \sum_{E \in \mathcal{T}_h} (K^{-1} \Pi (K \nabla p - \bar{K}_E \nabla q^1), \mathbf{v})_{Q,E} \\
&\quad + \sum_{E \in \mathcal{T}_h} ((K^{-1} \Pi \bar{K}_E \nabla q_1, \mathbf{v})_{Q,E} - (\nabla q_1, \mathbf{v})_E) \\
&\equiv I_4^a + I_4^b
\end{aligned} \tag{4.11}$$

Using (3.30) and (3.12), we estimate term  $I_4^a$  as

$$\begin{aligned}
|I_4^a| &\lesssim \sum_{E \in \mathcal{T}_h} \|K^{-1} \Pi (K \nabla p - \bar{K}_E \nabla q^1)\|_E \|\mathbf{v}\|_E \\
&\lesssim \sum_{E \in \mathcal{T}_h} (\|K \nabla p - \bar{K}_E \nabla q^1\|_E + h|K \nabla p - \bar{K}_E \nabla q^1|_{1,E}) \|\mathbf{v}\|_E \\
&\lesssim \sum_{E \in \mathcal{T}_h} (\|(K - \bar{K}_E) \nabla p\|_E + \|\bar{K}_E (\nabla p - \nabla q^1)\|_E + h|p|_{2,E}) \|\mathbf{v}\|_E \\
&\lesssim h\|p\|_2 \|\mathbf{v}\|,
\end{aligned} \tag{4.12}$$

where we used (3.24) and (3.20) in the last inequality. Using (3.16) (or (3.17) for 2D) and Lemma 2.4, we have that

$$I_4^b = (K^{-1} \bar{K}_E \nabla q_1, \mathbf{v})_Q - (\nabla q_1, \mathbf{v}) = 0. \tag{4.13}$$

A combination of (4.7)-(4.13) and (2.28) completes the proof.  $\square$

## 5 Numerical Experiments

In this section, we test both the symmetric and the non-symmetric MFMFE methods on  $h^2$ -parallelograms and  $h^2$ -parallelepipeds as well as on highly distorted quadrilaterals and hexahedra. The velocity error  $\|\Pi \mathbf{u} - \mathbf{u}_h\|$  is approximated by the trapezoidal quadrature rule. The face error  $\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{F}_h}$  is computed by a high order Gaussian quadrature rule. The pressure error  $\|p - p_h\|$  is the discrete  $L^2$ -norm computed by the midpoint quadrature rule:  $\|p - p_h\|^2 \equiv \sum_{E \in \mathcal{T}_h} |E| (p - p_h)^2(m_e)$ , where  $m_e$  is the center of mass of the element  $E$ . We also report convergence in the discrete flux error

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{F}_h}^2 \equiv \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} |E| \left( \frac{1}{|e|} \int_e \mathbf{u} \cdot \mathbf{n}_e - \frac{1}{|e|} \int_e \mathbf{u}_h \cdot \mathbf{n}_e \right)^2.$$

The resulting linear algebraic system is solved using the software HYPRE (high performance preconditioners) developed by researchers at Lawrence Livermore National Laboratory <sup>1</sup>.

---

<sup>1</sup><https://computation.llnl.gov/casc/hypre/software.html>

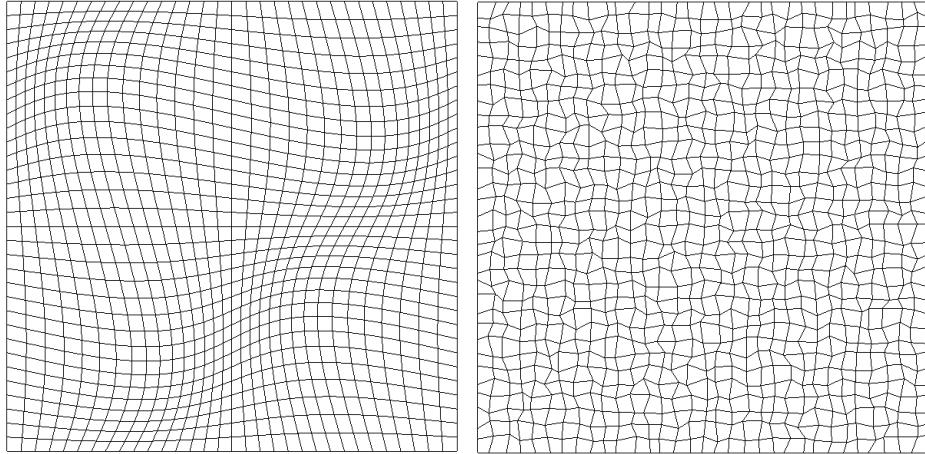


Figure 5: Smooth quadrilateral mesh (left) and randomly  $h$ -perturbed quadrilateral mesh (right).

### 5.1 2D examples

We consider two types of meshes - smooth and non-smooth. The smooth mesh is a  $C^\infty$  map of a uniform mesh on the unit square. The map is defined as

$$\begin{aligned} x &= \hat{x} + 0.06 \sin(2\pi\hat{x}) \sin(2\pi\hat{y}), \\ y &= \hat{y} - 0.05 \sin(2\pi\hat{x}) \sin(2\pi\hat{y}). \end{aligned}$$

The mesh is shown on the left in Fig. 5. It is easy to check that in this case the elements are  $h^2$ -parallelograms. The second mesh consists of highly distorted quadrilaterals generated by perturbing the uniform mesh points in a random direction by a distance of size  $\mathcal{O}(h)$ , see the right Fig. 5. The grids on the different levels of refinement are obtained by mapping or perturbing refinements of the uniform grid.

We consider problem (1.1)–(1.2) with a given analytical solution

$$p(x, y) = \sin(3\pi x)^2 \sin(3\pi y)^2,$$

and full permeability tensor

$$\mathbf{K} = \begin{pmatrix} 2 & 1.25 \\ 1.25 & 3 \end{pmatrix}.$$

The boundary conditions are of Dirichlet type.

We first test the convergence of  $\|\mathbf{u} - \Pi \mathbf{u}\|_{\mathcal{F}_h}$ . The results for  $\mathbf{u} = -K \nabla p$ , where  $p$  and  $K$  are defined above, are given in Table 1. We observe second order convergence on both the smooth grids and the rough grids, which confirms the theoretical results in Section 3.4. The slight variation in the convergence rate on the randomly perturbed grids is due to varying shape regularity constants on the different grid levels.

Next, we test the convergence of the MFMFE method. Table 2 shows the numerical results on  $h^2$ -parallelogram meshes. As the theory predicts for the symmetric MFMFE method [44] and the non-symmetric MFMFE method, we observe first order convergence

Table 1: Interpolation error  $\|\mathbf{u} - \Pi\mathbf{u}\|_{\mathcal{F}_h}$  on the 2D meshes

| $1/h$      | Mesh 1   |      | Mesh 2   |      |
|------------|----------|------|----------|------|
| <b>8</b>   | 3.20E+00 | —    | 3.43E+00 | —    |
| <b>16</b>  | 8.64E-01 | 1.89 | 9.30E-01 | 1.88 |
| <b>32</b>  | 2.20E-01 | 1.97 | 2.47E-01 | 1.91 |
| <b>64</b>  | 5.54E-02 | 1.99 | 5.34E-02 | 2.21 |
| <b>128</b> | 1.39E-02 | 1.99 | 1.38E-02 | 1.95 |
| <b>256</b> | 3.47E-03 | 2.00 | 3.80E-03 | 1.86 |

 Table 2: Convergence in the 2D example on  $h^2$ -parallelogram grids

| Symmetric MFMFE |               |      |                                    |      |   |      |
|-----------------|---------------|------|------------------------------------|------|---|------|
| $1/h$           | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>8</b>        | 4.70E-1       | —    | 1.02E+1                            | —    | 1.09E+1   | —    |
| <b>16</b>       | 2.03E-1       | 1.21 | 4.38E+0                            | 1.22 | 4.08E+0   | 1.42 |
| <b>32</b>       | 9.67E-2       | 1.07 | 2.19E+0                            | 1.00 | 1.82E+0   | 1.16 |
| <b>64</b>       | 4.79E-2       | 1.01 | 1.10E+0                            | 0.99 | 8.85E-1   | 1.04 |
| <b>128</b>      | 2.39E-2       | 1.00 | 5.50E-1                            | 1.00 | 4.40E-1   | 1.01 |
| <b>256</b>      | 1.19E-2       | 1.01 | 2.75E-1                            | 1.00 | 2.19E-1   | 1.01 |

| Non-symmetric MFMFE |               |      |                                    |      |   |      |
|---------------------|---------------|------|------------------------------------|------|---|------|
| $1/h$               | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>8</b>            | 4.70E-1       | —    | 1.03E+1                            | —    | 1.08E+1   | —    |
| <b>16</b>           | 2.03E-1       | 1.21 | 4.35E+0                            | 1.24 | 4.02E+0   | 1.43 |
| <b>32</b>           | 9.67E-2       | 1.07 | 2.17E+0                            | 1.00 | 1.80E+0   | 1.16 |
| <b>64</b>           | 4.79E-2       | 1.01 | 1.09E+0                            | 0.99 | 8.77E-1   | 1.04 |
| <b>128</b>          | 2.39E-2       | 1.00 | 5.45E-1                            | 1.00 | 4.36E-1   | 1.01 |
| <b>256</b>          | 1.19E-2       | 1.01 | 2.73E-1                            | 1.00 | 2.17E-1   | 1.01 |

| Symmetric (left) and Non-symmetric (right) MFMFE |               |      |   |      |               |   |
|--|---------------|------|---|------|---------------|---|
| $1/h$  | $\ p - p_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      | $\ p - p_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |
| <b>8</b>   | 2.79E-1       | —    | 8.24E+0   | —    | 2.78E-1       | —   |
| <b>16</b>  | 5.24E-2       | 2.41 | 2.40E+0   | 1.78 | 5.25E-2       | 2.40  |
| <b>32</b>  | 1.21E-2       | 2.11 | 6.17E-1   | 1.96 | 1.21E-2       | 2.12  |
| <b>64</b>  | 2.97E-3       | 2.03 | 1.56E-1   | 1.98 | 2.98E-3       | 2.02  |
| <b>128</b>                                       | 7.40E-4       | 2.00 | 3.92E-2   | 1.99 | 7.41E-4       | 2.01  |
| <b>256</b>                                       | 1.85E-4       | 2.00 | 9.82E-3   | 2.00 | 1.85E-4       | 2.00  |

Table 3: Convergence in the 2D example on randomly  $h$ -perturbed quadrilateral grids

| Symmetric MFMFE |               |      |                                    |      |   |      |
|-----------------|---------------|------|------------------------------------|------|---|------|
| $1/h$           | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>8</b>        | 6.13E-1       | –    | 1.13E+1                            | –    | 1.30E+1   | –    |
| <b>16</b>       | 2.14E-1       | 1.52 | 4.82E+0                            | 1.23 | 4.53E+0   | 1.52 |
| <b>32</b>       | 9.97E-2       | 1.10 | 2.76E+0                            | 0.80 | 2.46E+0   | 0.88 |
| <b>64</b>       | 4.96E-2       | 1.01 | 1.94E+0                            | 0.51 | 1.48E+0   | 0.73 |
| <b>128</b>      | 2.51E-2       | 0.98 | 1.69E+0                            | 0.20 | 1.34E+0   | 0.14 |
| <b>256</b>      | 1.33E-2       | 0.92 | 1.60E+0                            | 0.08 | 1.40E+0   | < 0  |

| Non-symmetric MFMFE |               |      |                                    |      |   |      |
|---------------------|---------------|------|------------------------------------|------|---|------|
| $1/h$               | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>8</b>            | 6.15E-1       | –    | 1.14E+1                            | –    | 1.26E+1   | –    |
| <b>16</b>           | 2.18E-1       | 1.50 | 4.74E+0                            | 1.27 | 4.18E+0   | 1.59 |
| <b>32</b>           | 1.00E-1       | 1.12 | 2.38E+0                            | 0.99 | 1.95E+0   | 1.10 |
| <b>64</b>           | 4.97E-2       | 1.01 | 1.18E+0                            | 1.01 | 8.02E-1   | 1.28 |
| <b>128</b>          | 2.48E-2       | 1.00 | 5.92E-1                            | 1.00 | 4.07E-1   | 0.98 |
| <b>256</b>          | 1.24E-2       | 1.00 | 2.97E-1                            | 1.00 | 2.24E-1   | 0.86 |

| Symmetric (left) and Non-symmetric (right) MFMFE |               |      |   |      |               |   |
|--|---------------|------|---|------|---------------|---|
| $1/h$  | $\ p - p_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      | $\ p - p_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |
| <b>8</b>   | 3.67E-1       | –    | 1.15E+1   | –    | 3.62E-1       | –   |
| <b>16</b>  | 6.88E-2       | 2.42 | 3.10E+0   | 1.89 | 7.54E-2       | 2.26  |
| <b>32</b>  | 1.46E-2       | 2.24 | 1.18E+0   | 1.39 | 1.41E-2       | 2.42  |
| <b>64</b>  | 5.59E-3       | 1.39 | 5.89E-1   | 1.00 | 3.19E-3       | 2.14  |
| <b>128</b>                                       | 5.08E-3       | 0.14 | 4.85E-1   | 0.28 | 7.63E-4       | 2.06  |
| <b>256</b>                                       | 5.13E-3       | < 0  | 4.99E-1   | < 0  | 2.17E-4       | 1.81  |
|  |               |      |   |      |               |   |
|  |               |      |   |      |               |   |

for the pressure and the velocity. In addition, we obtain second order superconvergence for the face flux and the pressure at the cell centers.

Table 3 demonstrates the convergence behavior for the randomly perturbed grids of size  $\mathcal{O}(h)$ . The numerical results show that the non-symmetric MFMFE method has first order convergence for both the velocity and the pressure on these highly distorted grids. Clearly the convergence of the velocity and the pressure of the symmetric MFMFE method deteriorates.

## 5.2 3D examples

In the 3D examples we consider four hexahedral meshes. The first one is generated using a smooth map of a uniform grid given by

$$\begin{aligned} x &= \hat{x} + 0.03 \sin(3\pi\hat{x}) \cos(3\pi\hat{y}) \cos(3\pi\hat{z}), \\ y &= \hat{y} - 0.04 \cos(3\pi\hat{x}) \sin(3\pi\hat{y}) \cos(3\pi\hat{z}), \\ z &= \hat{z} + 0.05 \cos(3\pi\hat{x}) \cos(3\pi\hat{y}) \sin(3\pi\hat{z}). \end{aligned}$$

This mapping yields an  $h^2$ -parallelepiped mesh. We choose the second mesh to be an  $h$ -perturbed grid given in [38]. The third mesh is generated by random perturbation of uniform mesh points by a distance of size  $\mathcal{O}(h)$ . The fourth mesh is a  $h$ -perturbed fishbone-like mesh from [24]. Element faces are planar in the second mesh and non-planar in the other three meshes. The four meshes are shown in Fig. 6.

We consider problem (1.1)–(1.2) with a given analytical solution

$$p(x, y, z) = x^2(x-1)^2y^2(y-1)^2z^2(1-z)^2,$$

and a full permeability tensor

$$\mathbf{K} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The convergence of  $\|\mathbf{u} - \Pi\mathbf{u}\|_{\mathcal{F}_h}$ ,  $\mathbf{u} = -K\nabla p$ , on the four grids is presented in Table 4. We observe first order convergence for the  $h$ -perturbed grids, as predicted by the theory. Moreover, second order convergence is obtained for the smooth grids. This can be proven by a scaling argument using the fact that the BDDF<sub>1</sub> interpolant preserves the normal components of linear vectors on the faces of the reference element. The scaling estimates developed in [28] for  $h^2$ -perturbed grids allow for the scaling argument to be employed without loss of accuracy. We note that such scaling argument would give no convergence on rough grids.

Tables 5–8 demonstrate the convergence behavior of both the symmetric and the non-symmetric MFMFE methods on the four hexahedral meshes described above.

The non-symmetric MFMFE method developed in this paper is robust for all four meshes. In particular, it exhibits first order convergence for both velocity and pressure. These numerical results confirm the theory established in Theorems 3.1, 3.2, and 4.1. Furthermore, the computational results indicate second order superconvergence for pressure at the cell centers on all four meshes and for the face fluxes on the smooth mesh.

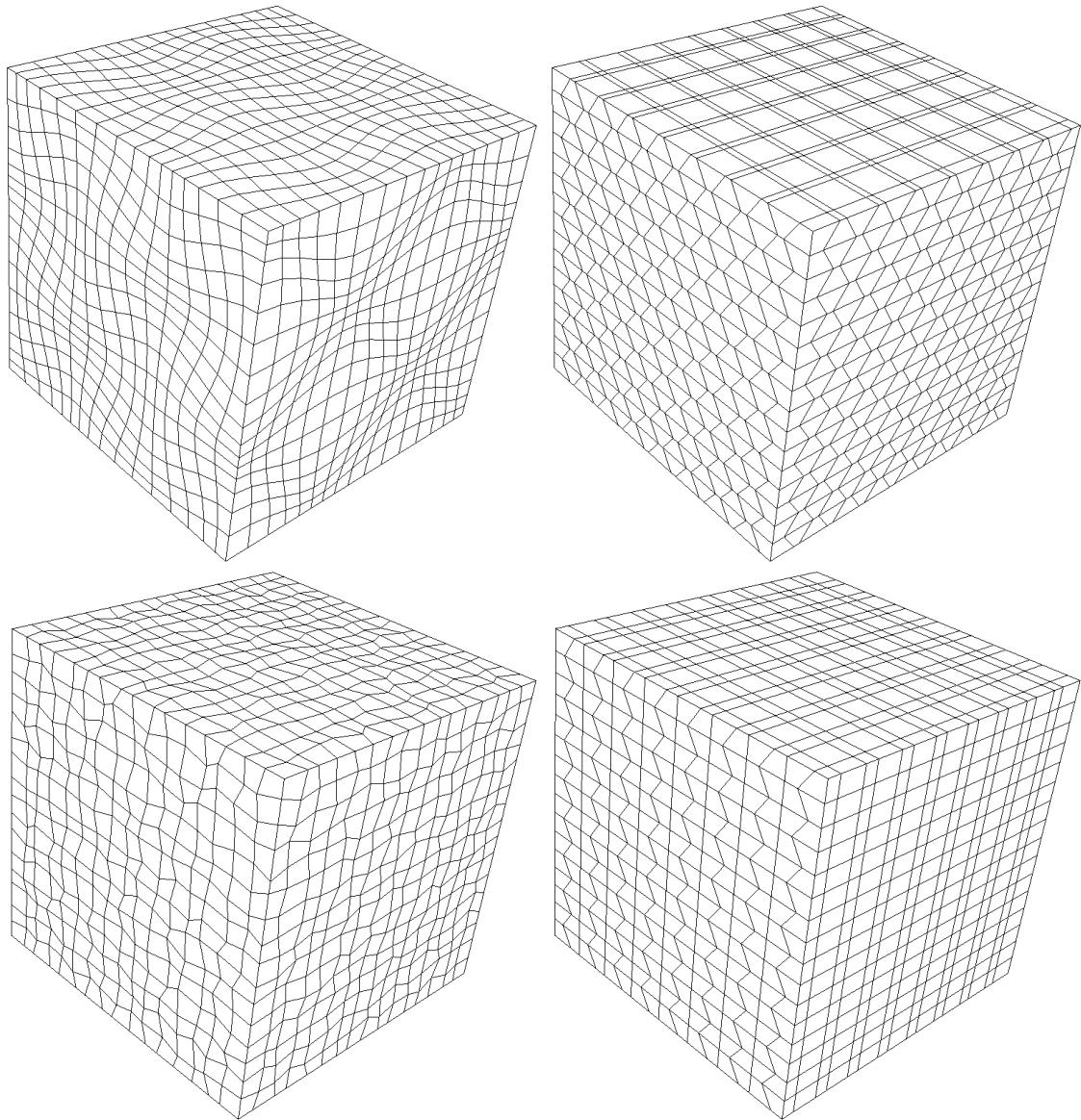


Figure 6: The four 3D meshes:  $h^2$ -perturbed hexahedral mesh (top-left),  $h$ -perturbed mesh [38] (top-right), randomly  $h$ -perturbed hexahedral mesh (bottom-left), and  $h$ -perturbed mesh [24] (bottom-right).

Table 4: Interpolation error  $\|\mathbf{u} - \Pi\mathbf{u}\|_{\mathcal{F}_h}$  on the 3D meshes

| $1/h$      | Mesh 1   |      | Mesh 2   |      | Mesh 3   |      | Mesh 4   |      |
|------------|----------|------|----------|------|----------|------|----------|------|
| <b>4</b>   | 1.21E-04 | –    | 1.67E-04 | –    | 1.36E-04 | –    | 1.60E-04 | –    |
| <b>8</b>   | 3.20E-05 | 1.92 | 7.20E-05 | 1.21 | 4.04E-05 | 1.75 | 5.65E-05 | 1.50 |
| <b>16</b>  | 7.26E-06 | 2.14 | 3.40E-05 | 1.08 | 1.57E-05 | 1.36 | 2.29E-05 | 1.30 |
| <b>32</b>  | 1.72E-06 | 2.08 | 1.69E-05 | 1.01 | 8.57E-06 | 0.87 | 1.06E-05 | 1.11 |
| <b>64</b>  | 4.22E-07 | 2.03 | 8.51E-06 | 0.99 | 3.98E-06 | 1.11 | 5.21E-06 | 1.02 |
| <b>128</b> | 1.05E-07 | 2.01 | 4.28E-06 | 0.99 | 2.04E-06 | 0.96 | 2.59E-06 | 1.01 |

Table 5: Convergence in the first 3D example on smooth hexahedral grids

| Symmetric MFMFE |               |      |                                    |      |   |      |
|-----------------|---------------|------|------------------------------------|------|---|------|
| $1/h$           | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>4</b>        | 5.05E-5       | –    | 4.36E-4                            | –    | 4.41E-4   | –    |
| <b>8</b>        | 2.49E-5       | 1.02 | 2.29E-4                            | 0.93 | 1.83E-4   | 1.27 |
| <b>16</b>       | 1.23E-5       | 1.02 | 1.18E-4                            | 0.96 | 7.78E-5   | 1.23 |
| <b>32</b>       | 6.16E-6       | 1.00 | 5.97E-5                            | 0.98 | 3.59E-5   | 1.12 |
| <b>64</b>       | 3.08E-6       | 1.00 | 3.00E-5                            | 0.99 | 1.75E-5   | 1.04 |

| Non-symmetric MFMFE |               |      |                                    |      |   |      |
|---------------------|---------------|------|------------------------------------|------|---|------|
| $1/h$               | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
| <b>4</b>            | 5.05E-5       | –    | 4.37E-4                            | –    | 4.43E-4   | –    |
| <b>8</b>            | 2.51E-5       | 1.01 | 2.07E-4                            | 1.08 | 1.68E-4   | 1.40 |
| <b>16</b>           | 1.24E-5       | 1.02 | 1.01E-4                            | 1.04 | 6.65E-5   | 1.34 |
| <b>32</b>           | 6.16E-6       | 1.01 | 5.03E-5                            | 1.01 | 3.01E-5   | 1.14 |
| <b>64</b>           | 3.08E-6       | 1.00 | 2.51E-5                            | 1.00 | 1.46E-5   | 1.04 |

| Symmetric (left) and Non-symmetric (right) MFMFE |               |      |   |      |               |   |
|--|---------------|------|---|------|---------------|---|
| $1/h$  | $\ p - p_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      | $\ p - p_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |
| <b>4</b>   | 1.61E-5       | –    | 3.23E-4   | –    | 1.61E-5       | –   |
| <b>8</b>   | 4.31E-6       | 1.90 | 9.79E-5   | 1.72 | 4.26E-6       | 1.92  |
| <b>16</b>  | 1.19E-6       | 1.86 | 2.69E-5   | 1.86 | 1.11E-6       | 1.94  |
| <b>32</b>  | 3.12E-7       | 1.93 | 6.87E-6   | 1.97 | 2.83E-7       | 1.97  |
| <b>64</b>  | 7.91E-8       | 1.98 | 1.73E-6   | 1.99 | 7.10E-8       | 1.99  |

Table 6: Second 3D example on an  $h$ -perturbed grid

| Symmetric MFMFE |               |                                    |   |      |         |      |
|-----------------|---------------|------------------------------------|---|------|---------|------|
| $1/h$           | $\ p - p_h\ $ | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |         |      |
| <b>4</b>        | 6.55E-5       | —                                  | 5.49E-4   | —    | 4.83E-4 | —    |
| <b>8</b>        | 3.26E-5       | 1.01                               | 3.13E-4   | 0.81 | 2.12E-4 | 1.19 |
| <b>16</b>       | 1.63E-5       | 1.00                               | 1.62E-4   | 0.95 | 1.02E-4 | 1.06 |
| <b>32</b>       | 8.16E-6       | 1.00                               | 8.46E-5   | 0.94 | 5.16E-5 | 0.98 |
| <b>64</b>       | 4.08E-6       | 1.00                               | 4.35E-5   | 0.96 | 2.63E-5 | 0.97 |

| Non-symmetric MFMFE |               |                                    |   |      |         |      |
|---------------------|---------------|------------------------------------|---|------|---------|------|
| $1/h$               | $\ p - p_h\ $ | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |         |      |
| <b>4</b>            | 6.10E-5       | —                                  | 5.90E-4   | —    | 4.72E-4 | —    |
| <b>8</b>            | 3.22E-5       | 0.92                               | 3.55E-4   | 0.73 | 2.25E-4 | 1.07 |
| <b>16</b>           | 1.63E-5       | 0.98                               | 1.87E-4   | 0.92 | 1.13E-4 | 0.99 |
| <b>32</b>           | 8.15E-6       | 1.00                               | 9.64E-5   | 0.96 | 5.73E-5 | 0.98 |
| <b>64</b>           | 4.07E-6       | 1.00                               | 4.89E-5   | 0.98 | 2.89E-5 | 0.99 |

| Symmetric (left) and Non-symmetric (right) MFMFE |               |   |               |   |         |      |
|--|---------------|---|---------------|---|---------|------|
| $1/h$  | $\ p - p_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ | $\ p - p_h\ $ | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |         |      |
| <b>4</b>   | 2.09E-5       | —   | 3.71E-4       | —   | 1.76E-5 | —    |
| <b>8</b>   | 5.38E-6       | 1.96  | 1.23E-4       | 1.59  | 4.63E-6 | 1.93 |
| <b>16</b>  | 1.36E-6       | 1.98  | 4.72E-5       | 1.38  | 1.18E-6 | 1.97 |
| <b>32</b>  | 3.45E-7       | 1.98  | 2.12E-5       | 1.15  | 2.98E-7 | 1.99 |
| <b>64</b>  | 8.69E-8       | 1.99  | 1.03E-5       | 1.04  | 7.47E-8 | 2.00 |

Table 7: Third 3D example on a randomly  $h$ -perturbed grid

Symmetric MFMFE

| $1/h$     | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|-----------|---------------|------|------------------------------------|------|---|------|
| <b>4</b>  | 5.29E-5       | –    | 4.96E-4                            | –    | 4.88E-4   | –    |
| <b>8</b>  | 2.59E-5       | 1.03 | 2.55E-4                            | 0.96 | 1.92E-4   | 1.35 |
| <b>16</b> | 1.27E-5       | 1.03 | 1.69E-4                            | 0.59 | 1.19E-4   | 0.69 |
| <b>32</b> | 6.48E-6       | 0.97 | 1.35E-4                            | 0.32 | 1.12E-4   | 0.09 |
| <b>64</b> | 3.51E-6       | 0.88 | 1.27E-4                            | 0.09 | 1.01E-4   | 0.15 |

Non-symmetric MFMFE

| $1/h$     | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|-----------|---------------|------|------------------------------------|------|---|------|
| <b>4</b>  | 5.31E-5       | –    | 5.03E-4                            | –    | 4.84E-4   | –    |
| <b>8</b>  | 2.62E-5       | 1.02 | 2.31E-4                            | 1.12 | 1.73E-4   | 1.48 |
| <b>16</b> | 1.28E-5       | 1.03 | 1.16E-4                            | 0.99 | 8.00E-5   | 1.11 |
| <b>32</b> | 6.39E-6       | 1.00 | 5.85E-5                            | 0.99 | 4.72E-5   | 0.76 |
| <b>64</b> | 3.19E-6       | 1.00 | 2.94E-5                            | 0.99 | 2.27E-5   | 1.06 |

Symmetric (left) and Non-symmetric (right) MFMFE

| $1/h$     | $\ p - p_h\ $ |      |         | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |         |      | $\ p - p_h\ $ |      |         | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |         |      |
|-----------|---------------|------|---------|---|---------|------|---------------|------|---------|---|---------|------|
| <b>4</b>  | 1.62E-5       | –    | 3.68E-4 | –   | 1.67E-5 | –    | 3.61E-4       | –    | 3.61E-4 | –   | 3.61E-4 | –    |
| <b>8</b>  | 4.31E-6       | 1.91 | 1.06E-4 | 1.80  | 4.23E-6 | 1.98 | 9.47E-5       | 1.93 | 9.47E-5 | 1.93  | 9.47E-5 | 1.93 |
| <b>16</b> | 1.82E-6       | 1.24 | 5.43E-5 | 0.97  | 1.11E-6 | 1.93 | 3.30E-5       | 1.52 | 3.30E-5 | 1.52  | 3.30E-5 | 1.52 |
| <b>32</b> | 1.52E-6       | 0.26 | 4.90E-5 | 0.15  | 2.81E-7 | 1.98 | 1.70E-5       | 0.96 | 1.70E-5 | 0.96  | 1.70E-5 | 0.96 |
| <b>64</b> | 1.54E-6       | <0   | 4.39E-5 | 0.16  | 7.06E-8 | 1.99 | 7.82E-6       | 1.12 | 7.82E-6 | 1.12  | 7.82E-6 | 1.12 |

Table 8: Fourth 3D example on a  $h$ -perturbed grid

Symmetric MFMFE

| $1/h$     | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|-----------|---------------|------|------------------------------------|------|---|------|
| <b>4</b>  | 5.99E-5       | —    | 5.34E-4                            | —    | 5.42E-4   | —    |
| <b>8</b>  | 2.73E-5       | 1.13 | 2.43E-4                            | 1.14 | 2.21E-4   | 1.29 |
| <b>16</b> | 1.33E-5       | 1.04 | 1.20E-4                            | 1.02 | 1.03E-4   | 1.10 |
| <b>32</b> | 6.63E-6       | 1.00 | 6.01E-5                            | 1.00 | 5.01E-5   | 1.04 |
| <b>64</b> | 3.31E-6       | 1.00 | 3.00E-5                            | 1.00 | 2.49E-5   | 1.01 |

Non-symmetric MFMFE

| $1/h$     | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|-----------|---------------|------|------------------------------------|------|---|------|
| <b>4</b>  | 5.91E-5       | —    | 5.70E-4                            | —    | 5.51E-4   | —    |
| <b>8</b>  | 2.72E-5       | 1.12 | 2.57E-4                            | 1.15 | 2.26E-4   | 1.29 |
| <b>16</b> | 1.33E-5       | 1.03 | 1.28E-4                            | 1.01 | 1.05E-4   | 1.11 |
| <b>32</b> | 6.63E-6       | 1.00 | 6.36E-5                            | 1.01 | 5.14E-5   | 1.03 |
| <b>64</b> | 3.31E-6       | 1.00 | 3.18E-5                            | 1.00 | 2.55E-5   | 1.01 |

Symmetric (left) and Non-symmetric (right) MFMFE

| $1/h$     | $\ p - p_h\ $ |      |         | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |         |      | $\ p - p_h\ $ |      |         | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |
|-----------|---------------|------|---------|---|---------|------|---------------|------|---------|---|
| <b>4</b>  | 1.79E-5       | —    | 4.03E-4 | —   | 1.70E-5 | —    | 4.02E-4       | —    | 4.02E-4 | —   |
| <b>8</b>  | 4.81E-6       | 1.90 | 1.24E-4 | 1.70  | 4.71E-6 | 1.85 | 1.24E-4       | 1.70 | 1.24E-4 | 1.70  |
| <b>16</b> | 1.22E-6       | 1.98 | 4.32E-5 | 1.52  | 1.20E-6 | 1.97 | 4.22E-5       | 1.56 | 4.22E-5 | 1.56  |
| <b>32</b> | 3.06E-7       | 2.00 | 1.81E-5 | 1.26  | 3.02E-7 | 1.99 | 1.74E-5       | 1.28 | 1.74E-5 | 1.28  |
| <b>64</b> | 7.65E-8       | 2.00 | 8.46E-6 | 1.10  | 7.56E-8 | 2.00 | 8.07E-6       | 1.11 | 8.07E-6 | 1.11  |

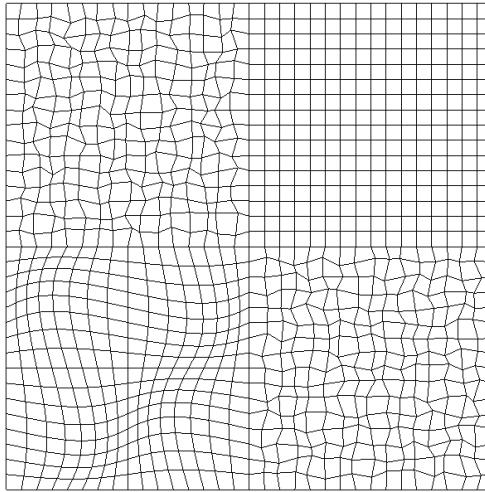


Figure 7: A “coupled” mesh consists of  $h^2$ -perturbed and randomly  $h$ -perturbed quadrilaterals, and uniform rectangles

The symmetric MFMFE method on  $h^2$ -parallelepipeds has been proven to have first order convergence for velocity and pressure [28]. Table 5 confirms this theoretical result. We also observe similar convergence on the second and fourth meshes, see Table 6 and Table 8, respectively. However, on the randomly perturbed mesh, the convergence of the velocity and the pressure in the symmetric MFMFE method deteriorates, see Table 7.

### 5.3 A coupled scheme

Both the theory and the numerical examples confirm that the symmetric MFMFE method is accurate on  $h^2$ -parallelograms and  $h^2$ -parallelepipeds, and the non-symmetric MFMFE method is accurate on general quadrilateral and hexahedral grids. Recall that the distinction between these two methods lies in the numerical quadrature, which is defined at the element level. Thus the symmetric and the non-symmetric MFMFE methods can be easily coupled element by element. In particular, one can select the symmetric quadrature rule for  $h^2$ -perturbed elements and the non-symmetric quadrature rule for highly distorted elements. To test this concept, we choose the “coupled” mesh as shown in Fig. 7 consisting of the randomly  $h$ -perturbed and the  $h^2$ -perturbed meshes from Fig. 5, as well as a uniform mesh. The analytical solution, permeability, and boundary condition are the same as in Section 5.1. Table 9 demonstrates that the coupled method has first order convergence for the pressure and the velocity and has second order convergence for the pressure at the cell centers.

## 6 Conclusions

In this paper, a non-symmetric MFMFE method on both quadrilaterals and hexahedra is developed based on  $\text{BDM}_1$  and enhanced  $\text{BDDF}_1$  spaces, and a special quadrature rule. This method gives a cell-centered pressure stencil by locally eliminating the velocity degrees

Table 9: Symmetric MFMFE coupled with Non-symmetric MFMFE

| $1/h$      | $\ p - p_h\ $ |      | $\ \Pi\mathbf{u} - \mathbf{u}_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|------------|---------------|------|------------------------------------|------|---|------|
| <b>4</b>   | 3.40E+0       | –    | 4.52E+1                            | –    | 5.86E+1   | –    |
| <b>8</b>   | 5.96E-1       | 2.51 | 1.02E+1                            | 2.15 | 1.18E+1   | 2.31 |
| <b>16</b>  | 2.03E-1       | 1.55 | 4.37E+0                            | 1.22 | 4.00E+0   | 1.56 |
| <b>32</b>  | 9.72E-2       | 1.06 | 2.16E+0                            | 1.02 | 1.75E+0   | 1.19 |
| <b>64</b>  | 4.79E-2       | 1.02 | 1.11E+0                            | 0.96 | 8.58E-1   | 1.03 |
| <b>128</b> | 2.39E-2       | 1.00 | 5.53E-1                            | 1.01 | 4.25E-1   | 1.01 |
| <b>256</b> | 1.20E-2       | 0.99 | 2.77E-1                            | 1.00 | 2.13E-1   | 1.00 |

| $1/h$      | $\ p - p_h\ $ |      | $\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{F}_h}$ |      |
|------------|---------------|------|---|------|
| <b>4</b>   | 3.04E+0       | –    | 5.47E+1   | –    |
| <b>8</b>   | 3.96E-1       | 2.94 | 9.82E+0   | 2.48 |
| <b>16</b>  | 4.77E-2       | 3.05 | 2.49E+0   | 1.98 |
| <b>32</b>  | 1.17E-2       | 2.03 | 7.07E-1   | 1.82 |
| <b>64</b>  | 2.91E-3       | 2.01 | 2.39E-1   | 1.56 |
| <b>128</b> | 7.73E-4       | 1.91 | 1.00E-1   | 1.26 |
| <b>256</b> | 1.84E-4       | 2.07 | 4.75E-2   | 1.07 |

of freedom. On highly distorted quadrilateral and hexahedral grids, the non-symmetric MFMFE method gives better approximation than the symmetric formulation from [44, 28]. In particular, the non-symmetric MFMFE method exhibits first order convergence for pressure and normal velocities on faces, as well as second order superconvergence for the pressure at the cell centers. Furthermore, second order superconvergence is observed for the face fluxes on smooth meshes. The method can handle non-planar hexahedral faces. Since the symmetric and non-symmetric MFMFE methods are locally defined, they can be coupled so that the non-symmetric method is applied to highly distorted elements. Affine elements can also be coupled with quadrilateral and hexahedral elements. We remark that based on accurate velocities on faces one can obtain accurate velocity inside elements by applying post-processing techniques.

## References

- [1] I. AAVATSMARK, *An introduction to multipoint flux approximations for quadrilateral grids*, Comp. Geosci., 6 (2002), pp. 405–432.
- [2] I. AAVATSMARK, T. BARKVE, O. BOE, AND T. MANNSETH, *Discretization on unstructured grids for inhomogeneous, anisotropic media. Part II: Discussion and numerical results*, SIAM J. Sci. Comput., 19 (1998), pp. 1717–1736.

- [3] S. AGMON, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, NJ, 1965.
- [4] T. ARBOGAST, C. N. DAWSON, P. T. KEENAN, M. WHEELER, AND I. YOTOV, *Enhanced cell-centered finite differences for elliptic equations on general geometry*, SIAM J. Sci. Comput., 19 (1998), pp. 404–425.
- [5] T. ARBOGAST, M. WHEELER, AND I. YOTOV, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.
- [6] D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.
- [7] D. N. ARNOLD, D. BOFFI, AND R. S. FALK, *Approximation by quadrilateral finite elements*, Math. Comp., 71 (2002), pp. 909–922.
- [8] ———, *Quadrilateral  $H(\text{div})$  finite elements*, SIAM J. Numer. Anal., 42 (2005), pp. 2429–2451.
- [9] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates*, Math. Mod. Numer. Anal., 19 (1985), pp. 7–32.
- [10] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Texts in Applied Mathematics, Springer-Verlag, 2007.
- [11] F. BREZZI, J. DOUGLAS, R. DURAN, AND M. FORTIN, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.
- [12] F. BREZZI, J. DOUGLAS, AND L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.
- [13] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
- [14] F. BREZZI, M. FORTIN, AND L. D. MARINI, *Error analysis of piecewise constant pressure approximations of Darcy’s law*, Comput. Methods Appl. Mech. Eng., 195 (2006), pp. 1547–1559.
- [15] F. BREZZI, K. LIPNIKOV, AND M. SHASHKOV, *Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes*, SIAM J. Numer. Anal., 43 (2005), pp. 1872–1896.
- [16] F. BREZZI, K. LIPNIKOV, AND M. SHASHKOV, *Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes with curved faces*, Math. Mod. Methods Appl. Sci., 16 (2006), pp. 275–297.
- [17] Z. CHEN AND J. DOUGLAS, *Prismatic mixed finite elements for second order elliptic problems*, Calcolo, 26 (1989), pp. 135–148.

- [18] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, Stud. Math. Appl. 4, North-Holland, Amsterdam, 1978; reprinted, SIAM, Philadelphia, 2002.
- [19] J. DOUGLAS AND J. WANG, *A new family of mixed finite element spaces over rectangles*, Math. Appl. Comput., 12 (1993), pp. 183–197.
- [20] M. G. EDWARDS, *Unstructured control-volume distributed, full-tensor finite-volume schemes with flow based grids*, Comp. Geosci., 6 (2002), pp. 433–452.
- [21] M. G. EDWARDS AND C. F. ROGERS, *Finite volume discretization with imposed flux continuity for the general tensor pressure equation*, Comp. Geosci., 2 (1998), pp. 259–290.
- [22] R. E. EWING, M. M. LIU, AND J. WANG, *Superconvergence of mixed finite element approximations over quadrilaterals*, SIAM. J. Numer. Anal., 36 (1999), pp. 772–787.
- [23] R. EYMARD, T. GALLOUËT, AND R. HERBIN, *Finite volume methods*, in Handbook of numerical analysis, Vol. VII, Handb. Numer. Anal., VII, North-Holland, Amsterdam, 2000, pp. 713–1020.
- [24] R. S. FALK, P. GATTO, AND P. MONK, *Hexahedral  $H(\text{div})$  and  $H(\text{curl})$  finite elements*, To appear in Math. Mod. Numer. Anal., (2010).
- [25] G. P. GALDI, *An Introduction to the Mathematical Theory of the Navier-Stokes equations*, vol. I, Springer-Verlag, New York, 1994. Linearized steady problems.
- [26] V. GIRAUT AND P. A. RAVIART, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer-Verlag, 1986.
- [27] J. HYMAN, M. SHASHKOV, AND S. STEINBERG, *The numerical solution of diffusion problem in strongly heterogeneous non-isotropic materials*, J. Comp. Phys., 132 (1997), pp. 130–148.
- [28] R. INGRAM, M. WHEELER, AND I. YOTOV, *A multipoint flux mixed finite element method on hexahedra*, To appear in SIAM J. Numer. Anal., (2010).
- [29] R. A. KLAUSEN AND R. WINther, *Convergence of multipoint flux approximations on quadrilateral grids*, Numer. Methods Partial Differential Equations, 22 (2006), pp. 1438–1454.
- [30] R. A. KLAUSEN AND R. WINther, *Robust convergence of multi point flux approximation on rough grids*, Numer. Math., 104 (2006), pp. 317–337.
- [31] Y. KUZNETSOV AND S. REPIN, *Convergence analysis and error estimates for mixed finite element method on distorted meshes*, J. Numer. Math., 13 (2005), pp. 33–51.
- [32] K. LIPNIKOV, M. SHASHKOV, AND I. YOTOV, *Local flux mimetic finite difference methods*, Numer. Math., 112 (2009), pp. 115–152.
- [33] J. E. MARSDEN AND T. J. HUGHES, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, NJ, 1983.

- [34] R. NAFF, T. RUSSELL, AND J. D. WILSON, *Shape functions for velocity interpolation in general hexahedral cells*, Comp. Geosci., 6 (2002), pp. 285–314.
- [35] J. C. NEDELEC, *Mixed finite elements in  $R^3$* , Numer. Math., 35 (1980), pp. 315–341.
- [36] P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2-nd order elliptic problems*, in Mathematical aspects of the Finite Elements Method, I. Galligani and E. Magenes, eds., Lectures Notes in Math. 606, Springer, Berlin, 1977, pp. 292–315.
- [37] T. RUSSELL AND M. WHEELER, *Finite element and finite difference method for continuous flows in porous media*, Frontiers in Applied Mathematics, 1 (1983), pp. 35–106.
- [38] A. SBOUI, J. JAFFRE, AND J. ROBERTS, *A composite mixed finite elements for hexahedral grids*, SIAM. J. Sci. Comput., 31 (2009), pp. 2623–2645.
- [39] J. SHEN, *Mixed finite element methods on distorted rectangular grids*, Technical Report, Texas A & M University, (1994).
- [40] J. M. THOMAS, *These de Doctorat d'etat*, ‘a l’Universite Pierre et Marie Curie, Paris, 1977.
- [41] J. WANG AND T. MATHEW, *Mixed finite element method over quadrilaterals*, Advances in numerical methods and applications: proceedings of third international conference. River edge, NJ: World Scientific, (1994), pp. 203–214.
- [42] A. WEISER AND M. WHEELER, *On convergence of block-centered finite differences for elliptic problems*, SIAM J. Numer. Anal., 25 (1988), pp. 351–375.
- [43] M. WHEELER, G. XUE, AND I. YOTOV, *A multiscale mortar multipoint flux mixed finite element method*, Submitted, (2010).
- [44] M. WHEELER AND I. YOTOV, *A multipoint flux mixed finite element method*, SIAM. J. Numer. Anal., 44 (2006), pp. 2082–2106.