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Variational formulations for time–harmonic Maxwell equations

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Abstract

The subject of this article is a review of all possible transmission problems for electromagnetic phenomena. In particular, we study the case of a perfect dielectric and a perfect conductor via a (formal) limit with conductivity approaching zero or infinity, and discuss the expected regularity of the involved unknowns. Finally, we formulate equivalent variational formulations for each considered problem.

Contents

1 Overview

The paper is organized as follows. After having introduced the necessary notational framework, we turn our attention to electromagnetic phenomena and the partial differential equations describing them: Maxwell's equations. This is done in section 2. Section 3 and section 4 focus on the analysis of transmission problems. The goal of each section is to obtain an equivalent variational formulation for as few unknowns as possible. We start with the natural transmission problem between two conducting materials and deduce the equations describing the other transmission problems out of this. From a mathematical point of view, the governing equations modeling a perfect electric conductor or perfectly dielectric media may then be obtained via a limiting process.

Before we look at mathematics, we need to agree on some notations. We define the following vector spaces of L^2-integrable functions

$$
\begin{aligned}
H^1_{loc}(\mathbb{R}^3) &= \left\{v \in L^2_{loc}(\mathbb{R}^3) : \ \ \nabla v \in L^2_{loc}(\mathbb{R}^3)^3\right\} &(1)\\
H_{loc}(\nabla\times, \mathbb{R}^3) &= \left\{\boldsymbol{v} \in L^2_{loc}(\mathbb{R}^3)^3 : \ \ \nabla \times \boldsymbol{v} \in L^2_{loc}(\mathbb{R}^3)^3\right\} &(2)\\
H_{loc}(\nabla\cdot, \mathbb{R}^3) &= \left\{\boldsymbol{v} \in L^2_{loc}(\mathbb{R}^3)^3 : \ \ \nabla \cdot \boldsymbol{v} \in L^2_{loc}(\mathbb{R}^3)\right\} . &(3)
\end{aligned}
$$

We need further the following two definitions

Definition 1.1 *1. Angular brackets $\langle \cdot\, , \, \cdot \rangle$ denote a duality pairing between the space of test functions $\mathscr{D}(\mathbb{R}^3)$ and the its topological dual - the space of distribution $\mathscr{D}(\mathbb{R}^3)'$. By definition, $\mathscr{D}(\mathbb{R}^3)'$ comprises all continuous functionals on $\mathscr{D}(\mathbb{R}^3)$. If the functional can be identified with an L^2_{loc}-function, the duality pairing reduces to the integral,*

$$
v \in L^2_{loc}(\mathbb{R}^3), \varphi \in \mathscr{D}(\mathbb{R}^3) : \qquad \langle v, \varphi\rangle = \int_{\mathbb{R}^3} v\varphi \, d\boldsymbol{x} = \int_{\Omega_1} v\varphi \, d\boldsymbol{x} + \int_{\Omega_2} v\varphi \, d\boldsymbol{x} \,,
$$

for an arbitrary decomposition $\mathbb{R}^3 = \Omega_1 \cup \Gamma \cup \Omega_2$.

2. Let $\boldsymbol{n}$ be the outer normal of the boundary of a Lipschitz domain $\Omega_1 \subset \mathbb{R}^3$ and $\mathbb{R}^3 = \Omega_1 \cup \Gamma \cup \Omega_2$. For $\boldsymbol{u} \in H(\nabla\cdot, \mathbb{R}^3)$ we define

$$
\boldsymbol{u}_1 = \boldsymbol{u}|_{\Omega_1} \qquad \boldsymbol{u}_2 = \boldsymbol{u}|_{\Omega_2} \,.
$$

We define the jump in the normal trace as

$$
[\gamma_{\boldsymbol{n}} \boldsymbol{u}] = \gamma_{\boldsymbol{n}} \boldsymbol{u}_1 - \gamma_{\boldsymbol{n}} \boldsymbol{u}_2 \,.
$$

Analogously, we define for $\boldsymbol{u} \in H(\nabla\times, \mathbb{R}^3)$

$$
[\gamma_D \boldsymbol{u}] = \boldsymbol{n} \times \boldsymbol{u}_1 - \boldsymbol{n} \times \boldsymbol{u}_2 \,.
$$

2 Maxwell's equations

Maxwell's equations are a set of eight partial differential equations that describe the interaction of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ within arbitrary medium and relate them to their sources, charge density and current density.

$$
\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}\\
\nabla \times \boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} + \boldsymbol{J}^{imp}\\
\nabla \cdot \boldsymbol{D} &= \rho + \rho^{imp}\\
\nabla \cdot \boldsymbol{B} &= 0 \,.
\end{aligned}
$$

The equations are known as Faraday's law of induction, Ampère's law with Maxwell's correction and the two Gauß' laws for the electric and the magnetic field.

We restrict our analysis to time harmonic electromagnetic waves where the time dependency of all quantities is harmonic

$$
\boldsymbol{v}(\boldsymbol{x}, t) = \tilde{\boldsymbol{v}}(\boldsymbol{x}) e^{j\omega t}
$$

and $\omega > 0$ denotes the angular frequency. In this case Maxwell's equations simplify to

$$
\begin{aligned}
\nabla \times \tilde{\boldsymbol{E}} &= -j\omega \tilde{\boldsymbol{B}}\\
\nabla \times \tilde{\boldsymbol{H}} &= j\omega \tilde{\boldsymbol{D}} + \tilde{\boldsymbol{J}} + \tilde{\boldsymbol{J}}^{imp}\\
\nabla \cdot \tilde{\boldsymbol{D}} &= \tilde{\rho} + \tilde{\rho}^{imp}\\
\nabla \cdot \tilde{\boldsymbol{B}} &= 0 \,.
\end{aligned}
$$

To simplify the notation, we will drop the tildes in what follows.

The Maxwell's equations are completed with relations between $\boldsymbol{D}$ and $\boldsymbol{E}$, and $\boldsymbol{B}$ and $\boldsymbol{H}$. The so called constitutive relations correspond physically to specifying the response of bound charge and bound current to the fields. They describe how much polarization and magnetization a material acquires in the presence of an electromagnetic field [5].

In a linear, isotropic, nondispersive, uniform material, the relations are:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \qquad\qquad \boldsymbol{B} = \mu \boldsymbol{H}\,.$$

A lossy medium comprises free, mobile charge. The electric current caused by these moving electrons depends on material properties and on the present electric field. For linear material one assumes the validity of Ohm's law:

$$\boldsymbol{J} = \sigma \boldsymbol{E}\,.$$

The constitutive laws yield

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} + \boldsymbol{J}^{imp} \\ \nabla \cdot \varepsilon \boldsymbol{E} &= \rho + \rho^{imp} \\ \nabla \cdot \mu \boldsymbol{H} &= 0 \\ \boldsymbol{J} &= \sigma \boldsymbol{E}\,. \end{aligned}$$

Notice that we have used the constitutive equations to eliminate the electric and magnetic fluxes but have included the Ohm's law explicitly. This reflects the plan to keep the permittivity ε and permeability μ fixed, but let conductivity $\sigma \to 0$ or $\sigma \to \infty$.

What makes Maxwell's equations difficult is that none of these equations is independent. To end up with as many equations as unknowns we need to clarify their mutual dependence.

In what concerns the impressed sources we assume the so-called continuity equation to hold true

$$-\nabla \cdot \boldsymbol{J}^{imp} = j\omega\rho^{imp}\,.$$

Thus, we notice that the validity of

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega\mu \boldsymbol{H} && (4) \\ \nabla \times \boldsymbol{H} &= j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} + \boldsymbol{J}^{imp} && (5) \\ \nabla \cdot \varepsilon \boldsymbol{E} &= \rho + \rho^{imp} && (6) \\ \boldsymbol{J} &= \sigma \boldsymbol{E} && (7) \end{aligned}$$

yield formally Gauß' law for the magnetic field and the continuity equation that links the unknowns ρ and $\boldsymbol{J}$:

$$\begin{aligned} \nabla \cdot \mu \boldsymbol{H} &= 0 && (8) \\ -\nabla \cdot \boldsymbol{J} &= j\omega\rho\,. && (9) \end{aligned}$$

The continuity equation ties together dynamics and statics, it says that the only source of an electric current $\boldsymbol{J}$ is a variation of a free charge density ρ in time.

Although we complete our list of equations always by (8) and (9) we keep in mind that (4)-(7) yield 10 (scalar) equations for (scalar) 10 unknowns $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \rho$.

As every media is lossy, it is reasonable to analyze Maxwell's equations in lossy media first and deduce all other scenarios from this case.

Let $\mathbb{R}^3$ be partitioned into two disjoint, open domains Ω_1, Ω_2 with common boundary Γ, $\mathbb{R}^3 = \Omega_1 \cup \Gamma \cup \Omega_2$. Unless otherwise stated, we assume the permeability μ as well as the permittivity ε to piecewise continuous, bounded and bounded away from zero:

$$\begin{aligned} \mu_i &\in \mathscr{C}(\Omega_i), \quad 0 < \mu_0 \le \mu_i \le \mu_1 < \infty, \quad i = 1,2 \\ \varepsilon_i &\in \mathscr{C}(\Omega_i), \quad 0 < \varepsilon_0 \le \varepsilon_i \le \varepsilon_1 < \infty, \quad i = 1,2\,. \end{aligned} \qquad (10)$$

The conductivity is supposed to be piecewise constant such that

$$\sigma = \begin{cases} 0 < \sigma_1 < \infty, & \Omega_1 \\ 0 < \sigma_2 < \infty, & \Omega_2 \end{cases} \quad . \tag{11}$$

Thus, we are looking for

$$\boldsymbol{E} = \begin{cases} \boldsymbol{E}_1, & \Omega_1 \\ \boldsymbol{E}_2, & \Omega_2 \end{cases} \qquad \boldsymbol{H} = \begin{cases} \boldsymbol{H}_1, & \Omega_1 \\ \boldsymbol{H}_2, & \Omega_2 \end{cases} \qquad \boldsymbol{J} = \begin{cases} \boldsymbol{J}_1, & \Omega_1 \\ \boldsymbol{J}_2, & \Omega_2 \end{cases} \qquad \rho = \begin{cases} \rho_1, & \Omega_1 \\ \rho_2, & \Omega_2 \end{cases} \tag{12}$$

in all over $\mathbb{R}^3$ with:

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} + \boldsymbol{J}^{imp} \\ \nabla \cdot \varepsilon\boldsymbol{E} &= \rho + \rho^{imp} \\ \boldsymbol{J} &= \sigma\boldsymbol{E} \\ \\ \nabla \cdot \mu\boldsymbol{H} &= 0 \\ -\nabla \cdot \boldsymbol{J} &= j\omega\rho \,, \end{aligned} \tag{13}$$

where the two last equations are implicitly satisfied.

We will begin with the fundamental questions:

- In what function spaces do we look for a solution?
- What do we call a solution?
- Is it possible to further reduce the number of unknowns?

2.1 Mathematical Setting

Before we can formulate a mathematical task we have to find a functional setting that suits the problem. In what concerns the impressed sources we assume throughout the paper

$$\rho^{imp} \in L^2_{loc}(\mathbb{R}^3)\,, \quad \boldsymbol{J}^{imp} \in L^2_{loc}(\mathbb{R}^3)^3 \quad \text{with} \quad -\nabla \cdot \boldsymbol{J}^{imp} = j\omega\rho^{imp}, \quad [\gamma_n \boldsymbol{J}^{imp}] = 0\,. \tag{14}$$

Finite energy considerations of the electromagnetic field lead to the following requirement [8]

$$\forall \text{ compact} K \subset \mathbb{R}^3 : \qquad \left| \int_K \boldsymbol{E} \times \boldsymbol{H} \, d\boldsymbol{x} \right| < \infty\,.$$

Thus, it is natural to require L^2_{loc}-integrability of $\boldsymbol{E}$ and $\boldsymbol{H}$. Faraday's law implies then the L^2_{loc}-integrability of $\nabla \times \boldsymbol{E}$. We look thus for a solution in the following function spaces:

$$(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \rho) \in \Big(H_{loc}(\nabla\times, \mathbb{R}^3) \times L^2_{loc}(\mathbb{R}^3)^3 \times \left(\mathscr{D}(\mathbb{R}^3)^3\right)' \times \mathscr{D}(\mathbb{R}^3)' \Big)\,. \tag{15}$$

As long as the conductivities are bounded $\sigma_1, \sigma_2 < \infty$ our functional setting allows for more conclusions:

1. Ohm's law tells us that $\boldsymbol{J}$ is in $L^2_{loc}(\mathbb{R}^3)^3$, and the magnetic field is as regular as the electric field $\boldsymbol{H} \in H(\nabla\times, \mathbb{R}^3)$.
2. Moreover, due to Faraday's law we know implicitly

$$\text{for a.e.} \quad \boldsymbol{x} \in \mathbb{R}^3 : \quad \nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} \quad \Rightarrow \; \mu\boldsymbol{H} \in H_{loc}(\nabla\cdot, \mathbb{R}^3)\,.$$

We summarize: a solution to (13) satisfies for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$, $\psi \in \mathscr{D}(\mathbb{R}^3)$

$$\begin{aligned}
\int_{\mathbb{R}^3} (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{F}\, d\boldsymbol{x} + \langle [\gamma_D \boldsymbol{E}]\delta_\Gamma, \boldsymbol{F}\rangle &= -j\omega \int_{\mathbb{R}^3} (\mu \boldsymbol{H}) \cdot \boldsymbol{F}\, d\boldsymbol{x} \\
\int_{\mathbb{R}^3} (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{F}\, d\boldsymbol{x} + \langle [\gamma_D \boldsymbol{H}]\delta_\Gamma, \boldsymbol{F}\rangle &= j\omega \int_{\mathbb{R}^3} (\varepsilon \boldsymbol{E}) \cdot \boldsymbol{F}\, d\boldsymbol{x} + \int_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{F}\, d\boldsymbol{x} + \int_{\mathbb{R}^3} \boldsymbol{J}^{imp} \cdot \boldsymbol{F}\, d\boldsymbol{x} \\
-\int_{\mathbb{R}^3} (\varepsilon \boldsymbol{E}) \cdot \nabla\psi\, d\boldsymbol{x} &= \langle \rho, \psi\rangle + \int_{\mathbb{R}^3} \rho^{imp} \psi\, d\boldsymbol{x} \\
\int_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{F}\, d\boldsymbol{x} &= \int_{\mathbb{R}^3} \sigma \boldsymbol{E} \cdot \boldsymbol{F}\, d\boldsymbol{x} \\
\int_{\mathbb{R}^3} (\nabla \cdot \mu \boldsymbol{H})\, \psi\, d\boldsymbol{x} - \langle [\gamma_{\boldsymbol{n}} \mu \boldsymbol{H}]\delta_\Gamma, \psi\rangle &= 0 \\
\int_{\mathbb{R}^3} \boldsymbol{J} \cdot \nabla\psi\, d\boldsymbol{x} &= j\omega \langle \rho, \psi\rangle\,.
\end{aligned}$$

Matching terms yields the transmission conditions

$$\begin{array}{lrclcrcl}
\forall\, \boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3: & \langle [\gamma_D \boldsymbol{E}]\delta_\Gamma, \boldsymbol{F}\rangle &=& 0 & \Rightarrow & [\gamma_D \boldsymbol{E}] &=& \boldsymbol{0} \\
\forall\, \boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3: & \langle [\gamma_D \boldsymbol{H}]\delta_\Gamma, \boldsymbol{F}\rangle &=& 0 & \Rightarrow & [\gamma_D \boldsymbol{E}] &=& \boldsymbol{0} \\
\forall\, \psi \in \mathscr{D}(\mathbb{R}^3): & \langle [\gamma_{\boldsymbol{n}} \mu \boldsymbol{H}]\delta_\Gamma, \psi\rangle &=& 0 & \Rightarrow & [\gamma_{\boldsymbol{n}} \mu \boldsymbol{H}] &=& 0\,.
\end{array} \tag{16}$$

We emphasize that there is no information about the regularity of $\nabla \cdot (\varepsilon \boldsymbol{E})$. Consequently, Gauß' law has to be understood in distributional sense and we cannot conclude continuity of the normal component of the electric flux $\varepsilon \boldsymbol{E}$. This will be illustrated with simple examples at the end of section 3. In general, the normal component of the electric flux may jump at the boundary

$$\forall\, \psi \in \mathscr{D}(\mathbb{R}^3): \qquad \langle [\gamma_{\boldsymbol{n}} \varepsilon \boldsymbol{E}]\delta_\Gamma, \psi\rangle \neq 0 \quad \Rightarrow \quad [\gamma_{\boldsymbol{n}} \varepsilon \boldsymbol{E}] = \rho_\Gamma\,. \tag{17}$$

and therefore $\rho \in \mathscr{D}(\mathbb{R}^3)'$ exhibits, in general, a surface charge contribution.

Due to the functional setting, some of the equations in (13) are going to be satisfied in a weak sense and others pointwise [4]. We rewrite setting (15) in a more compact form,

$$\begin{aligned}
\langle \nabla \times \boldsymbol{E}, \boldsymbol{F}\rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F}\rangle, \quad [\gamma_D \boldsymbol{E}] = \boldsymbol{0} \\
\langle \boldsymbol{H}, \nabla \times \boldsymbol{F}\rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F}\rangle + \langle \boldsymbol{J}, \boldsymbol{F}\rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle \\
-\langle \varepsilon \boldsymbol{E}, \nabla\psi\rangle &= \langle \rho, \psi\rangle + \langle \rho^{imp}, \psi\rangle \\
\langle \boldsymbol{J}, \boldsymbol{F}\rangle &= \langle \sigma \boldsymbol{E}, \boldsymbol{F}\rangle \\
\\
-\langle \mu \boldsymbol{H}, \nabla\psi\rangle &= 0 \\
\langle \boldsymbol{J}, \nabla\psi\rangle &= j\omega \langle \rho, \psi\rangle\,.
\end{aligned}$$

By spacing the two sets of equations, we emphasize that the first four equations imply automatically the last two.

3 Transmission between two conductors

We recall what we have learned about a Maxwell solution for the transmission problem between two conductors.

$$
\begin{array}{c}
0 < \sigma_1 < \infty,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ conductor } 1 \\
\hline
0 < \sigma_2 < \infty,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ conductor } 2
\end{array}
$$

Definition 3.1 *For the conductor/conductor problem with* $0 < \sigma_1, \sigma_2 < \infty$*, a Maxwell solution* $\boldsymbol{E} \in H_{loc}(\nabla\times, \mathbb{R}^3)$*,* $\boldsymbol{H}, \boldsymbol{J} \in L^2_{loc}(\mathbb{R}^3)^3, \rho \in \mathscr{D}(\mathbb{R}^3)'$ *satisfies for all* $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$*,* $\psi \in \mathscr{D}(\mathbb{R}^3)$ *the equations:*

$$
\begin{aligned}
\langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle & (18)\\
\langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle & (19)\\
-\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle &= \langle \rho, \psi \rangle + \langle \rho^{imp}, \psi \rangle & (20)\\
\langle \boldsymbol{J}, \boldsymbol{F} \rangle &= \langle \sigma \boldsymbol{E}, \boldsymbol{F} \rangle & (21)
\end{aligned}
$$

$$
\begin{aligned}
-\langle \mu \boldsymbol{H}, \nabla \psi \rangle &= 0 & (22)\\
\langle \boldsymbol{J}, \nabla \psi \rangle &= j\omega \langle \rho, \psi \rangle\,. & (23)
\end{aligned}
$$

We refer to them as Faraday's law (18)*, Ampère's law* (19)*, Gauß' law* (20)*, Ohm's law* (21)*, Gauß' law for the magnetic field* (22) *and the continuity equation* (23)*.*

The goal of this section is to set up a reduced system of equations for less unknowns, namely $\boldsymbol{E}, \boldsymbol{J}$ and ρ. A closer look at Faraday's law (18) and Ampère's law (19) clarifies where this idea comes from: we can substitute one equation into the other and eliminate one quantity [3], [7]. The new system is shown to yield a Maxwell solution in the sense of definition 3.1.

Our functional setting is such that we fulfill Faraday's law (18) almost everywhere. Provided $\omega > 0$, we can substitute the magnetic field

$$
\boldsymbol{H} = (-j\omega)^{-1} (\mu)^{-1} \nabla \times \boldsymbol{E}
$$

into Ampères's law (19) to obtain

$$
\int\limits_{\mathbb{R}^3} \left(\mu^{-1} \nabla \times \boldsymbol{E}\right) \cdot \nabla \times \boldsymbol{F}\, d\boldsymbol{x} = \omega^2 \int\limits_{\mathbb{R}^3} \varepsilon \boldsymbol{E} \cdot \boldsymbol{F}\, d\boldsymbol{x} - j\omega \int\limits_{\mathbb{R}^3} \boldsymbol{J} \cdot \boldsymbol{F}\, d\boldsymbol{x} - j\omega \int\limits_{\mathbb{R}^3} \boldsymbol{J}^{imp} \cdot \boldsymbol{F}\, d\boldsymbol{x}\,.
$$

In its abbreviated version the second order equation reads

$$
\langle \mu^{-1} \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \rangle = \omega^2 \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle - j\omega \langle \boldsymbol{J}, \boldsymbol{F} \rangle - j\omega \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle\,. \tag{24}
$$

Lemma 3.2 *If* $\boldsymbol{E} \in H_{loc}(\nabla\times, \mathbb{R}^3)$ *satisfies for all* $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$

$$
(-j\omega)^{-1} \langle \mu^{-1} \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \rangle = j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle - j\omega \langle \sigma \boldsymbol{E}, \boldsymbol{F} \rangle - j\omega \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle
$$

and we define $\boldsymbol{J}$ *and* ρ *by*

$$
\begin{aligned}
\forall\ \boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3 \quad \langle \sigma \boldsymbol{E}, \boldsymbol{F} \rangle &= \langle \boldsymbol{J}, \boldsymbol{F} \rangle\,,\\
\forall\ \varphi \in \mathscr{D}(\mathbb{R}^3) \quad j\omega \langle \rho, \psi \rangle &= \langle \boldsymbol{J}, \nabla \psi \rangle
\end{aligned}
$$

then $\boldsymbol{J} \in L^2_{loc}(\mathbb{R}^3)^3, \rho \in \mathscr{D}(\mathbb{R}^3)'$ *and* $(\boldsymbol{E}, \boldsymbol{J}, \rho)$ *is a Maxwell solution in the sense of definition* 3.1.

Proof: The only equation that is left to check is Gauß' law. The special choice of test functions $\boldsymbol{F} = \nabla \psi \in \mathscr{D}(\mathbb{R}^3)^3$ yields

$$
-\omega^2 \langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle + j\omega \langle \sigma \boldsymbol{E}, \nabla \psi \rangle + j\omega \langle \boldsymbol{J}^{imp}, \nabla \psi \rangle = 0\,.
$$

Under consideration of (14) and the definition of ρ, we end up with Gauß' law

$$
-\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle = \langle \rho, \psi \rangle + \langle \rho^{imp}, \psi \rangle\,.
$$

□

Example 3.3 *Let us assume the material data is piecewise constant and that the electric field has zero divergence, i.e.* $\nabla \cdot \boldsymbol{E}_i = 0$ *in* $\Omega_i, \quad \rho_i = \rho^{imp} = 0,\ i = 1, 2.$ *For the special choice of test functions* $\boldsymbol{F} = \nabla\psi \in \mathscr{D}(\mathbb{R}^3)^3$ *the second order equation* (24) *shrinks to*

$$j\omega\langle (j\omega\varepsilon + \sigma)\boldsymbol{E}, \nabla\psi\rangle = 0\,.$$

Thus, the following identity holds true

$$j\omega(\varepsilon_1 \boldsymbol{n}\cdot\boldsymbol{E}_1 - \varepsilon_2 \boldsymbol{n}\cdot\boldsymbol{E}_2) + (\sigma_1 \boldsymbol{n}\cdot\boldsymbol{E}_1 - \sigma_2 \boldsymbol{n}\cdot\boldsymbol{E}_2) = 0\,.$$

Unless the material data and the electric field fulfill

$$\begin{aligned} \frac{\varepsilon_1}{\varepsilon_2} &= \frac{\sigma_1}{\sigma_2}\,, \\ \boldsymbol{n}\cdot\boldsymbol{E}_1 &= \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{n}\cdot\boldsymbol{E}_2\,, \end{aligned} \tag{25}$$

the normal components of the electric flux and the normal components of the electric current jump

$$\rho_\Gamma = \varepsilon_1 \boldsymbol{n}\cdot\boldsymbol{E}_1 - \varepsilon_2 \boldsymbol{n}\cdot\boldsymbol{E}_2 \quad \Rightarrow \quad \sigma_1 \boldsymbol{n}\cdot\boldsymbol{E}_1 - \sigma_2 \boldsymbol{n}\cdot\boldsymbol{E}_2 = -j\omega\rho_\Gamma\,.$$

Example 3.4 *Appendix* A *is devoted to a detailed study of uniform plane waves. In section* A.3*, we use the analytical solution of the scattering problem to affirm what we learned form example* 3.3*: unless* (25) *holds, the oblique incidence of an parallel polarized plane wave causes an surface charge distribution on the interface between two conducting medias.*

4 Transmission problems

The transmission between two conductors serves now as a starting point for all other scenarios. We introduce the characteristic functions.

Definition 4.1 *Let* $\mathbb{R} = \Omega_1 \cup \Gamma \cup \Omega_2$.

$$\chi_1(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x}\in\Omega_1 \\ 0, & \boldsymbol{x}\in\Omega_2 \end{cases} \qquad \chi_2(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x}\in\Omega_1 \\ 1, & \boldsymbol{x}\in\Omega_2 \end{cases}\,.$$

Perfect dielectric. We say that domain Ω_2 is a perfect dielectric if $\sigma_2 \to 0$, say in L^∞-norm. If the corresponding electric field stays uniformly bounded in L^2_{loc}, then the corresponding current $J_2 \to 0$ in the same norm. This implies that the corresponding free charge $\rho_2 \to 0$ *in the sense of distributions*. This does not imply that that the corresponding free charge $\rho_2 \to 0$ in L^2_{loc}. For this, $J_2 \to 0$ must converge to zero in the stronger $H_{loc}(\nabla\cdot, \Omega_2)$-norm. For instance, function $f_n(x) = \frac{1}{n}\sin nx$ converges uniformly to zero, its derivative $f_n'(x) = \cos nx$ converges to zero in the sense of distributions but, obviously, not in the L^2_{loc} sense.

In the limit, $\boldsymbol{J} = \boldsymbol{J}_1\chi_1$. Under the additional assumption that $\boldsymbol{J}_1 \in H_{loc}(\nabla\cdot, \Omega_1)$, well known trace theorem allow us to conclude

$$\begin{aligned} \nabla\cdot\boldsymbol{J} &= j\omega\rho_1\chi_1 + j\omega\rho_\Gamma\delta_\Gamma\,, \\ j\omega\rho_\Gamma &= \boldsymbol{n}\cdot\boldsymbol{J}_1\,. \end{aligned}$$

Consulting the continuity equation for this case, we learn that $\nabla\cdot\boldsymbol{J}$ exhibits the surface charge density at the interface Γ

$$\forall\psi\in\mathscr{D}(\mathbb{R}^3): \quad \langle \boldsymbol{J}, \nabla\psi\rangle = j\omega\langle\rho_1\chi_1, \psi\rangle + j\omega\langle\rho_\Gamma\delta_\Gamma, \psi\rangle\,.$$

It is important to notice

1. The electromagnetic field does not vanish anywhere, neither in Ω_1 nor in Ω_2.
2. The unknown charge density reads $\rho = \rho_1\chi_1 + \rho_\Gamma\delta_\Gamma$.
3. Due to the regularity assumption about the current we also have $\varepsilon_1\boldsymbol{E}_1 \in H(\nabla\cdot, \Omega_1)$ and
$$0 = j\omega[\varepsilon E_n] + [J_n] = j\omega[\varepsilon E_n] - \boldsymbol{n}\cdot\boldsymbol{J}_1 \quad \Rightarrow \quad [\varepsilon E_n] = \rho_\Gamma\,. \tag{26}$$

Example 4.2 *We verify the assumptions for the special case of uniform plane waves propagating through dielectric material. This is done in subsection* A.1.

Perfect conductor. We say that domain Ω_1 is a perfect conductor, if $\sigma_1 \to \infty$. We shall assume that

1. $\sigma_1\boldsymbol{E}\chi_1$ converges in $L^2_{loc}(\Omega_1)$ to zero.
2. $\boldsymbol{E}\chi_1$ converges in $H_{loc}(\nabla\times, \Omega_1)$ to zero.

Due to Faraday's law the latter property implies that the magnetic field $\boldsymbol{H}$ converges in $L^2_{loc}(\Omega_1)$ to zero.

This limiting behavior impacts on all Maxwell equations. The electric field converges to step function $\boldsymbol{E} = \boldsymbol{E}_2\chi_2$. $\boldsymbol{E}_2\chi_2 \in H_{loc}(\nabla\times, \mathbb{R}^3)$ and, due to the continuity of the trace operator, γ_D [1], the transmission condition (16) translates into a boundary condition

$$\nabla\times\boldsymbol{E}_2 = -j\omega\mu_2\boldsymbol{H}_2 \quad \text{with} \quad \gamma_D\boldsymbol{E}_2 = \boldsymbol{0}\,.$$

As $\sigma_2 < \infty$ we know from Ampère's law that $\boldsymbol{H}_2 \in H(\nabla\times, \Omega_2)$. Under the additional assumption $\varepsilon_2\boldsymbol{E}_2 \in H(\nabla\cdot, \Omega_2)$, the step functions

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{H}_2\chi_2 \quad \in L^2_{loc}(\mathbb{R}^3)^3\,, \\ \varepsilon\boldsymbol{E} &= \varepsilon_2\boldsymbol{E}_2\chi_2 \in L^2_{loc}(\mathbb{R}^3)^3 \end{aligned}$$

result in the following distributions

$$\begin{aligned} \nabla\times\boldsymbol{H} &= \left(j\omega\varepsilon_2\boldsymbol{E}_2 + \boldsymbol{J}_2 + \boldsymbol{J}_2^{imp}\right)\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma\,, \\ \nabla\cdot\varepsilon\boldsymbol{E} &= \left(\rho_2 + \rho_2^{imp}\right)\chi_2 + \rho_\Gamma\delta_\Gamma\,, \end{aligned}$$

where the surface distributions correspond to the following traces

$$\rho_\Gamma = \boldsymbol{n}\cdot\varepsilon_2\boldsymbol{E}_2 \tag{27}$$
$$\boldsymbol{J}_\Gamma = \boldsymbol{n}\times\boldsymbol{H}_2\,. \tag{28}$$

Finally, let us analyze Ohm's law. Contrary to the perfect dielectric case, $\boldsymbol{J}$ itself picks up a surface contribution

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}_2\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma \\ &= \sigma_2\boldsymbol{E}_2\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma\,, \\ \nabla\cdot\boldsymbol{J} &= j\omega\rho_2\chi_2 + j\omega\rho_\Gamma\delta_\Gamma + \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma\delta_\Gamma\,. \end{aligned}$$

It is important to notice

1. The electromagnetic field vanishes in Ω_1.
2. The unknown charge density reads $\rho = \rho_1\chi_1 + \rho_\Gamma\delta_\Gamma$.
3. The unknown current density reads $\boldsymbol{J} = \boldsymbol{J}_2\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma$.

Example 4.3 *We verify the assumptions for the special case of uniform plane waves. This is done in subsection* A.2.

In the next sections we apply the reasoning for the perfect dielectric and perfect conductor to all possible transmission problems. Analogously to the conductor/conductor problem, we formulate equivalent systems of equations with less unknowns.

[1] [3]

4.1 Transmission between a conductor and a perfect dielectric

Let Ω_1 be filled with a conducting material with $0 < \sigma_1 < \infty$, and let Ω_2 be occupied by a perfect dielectric $\sigma_2 = 0$.

$$0 < \sigma_1 < \infty,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ conductor}$$

$$\sigma_2 = 0,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ perfect dielectric}$$

We are looking for

$$\begin{aligned} \boldsymbol{E} &\in H_{loc}(\nabla\times, \mathbb{R}^3) \\ \boldsymbol{H} &\in L^2_{loc}(\mathbb{R}^3)^3 \\ \boldsymbol{J} = \sigma_1 \boldsymbol{E}_1 \chi_1 &\in L^2_{loc}(\mathbb{R}^3)^3 \\ \rho &\in \mathscr{D}(\mathbb{R}^3)'\,. \end{aligned}$$

Definition 4.4 *$\boldsymbol{E} \in H_{loc}(\nabla\times, \mathbb{R}^3)$, $\boldsymbol{H}, \boldsymbol{J} \in L^2_{loc}(\mathbb{R}^3)^3, \rho \in \mathscr{D}(\mathbb{R}^3)'$ is a solution to the transmission problem between the conductor Ω_1 and the perfect dielectric Ω_2 if*

$$\begin{aligned} \langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle \\ \langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \\ -\langle \varepsilon \boldsymbol{E}, \nabla\psi \rangle &= \langle \rho, \psi \rangle + \langle \rho^{imp}, \psi \rangle \\ \langle \boldsymbol{J}, \boldsymbol{F} \rangle &= \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \boldsymbol{F} \rangle \\ \\ -\langle \mu \boldsymbol{H}, \nabla\psi \rangle &= 0 \\ \langle \boldsymbol{J}, \nabla\psi \rangle &= j\omega \langle \rho, \psi \rangle \end{aligned}$$

for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$, $\psi \in \mathscr{D}(\mathbb{R}^3)$.

More explicitly, if we represent the unknown quantities $\boldsymbol{J} = \boldsymbol{J}_1\chi_1$ and $\rho = \rho_1\chi_1 + \rho_\Gamma\delta_\Gamma$, we obtain

$$\begin{aligned} \langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle \\ \langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}_1\chi_1, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \\ -\langle \varepsilon \boldsymbol{E}, \nabla\psi \rangle &= \langle \rho_1\chi_1, \psi \rangle + \langle \rho^{imp}, \psi \rangle \qquad + \quad \langle \rho_\Gamma \delta_\Gamma, \psi \rangle \\ \langle \boldsymbol{J}_1\chi_1, \boldsymbol{F} \rangle &= \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \boldsymbol{F} \rangle \\ \\ -\langle \mu \boldsymbol{H}, \nabla\psi \rangle &= 0 \\ \langle \boldsymbol{J}_1\chi_1, \nabla\psi \rangle &= j\omega \langle \rho_1\chi_1, \psi \rangle \qquad + \quad j\omega\langle \rho_\Gamma \delta_\Gamma, \psi \rangle \end{aligned}$$

for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$, $\psi \in \mathscr{D}(\mathbb{R}^3)$.

Lemma 4.5 *If $\boldsymbol{E} \in H_{loc}(\nabla\times, \mathbb{R}^3)$ satisfies for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$*

$$\langle \mu^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \rangle = \omega^2 \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle - j\omega \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \boldsymbol{F} \rangle - j\omega \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle$$

and we define $\boldsymbol{J}$ and ρ by

$$\begin{aligned} \forall\ \boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3 \qquad \langle \boldsymbol{J}, \boldsymbol{F} \rangle &= \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \boldsymbol{F} \rangle\,, \\ \forall\ \psi \in \mathscr{D}(\mathbb{R}^3) \qquad j\omega\langle \rho, \psi \rangle &= \langle \boldsymbol{J}, \nabla\psi \rangle \end{aligned}$$

then $\boldsymbol{E}, \boldsymbol{J}, \rho$ is a solution to the transmission problem between a conductor in Ω_1 and a perfect dielectric in Ω_2 in the sense of definition 4.4.

Proof: The only equation that is left to check is Gauß' law for the electric field. The special choice of test functions $\boldsymbol{F} = \nabla\psi \in \mathscr{D}(\mathbb{R}^3)^3$ yields

$$-\omega^2 \langle \varepsilon \boldsymbol{E}, \nabla\psi \rangle = -j\omega \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \nabla\psi \rangle - j\omega \langle \boldsymbol{J}^{imp}, \nabla\psi \rangle\,.$$

The definition of ρ and (14) yield Gauß' law

$$-\langle \varepsilon \boldsymbol{E}, \nabla\psi \rangle = \langle \rho, \psi \rangle + \langle \rho^{imp}, \psi \rangle\,.$$

□

4.2 Transmission between a perfect conductor and a conductor

Let Ω_1 be occupied by a perfect conductor, and let Ω_2 be a conductor with $0 < \sigma_2 < \infty$.

$$\sigma_1 = \infty,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ perfect conductor}$$

$$0 < \sigma_2 < \infty,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ conductor}$$

We are looking for

$$\begin{array}{rcl} \boldsymbol{E} = \boldsymbol{E}_2\chi_2 & \in & H_{loc}(\nabla\times, \mathbb{R}^3) \\ \boldsymbol{H} = \boldsymbol{E}_2\chi_2 & \in & L^2_{loc}(\mathbb{R}^3)^3 \\ \boldsymbol{J} = \boldsymbol{J}_2\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma & \in & \left(\mathscr{D}(\mathbb{R}^3)^3\right)' \\ \rho = \rho_2\chi_2 + \rho_\Gamma\delta_\Gamma & \in & \mathscr{D}(\mathbb{R}^3)'\,. \end{array}$$

All equations are partial differential equations in Ω_2 only, and we drop the characteristic function to simplify the notations.

Definition 4.6 *A Maxwell solution satisfies*

$$\begin{array}{rcll} \langle \nabla \times \boldsymbol{E}_2, \boldsymbol{F}\rangle & = & -j\omega\langle \mu_2 \boldsymbol{H}_2, \boldsymbol{F}\rangle & \\ \langle \boldsymbol{H}_2, \nabla \times \boldsymbol{F}\rangle & = & j\omega\langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}, \boldsymbol{F}\rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle & \\ -\langle \varepsilon_2 \boldsymbol{E}_2, \nabla\psi\rangle & = & \langle \rho, \psi\rangle + \langle \rho^{imp}, \psi\rangle & \\ \langle \boldsymbol{J}, \boldsymbol{F}\rangle & = & \langle \sigma_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle & - \quad \langle \boldsymbol{J}_\Gamma\delta_\Gamma, \boldsymbol{F}\rangle \\ & & & \\ -\langle \mu_2 \boldsymbol{H}_2, \nabla\psi\rangle & = & 0 & \\ \langle \boldsymbol{J}, \nabla\psi\rangle & = & j\omega\langle \rho, \psi\rangle & \end{array}$$

for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$, $\psi \in \mathscr{D}(\mathbb{R}^3)$.

More explicitly, with $\boldsymbol{J} = \boldsymbol{J}_2\chi_2 + \boldsymbol{J}_\Gamma\delta_\Gamma$ and $\rho = \rho_2\chi_2 + \rho_\Gamma\delta_\Gamma$ we obtain

$$\begin{array}{rcll} \langle \nabla \times \boldsymbol{E}_2, \boldsymbol{F}\rangle & = & -j\omega\langle \mu_2 \boldsymbol{H}_2, \boldsymbol{F}\rangle & \\ \langle \boldsymbol{H}_2, \nabla \times \boldsymbol{F}\rangle & = & j\omega\langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle & + \quad \langle \boldsymbol{J}_\Gamma\delta_\Gamma, \boldsymbol{F}\rangle \\ -\langle \varepsilon_2 \boldsymbol{E}_2, \nabla\psi\rangle & = & \langle \rho_2, \psi\rangle + \langle \rho^{imp}, \psi\rangle & + \quad \langle \rho_\Gamma\delta_\Gamma, \psi\rangle \\ \langle \boldsymbol{J}_2, \boldsymbol{F}\rangle & = & \langle \sigma_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle & \\ & & & \\ -\langle \mu_2 \boldsymbol{H}_2, \nabla\psi\rangle & = & 0 & \\ \langle \boldsymbol{J}_2, \nabla\psi\rangle & = & j\omega\langle \rho_2, \psi\rangle & + \quad \langle (j\omega\rho_\Gamma + \nabla_\Gamma\cdot \boldsymbol{J}_\Gamma)\,\delta_\Gamma, \psi\rangle \end{array}$$

for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$, $\psi \in \mathscr{D}(\mathbb{R}^3)$.

Lemma 4.7 *If $\boldsymbol{E}_2 \in H_{loc}(\nabla\times, \mathbb{R}^3)$ satisfies for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3$*

$$\begin{array}{rcl} \langle {\mu_2}^{-1}\nabla \times \boldsymbol{E}_2, \nabla \times \boldsymbol{F}\rangle & = & \omega^2\langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle - j\omega\langle \sigma_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle - j\omega\langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle \\ & & -j\omega\langle \boldsymbol{J}_\Gamma\delta_\Gamma, \boldsymbol{F}\rangle \end{array}$$

and we define $\boldsymbol{J}$ and ρ by

$$\begin{array}{lrcl} \forall\ \boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3 & \langle \boldsymbol{J}_2, \boldsymbol{F}\rangle & = & \langle \sigma_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle\,, \\ \forall\ \psi \in \mathscr{D}(\mathbb{R}^3) & j\omega\langle \rho, \psi\rangle & = & \langle \boldsymbol{J}_2, \nabla\psi\rangle - \langle \nabla_\Gamma\cdot \boldsymbol{J}_\Gamma\delta_\Gamma, \psi\rangle \end{array}$$

then $\boldsymbol{E}_2 \in H_{loc}(\nabla\times, \mathbb{R}^3), \boldsymbol{J}_2 \in L^2_{loc}(\mathbb{R}^3), \rho \in \mathscr{D}(\mathbb{R}^3)'$ solve the transmission problem between a conductor in Ω_2 and a perfect conductor in Ω_1 in the sense of definition 4.6.

Proof: The only equation that is left to check is Gauß' law. The special choice of test functions $\boldsymbol{F} = \nabla\psi \in \mathscr{D}(\mathbb{R}^3)^3$ yields

$$-\omega^2\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle + j\omega\langle\sigma_2\boldsymbol{E}_2, \nabla\psi\rangle + j\omega\langle\boldsymbol{J}^{imp}, \nabla\psi\rangle + j\omega\langle\boldsymbol{J}_\Gamma\delta_\Gamma, \nabla\psi\rangle = 0\,.$$

The definition of ρ_2 and (14) yield Gauß' law

$$-\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle = \langle\rho_2, \psi\rangle + \langle\rho^{imp}, \psi\rangle + \langle\rho_\Gamma\delta_\Gamma, \psi\rangle\,.$$

□

4.3 Transmission between a perfect conductor and a dielectric

Let Ω_1 be occupied by a perfectly conducting material with $\sigma_1 = \infty$ and let Ω_2 be a perfect dielectric, i.e. $\sigma_2 = 0$.

$$\sigma_1 = \infty,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ perfect conductor}$$

$$\sigma_2 = 0,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ perfect dielectric}$$

The governing equations are found by a limiting process whose starting point is either transmission problem 4.1 or transmission problem 4.2.

1. The perfect dielectric occupies Ω_2 and the conductor occupies Ω_1. We recall the governing equations for this case:

$$\begin{aligned}
\langle\nabla\times\boldsymbol{E}, \boldsymbol{F}\rangle &= -j\omega\langle\mu\boldsymbol{H}, \boldsymbol{F}\rangle \\
\langle\boldsymbol{H}, \nabla\times\boldsymbol{F}\rangle &= j\omega\langle\varepsilon\boldsymbol{E}, \boldsymbol{F}\rangle + \langle\boldsymbol{J}_1\chi_1, \boldsymbol{F}\rangle + \langle\boldsymbol{J}^{imp}, \boldsymbol{F}\rangle \\
-\langle\varepsilon\boldsymbol{E}, \nabla\psi\rangle &= \langle\rho_1\chi_1, \psi\rangle + \langle\rho^{imp}, \psi\rangle + \langle\rho_\Gamma\delta_\Gamma, \psi\rangle \\
\langle\boldsymbol{J}_1\chi_1, \boldsymbol{F}\rangle &= \langle\sigma_1\boldsymbol{E}_1\chi_1, \boldsymbol{F}\rangle \\
\\
-\langle\mu\boldsymbol{H}, \nabla\psi\rangle &= 0 \\
\langle\boldsymbol{J}_1\chi_1, \nabla\psi\rangle &= j\omega\langle\rho_1\chi_1, \psi\rangle + j\omega\langle\rho_\Gamma\delta_\Gamma, \psi\rangle\,.
\end{aligned}$$

For $\sigma_1 \to \infty$, Ω_1 becomes a perfect conductor. Due to our assumptions Faraday's law turns into a variational formulation for a boundary value problem. so it does the Gauß' law for the magnetic field, while the volume current density $\boldsymbol{J}_1$ vanishes

$$\begin{aligned}
\langle\boldsymbol{J}_1, \boldsymbol{F}\rangle &= 0 \\
\langle\nabla\times\boldsymbol{E}_2, \boldsymbol{F}\rangle &= -j\omega\langle\mu_2\boldsymbol{H}_2, \boldsymbol{F}\rangle \\
-\langle\mu_2\boldsymbol{H}_2, \nabla\psi\rangle &= 0\,.
\end{aligned}$$

Due to the step properties of the magnetic field and the electric flux, Ampère's law and Gauß' law exhibit surface distributions

$$\begin{aligned}
\langle\boldsymbol{H}_2, \nabla\times\boldsymbol{F}\rangle &= j\omega\langle\varepsilon_2\boldsymbol{E}_2, \boldsymbol{F}\rangle + \langle\boldsymbol{J}^{imp}, \boldsymbol{F}\rangle + \langle\boldsymbol{J}_\Gamma\delta_\Gamma, \boldsymbol{F}\rangle \\
-\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle &= \langle\rho_2, \psi\rangle + \langle\rho^{imp}, \psi\rangle + \langle\rho_\Gamma\delta_\Gamma, \psi\rangle\,.
\end{aligned}$$

Consequently, the solution has to fulfill

$$0 = j\omega\langle\rho_2, \psi\rangle + \langle(j\omega\rho_\Gamma + \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma)\delta_\Gamma, \psi\rangle$$

for all $\psi \in \mathscr{D}(\Omega_1)$. This means that the charge density ρ_1 also vanishes.

2. The perfect conductor occupies Ω_1 and the conductor occupies Ω_2. The governing equations for this case look as follows.

$$\begin{array}{rclcl}
\langle \nabla \times \boldsymbol{E}_2, \boldsymbol{F}\rangle & = & -j\omega\langle \mu_2 \boldsymbol{H}_2, \boldsymbol{F}\rangle & & \\
\langle \boldsymbol{H}_2, \nabla \times \boldsymbol{F}\rangle & = & j\omega\langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle & + & \langle \boldsymbol{J}_\Gamma \delta_\Gamma, \boldsymbol{F}\rangle \\
-\langle \varepsilon_2 \boldsymbol{E}_2, \nabla\psi\rangle & = & \langle \rho_2, \psi\rangle + \langle \rho^{imp}, \psi\rangle & + & \langle \rho_\Gamma \delta_\Gamma, \psi\rangle \\
\langle \boldsymbol{J}_2, \boldsymbol{F}\rangle & = & \langle \sigma_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle & & \\
 & & & & \\
-\langle \mu_2 \boldsymbol{H}_2, \nabla\psi\rangle & = & 0 & & \\
\langle \boldsymbol{J}_2, \nabla\psi\rangle & = & j\omega\langle \rho_2, \psi\rangle & + & \langle (j\omega\rho_\Gamma + \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma)\,\delta_\Gamma, \psi\rangle
\end{array}$$

For $\sigma_2 \to 0$, Ω_2 turns in a dielectric. This limiting process impacts on Ohm's law in such a way that

$$\langle \boldsymbol{J}_2, \boldsymbol{F}\rangle = 0 \quad \Rightarrow \boldsymbol{J}_2 = \boldsymbol{0}\,.$$

The same arguments as before lead us to a vanishing volume charge density ρ_2.

In the end, we obtain the same set of equations from the different limiting processes:

$$\begin{array}{rclcl}
\langle \nabla \times \boldsymbol{E}_2, \boldsymbol{F}\rangle & = & -j\omega\langle \mu_2 \boldsymbol{H}_2, \boldsymbol{F}\rangle & & \\
\langle \boldsymbol{H}_2, \nabla \times \boldsymbol{F}\rangle & = & j\omega\langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F}\rangle & + & \langle \boldsymbol{J}_\Gamma \delta_\Gamma, \boldsymbol{F}\rangle \\
-\langle \varepsilon_2 \boldsymbol{E}_2, \nabla\psi\rangle & = & \langle \rho^{imp}, \psi\rangle & + & \langle \rho_\Gamma \delta_\Gamma, \psi\rangle \\
 & & & & \\
-\langle \mu_2 \boldsymbol{H}_2, \nabla\psi\rangle & = & 0 & & \\
0 & = & \langle (j\omega\rho_\Gamma + \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma)\delta_\Gamma, \psi\rangle\,. & &
\end{array}$$

The unknowns of the problem are

$$\begin{array}{rcl}
\boldsymbol{E} = \boldsymbol{E}_2\chi_2 & \in & H(\nabla\times, \mathbb{R}^3) \\
\boldsymbol{H} = \boldsymbol{H}_2\chi_2 & \in & L^2(\mathbb{R}^3)^3 \\
\boldsymbol{J}_\Gamma \delta_\Gamma & \in & \left(\mathscr{D}(\mathbb{R}^3)^3\right)' \\
\rho_\Gamma \delta_\Gamma & \in & \mathscr{D}(\mathbb{R}^3)'\,.
\end{array}$$

Definition 4.8 *Functions above is a Maxwell solution if*

$$\begin{array}{rclcl}
\langle \nabla \times \boldsymbol{E}_2, \boldsymbol{F}\rangle & = & -j\omega\langle \mu_2 \boldsymbol{H}_2, \boldsymbol{F}\rangle & & \\
\langle \boldsymbol{H}_2, \nabla \times \boldsymbol{F}\rangle & = & j\omega\langle \varepsilon_2 \boldsymbol{E}, \boldsymbol{F}\rangle & + & \langle \boldsymbol{J}_\Gamma \delta_\Gamma, \boldsymbol{F}\rangle \\
 & & & & \\
-\langle \varepsilon_2 \boldsymbol{E}_2, \nabla\psi\rangle & = & 0 & + & \langle \rho_\Gamma \delta_\Gamma, \psi\rangle \\
-\langle \mu_2 \boldsymbol{H}_2, \nabla\psi\rangle & = & 0 & &
\end{array}$$

The continuity equation is satisfied in the sense of distributions,

$$j\omega\langle \rho_\Gamma \delta_\Gamma, \psi\rangle = -\langle \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma \delta_\Gamma, \psi\rangle\,.$$

Lemma 4.9 *If* $\boldsymbol{E} \in H(\nabla\times, \mathbb{R}^3), \boldsymbol{J}_\Gamma \in \left(\mathscr{D}(\mathbb{R}^3)^3\right)'$ *satisfies for all* $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3,\ \psi \in \mathscr{D}(\mathbb{R}^3)$

$$\langle {\mu_2}^{-1} \nabla \times \boldsymbol{E}_2, \nabla \times \boldsymbol{F}\rangle = \omega^2 \langle \varepsilon_2 \boldsymbol{E}_2, \boldsymbol{F}\rangle - j\omega\langle \boldsymbol{J}_\Gamma \delta_\Gamma, \boldsymbol{F}\rangle$$

and we define the surface charge density by

$$j\omega\langle \rho_\Gamma \delta_\Gamma, \psi\rangle = \langle \boldsymbol{J}_\Gamma \delta_\Gamma, \nabla\psi\rangle\,.$$

$(\boldsymbol{E}, \rho_\Gamma)$ *is a solution to the transmission problem between a perfect conductor in* Ω_1 *and a perfect dielectric in* Ω_2 *in the sense of the previous definition.*

Proof: The only equation that is left to check is Gauß' law. The special choice of test functions $\boldsymbol{F} = \nabla\psi \in \mathscr{D}(\mathbb{R}^3)^3$ yields

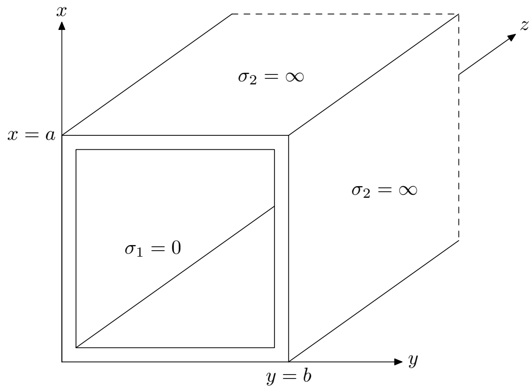
$$\begin{aligned} -\omega^2\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle - j\omega\langle\boldsymbol{J}_\Gamma\delta_\Gamma, \nabla\psi\rangle &= 0 \\ \Leftrightarrow -\omega\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle + j\langle\nabla_\Gamma\cdot\boldsymbol{J}_\Gamma\delta_\Gamma, \psi\rangle &= 0\,. \end{aligned}$$

Together with the definition of ρ_Γ we obtain

$$\langle\varepsilon_2\boldsymbol{E}_2, \nabla\psi\rangle = \langle\rho_\Gamma\delta_\Gamma, \psi\rangle\,.$$

□

Example 4.10 *We would like to conclude this section with an example that illustrates the continuity equation.*



The electromagnetic fields in a rectangular waveguide are given by

$$\begin{aligned} \boldsymbol{E} &= \begin{pmatrix} 0 \\ C\frac{\omega\mu a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t - \beta z) \\ 0 \end{pmatrix} \\ \boldsymbol{H} &= \begin{pmatrix} -C\frac{\beta a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t - \beta z) \\ 0 \\ C\cos\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta z) \end{pmatrix}. \end{aligned}$$

The walls of the waveguide are assumed to be perfect conductors. The electromagnetic field above satisfies Maxwell's equations with boundary conditions

$$\begin{aligned} \nabla\times\boldsymbol{E} &= -\mu\tfrac{\partial\boldsymbol{H}}{\partial t} & \gamma_D\boldsymbol{E} &= \boldsymbol{0} \\ \nabla\times\boldsymbol{H} &= \varepsilon\tfrac{\partial\boldsymbol{E}}{\partial t} & \gamma_D\boldsymbol{H} &= \boldsymbol{J}_\Gamma \\ \nabla\cdot(\varepsilon\boldsymbol{E}) &= 0 & \gamma_{\boldsymbol{n}}(\varepsilon\boldsymbol{E}) &= \rho_\Gamma \\ \nabla\cdot(\mu\boldsymbol{H}) &= 0 & \gamma_{\boldsymbol{n}}(\mu\boldsymbol{H}) &= 0 \end{aligned}$$

if the coefficients of the solution are such that

$$\beta^2\frac{a}{\pi} + \frac{\pi}{a} = \omega^2\varepsilon\mu\frac{a}{\pi}\,. \tag{29}$$

As the electromagnetic field is explicitly known, we can determine the unknown boundary data ρ_Γ

and $\boldsymbol{J}_\Gamma$:

$$
\begin{aligned}
y=0: \quad \boldsymbol{J}_\Gamma &= -H_x\Big|_{y=0}\boldsymbol{e}_z + H_z\Big|_{y=0}\boldsymbol{e}_x \\
&= C\frac{\beta a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t-\beta z)\boldsymbol{e}_z + C\cos\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z)\boldsymbol{e}_x \\
y=b: \quad \boldsymbol{J}_\Gamma &= H_x\Big|_{y=b}\boldsymbol{e}_z - H_z\Big|_{y=b}\boldsymbol{e}_x \\
&= -C\frac{\beta a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t-\beta z)\boldsymbol{e}_z - C\cos\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z)\boldsymbol{e}_x \\
x=a: \quad \boldsymbol{J}_\Gamma &= H_z\Big|_{x=a}\boldsymbol{e}_y = -C\cos(\omega t-\beta z)\boldsymbol{e}_y \\
x=0: \quad \boldsymbol{J}_\Gamma &= -H_z\Big|_{x=0}\boldsymbol{e}_y = -C\cos(\omega t-\beta z)\boldsymbol{e}_y\,.
\end{aligned}
$$

The surface charge density on the walls is numerically equal to the component of $\varepsilon\boldsymbol{E}$ normal to the wall. Along the walls

$$
\begin{aligned}
y=0: \quad \rho_\Gamma = \varepsilon E_y|_{y=0} &= C\varepsilon\mu\frac{\omega a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t-\beta z) && (30)\\
y=b: \quad \rho_\Gamma = -\varepsilon E_y|_{y=b} &= -C\varepsilon\mu\frac{\omega a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t-\beta z) && (31)\\
x=a: \quad \rho_\Gamma = -\varepsilon E_y|_{x=a} &= 0 && (32)\\
x=0: \quad \rho_\Gamma = -\varepsilon E_y|_{x=0} &= 0\,. && (33)
\end{aligned}
$$

Now, we check if we get the same result from the continuity equation. The surface divergence of $\boldsymbol{J}_\Gamma$ corresponds to a differentiation with respect to z at at the left and to x at the right wall of the waveguide, on the other two walls it is a differentiation with respect to z, y

$$
\begin{aligned}
y=0: \quad \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= C\frac{-\beta^2 a}{\pi}\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) - C\frac{\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
&= -C\left(\frac{\beta^2 a}{\pi}+\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
y=b: \quad \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= C\frac{\beta^2 a}{\pi}\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) + C\frac{\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
&= C\left(\frac{\beta^2 a}{\pi}+\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
x=a: \quad \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= 0 \\
x=0: \quad \nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= 0\,.
\end{aligned}
$$

We apply the continuity equation to obtain the surface charge distribution:

$$
\begin{aligned}
y=0: \quad \rho_\Gamma = \frac{1}{\omega}\nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= -\frac{C}{\omega}\left(\frac{\beta^2 a}{\pi}+\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
y=b: \quad \rho_\Gamma = \frac{1}{\omega}\nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= \frac{C}{\omega}\left(\frac{\beta^2 a}{\pi}+\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos(\omega t-\beta z) \\
x=a: \quad \rho_\Gamma = \frac{1}{\omega}\nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= 0 \\
x=0: \quad \rho_\Gamma = \frac{1}{\omega}\nabla_\Gamma\cdot\boldsymbol{J}_\Gamma &= 0\,.
\end{aligned}
$$

This corresponds to (30)-(33) *if we consider* (29). □

4.4 Transmission between two dielectrics

We still miss the transmission problem between two dielectrics.

$$\sigma_1 = 0,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ perfect dielectric 1}$$

$$\sigma_2 = 0,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ perfect dielectric 2}$$

The unknowns of the problem are only the electromagnetic field components

$$\begin{aligned} \boldsymbol{E} &\in H(\nabla\times, \mathbb{R}^3) \\ \boldsymbol{H} &\in L^2(\mathbb{R}^3)^3 \,. \end{aligned}$$

The governing equations are found again by the limiting process. A starting point is again transmission problem 4.1:

$$\begin{aligned} \langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle \\ \langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}_1 \chi_1, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \\ -\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle &= \langle \rho_1 \chi_1, \psi \rangle + \langle \rho^{imp}, \psi \rangle \qquad + \quad \langle \rho_\Gamma \delta_\Gamma, \psi \rangle \\ \langle \boldsymbol{J}_1 \chi_1, \boldsymbol{F} \rangle &= \langle \sigma_1 \boldsymbol{E}_1 \chi_1, \boldsymbol{F} \rangle \\ \\ -\langle \mu \boldsymbol{H}, \nabla \psi \rangle &= 0 \\ \langle \boldsymbol{J}_1 \chi_1, \nabla \psi \rangle &= j\omega \langle \rho_1 \chi_1, \psi \rangle \qquad + \quad j\omega \langle \rho_\Gamma \delta_\Gamma, \psi \rangle \end{aligned}$$

For $\sigma_1 \to 0$, Ω_1 becomes a perfect dielectric which means that Ohm's law becomes

$$\begin{aligned} \langle \boldsymbol{J}_1 \chi_1, \boldsymbol{F} \rangle &= 0 \\ \Rightarrow 0 &= j\omega \langle \rho_1 \chi_1, \psi \rangle + j\omega \langle \rho_\Gamma \delta_\Gamma, \psi \rangle \,. \end{aligned}$$

The governing equations read

$$\begin{aligned} \langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle \\ \langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \\ \\ -\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle &= \langle \rho^{imp}, \psi \rangle \\ -\langle \mu \boldsymbol{H}, \nabla \psi \rangle &= 0 \,. \end{aligned}$$

Definition 4.11 *A Maxwell solution is a functional on $\mathscr{D}(\mathbb{R}^3)^3$, $\mathscr{D}(\mathbb{R}^3)$, with*

$$\begin{aligned} \langle \nabla \times \boldsymbol{E}, \boldsymbol{F} \rangle &= -j\omega \langle \mu \boldsymbol{H}, \boldsymbol{F} \rangle \\ \langle \boldsymbol{H}, \nabla \times \boldsymbol{F} \rangle &= j\omega \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle + \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \\ \\ -\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle &= \langle \rho^{imp}, \psi \rangle \\ -\langle \mu \boldsymbol{H}, \nabla \varphi \rangle &= 0 \,. \end{aligned}$$

Lemma 4.12 *If $\boldsymbol{E} \in H(\nabla\times, \mathbb{R}^3)$ satisfies for all $\boldsymbol{F} \in \mathscr{D}(\mathbb{R}^3)^3,\ \psi \in \mathscr{D}(\mathbb{R}^3)$*

$$\langle \mu^{-1} \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \rangle = \omega^2 \langle \varepsilon \boldsymbol{E}, \boldsymbol{F} \rangle - j\omega \langle \boldsymbol{J}^{imp}, \boldsymbol{F} \rangle \,,$$

it is a solution to the transmission problem between two perfect dielectric in Ω_1 and Ω_2 in the sense of the previous definition.

Proof: The only equation that is left to check is Gauß' law. The special choice of test functions $\boldsymbol{F} = \nabla \psi \in \mathscr{D}(\mathbb{R}^3)^3$ yields

$$-\langle \varepsilon \boldsymbol{E}, \nabla \psi \rangle = \langle \rho^{imp}, \psi \rangle \,.$$

□

5 Conclusion

The note discusses all possible transmition problems for the Maxwell equations. The case of a perfect dielectric or a perfect conductor is interpreted through a limiting process corresponding to conductivity $\sigma \rightarrow 0$ or $\sigma \rightarrow \infty$. The discussion is only formal, an actual analysis would require "hard estimates" and investigation of the limits. The obtained limiting cases are illustrated with the case of plane waves presented in the Appendix. The presented discussion is intended to elucidate a common phrase that many technical papers begin with: *we shall understand the Maxwell equations in the distributional sense...*. A related problem concerns the equivalence of Maxwell equations understood in the distributional sense discussed here and the integral form of the Maxwell's equations involving line and surface integrals, see e.g. [8], [2]. The two formulations are equivalent in the sense that, *with additional regularity assumptions*, they yield the same classical equations and the same interface conditions. To our best knowledge, we are not aware of an equivalence proof that would not use the additional regularity assumptions, like for the grad-div case discussed in [1].

All discussed cases admit the standard variational formulation in terms of electric field only [6] which is fully equivalent to the whole Maxwell system *without* any extra regularity assumptions.

A Uniform plane waves

The introduction here follows more or less the lines of [8]. The goal of this section is to derive the so-called uniform plane waves. These functions are smooth solutions to the harmonic Maxwell equations in a linear, isotropic, homogeneous medium that comprises no net free charge: $\rho = 0$. These types of regions are quite general ones and include the practical cases of free space ($\sigma = 0$) as well as most conductors and dielectrics. Maxwell's equations for this region become

$$\begin{aligned}
\nabla \times \boldsymbol{E} &= -\mu \frac{\partial \boldsymbol{H}}{\partial t} \\
\nabla \times \boldsymbol{H} &= \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J} \\
\nabla \cdot \boldsymbol{H} &= 0 \\
\nabla \cdot \boldsymbol{E} &= 0 \\
\boldsymbol{J} &= \sigma \boldsymbol{E} \,.
\end{aligned}$$

We restrict ourselves to solutions with harmonic time dependency

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x}, t) &= \tilde{\boldsymbol{E}}(\boldsymbol{x}) e^{j\omega t} \\
\boldsymbol{H}(\boldsymbol{x}, t) &= \tilde{\boldsymbol{H}}(\boldsymbol{x}) e^{j\omega t} \,.
\end{aligned}$$

Thus, the above system reduces to equations in terms of the phasors only

$$\nabla \times \tilde{\boldsymbol{E}} = -\mu\omega \tilde{\boldsymbol{H}} \tag{34}$$

$$\nabla \times \tilde{\boldsymbol{H}} = \varepsilon\omega \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{J}} \tag{35}$$

$$\nabla \cdot \tilde{\boldsymbol{H}} = 0 \tag{36}$$

$$\nabla \cdot \tilde{\boldsymbol{E}} = 0 \tag{37}$$

$$\tilde{\boldsymbol{J}} = \sigma \tilde{\boldsymbol{E}} \,. \tag{38}$$

Whenever it is clear that we are dealing with the phasors, we will skip the tildes from the notations. We assume the electric field vector to have the following representation

$$\boldsymbol{E}(x, y, z) = E_x(z) \boldsymbol{e}_x \,, \tag{39}$$

where the complex phasor is supposed to be smooth. Due to

$$\nabla \times E_x(z) \boldsymbol{e}_x = \frac{dE_x(z)}{dz} \boldsymbol{e}_y$$

Faraday's law yield the magnetic field vector to be

$$\boldsymbol{H}(x, y, z) = H_y(z) \boldsymbol{e}_y \,. \tag{40}$$

Obviously the Gauß' laws (36)+(37) are automatically fulfilled

$$\begin{aligned}
\frac{dH_y(z)}{dy} &= 0 \\
\frac{dE_x(z)}{dy} &= 0 \,.
\end{aligned}$$

They do not contribute to the solution any further. Thus, the ansatz (39) turns (34)-(35) into ordinary scalar valued differential equations:

$$\frac{dE_x(z)}{dz} = -\mu\omega H_y(z) \tag{41}$$

$$\frac{dH_y(z)}{dz} = \varepsilon\omega E_x(z) + \sigma E_x(z) \,. \tag{42}$$

As we restrict ourselves to smooth solutions, instead of solving (41)-(43) we can consider the second order equations

$$\frac{d^2E_x(z)}{dz^2} = -\mu\omega(\varepsilon\omega+\sigma)E_x(z) \tag{43}$$

$$\frac{d^2H_y(z)}{dz^2} = -\mu\omega(\varepsilon\omega+\sigma)H_y(z) \tag{44}$$

that are deduced by another differentiation and substitution. We will use a special symbol γ^2 for the quantity $j\omega(\mu\sigma + j\omega\mu\varepsilon)$ such that

$$\begin{aligned}\gamma^2 &= j\omega(\mu\sigma+j\omega\mu\varepsilon)\\ &= -\omega^2\mu\varepsilon + j\omega\mu\sigma\,. \end{aligned}\tag{45}$$

The positive square root of γ^2, γ, will be referred to as the propagation constant of the medium for reasons that will become clear in the following section. Since γ^2 is a complex number, the square root of γ^2 will also be a complex number, which we write as

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma+j\omega\varepsilon)}\,. \tag{46}$$

It will be of particular interest to exploit the dependency $\gamma(\sigma)$ that is why we need explicit expressions for α and β. They are easily found by solving a quadratic equation and ignoring possible complex solutions:

$$\alpha = \left(-\frac{1}{2}\left(\omega^2\mu\varepsilon - \omega\mu\sqrt{\omega^2\varepsilon^2+\sigma^2}\right)\right)^{\frac{1}{2}} \tag{47}$$

$$\beta^2 = \omega\mu\varepsilon - \alpha^2\,. \tag{48}$$

The general complex valued solution to (43) and (44) are of the form

$$E_x = E_c^+e^{-\gamma z} + E_c^-e^{\gamma z} = E_c^+e^{-\alpha z}e^{-j\beta z} + E_c^-e^{\alpha z}e^{j\beta z} \tag{49}$$

$$H_y = H_c^+e^{-\gamma z} + H_c^-e^{\gamma z} = H_c^+e^{-\alpha z}e^{-j\beta z} + H_c^-e^{\alpha z}e^{j\beta z}\,, \tag{50}$$

where the phasors $H_c^+, H_c^-, E_c^+, E_c^-$ are undetermined complex constants. Before interpreting these solutions, we notice that from (41) it follows that the phasors are related:

$$\frac{E_c^+}{H_c^+} = \frac{j\omega\mu}{\gamma} = \hat{\eta} \tag{51}$$

$$\frac{E_c^-}{H_c^-} = -\frac{j\omega\mu}{\gamma} = -\hat{\eta}\,. \tag{52}$$

The quantity $j\omega\mu/\gamma$ has the units of ohms since it is a ratio of electric field intensity (volts per meter) to magnetic field intensity (amperes per meter). It will be called the intrinsic impedance of the medium and denoted by the symbol $\hat{\eta}$. The intrinsic impedance $\hat{\eta}$ as well as E_c^+, E_c^- are complex numbers and we introduce the following notation for its magnitude and angle

$$\hat{\eta} = \eta e^{j\theta_\eta} \tag{53}$$

$$E_c^+ = E^+e^{j\theta^+} \tag{54}$$

$$E_c^- = E^+e^{j\theta^-}\,. \tag{55}$$

The plane wave solutions to (34)-(38) become

$$\begin{aligned}\boldsymbol{E}(z,t) &= E_x(z)e^{j\omega t}\boldsymbol{e}_x\\ &= E^+e^{-\alpha z}e^{j(-\beta z+\theta^+)}e^{j\omega t}\boldsymbol{e}_x + E^-e^{\alpha z}e^{j(\beta z+\theta^-)}e^{j\omega t}\boldsymbol{e}_x \end{aligned}\tag{56}$$

$$\begin{aligned}\boldsymbol{H}(z,t) &= H_y(z)e^{j\omega t}\boldsymbol{e}_y\\ &= \frac{E^+}{\eta}e^{-\alpha z}e^{j(-\beta z+\theta^+-\theta_\eta)}e^{j\omega t}\boldsymbol{e}_y - \frac{E^-}{-\eta}e^{\alpha z}e^{j(\beta z+\theta^--\theta_\eta)}e^{j\omega t}\boldsymbol{e}_y\,. \end{aligned}\tag{57}$$

Let us analyze its geometrical character:

- The fields have got wave character in both, time and space.
- The field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ at each point in space lie in the xy plane.
- The spatial part of the field vectors (magnitude and phase) is independent of position in each of these planes. This is why the waves are called uniform.
- The energy propagates due to the cross product $\boldsymbol{E} \times \boldsymbol{H}$ in z direction.

The physical interpretation of the time-domain results (55), (56) is particularly important because we find estimates for the limiting cases $\sigma \to 0$, $\sigma \to \infty$.

A.1 Plane waves in lossless medium

In this section we concentrate on a uniform forward traveling plane wave

$$\begin{aligned} \boldsymbol{E}(z,t) &= E^+ e^{-\alpha z} e^{j(-\beta z+\theta^+)} e^{j\omega t} \boldsymbol{e}_x \\ \boldsymbol{H}(z,t) &= \frac{E^+}{\eta} e^{-\alpha z} e^{j(-\beta z+\theta^+-\theta_\eta)} e^{j\omega t} \boldsymbol{e}_y \end{aligned}$$

and analyze its behavior for $\sigma \to 0$. The limiting case $\sigma = 0$ characterizes the material that has no conductivity, i.e. a perfect dielectric. We recall

$$\begin{aligned} \gamma(\sigma) &= \alpha(\sigma) + j\beta(\sigma) \\ \alpha &= \left(-\frac{1}{2} \left(\omega^2 \mu \varepsilon - \omega\mu \sqrt{\omega^2 \varepsilon^2 + \sigma^2} \right) \right)^{\frac{1}{2}} \\ \beta^2 &= \omega\mu\varepsilon - \alpha^2 \,. \end{aligned}$$

α depends continuously from σ so that

$$\lim_{\sigma \to 0} \alpha(\sigma) = 0 \quad \text{and} \quad \beta = \omega\sqrt{\mu\varepsilon}\,. \tag{58}$$

For a forward traveling plane wave we gain the following estimates for $z \in I = [a,b], 0 < a < b$

$$\begin{aligned} \lim_{\sigma \to 0} |\boldsymbol{E}(z,t)| &\leq \lim_{\sigma \to 0} |E^+| \left| e^{-\alpha a} \right| \left| e^{j(-\beta z+\theta^+)} e^{j\omega t} \right| \\ &\leq |E^+| \lim_{\sigma \to 0} \left| e^{-\alpha a} \right| = |E^+| \,. \end{aligned} \tag{59}$$

This means that the electromagnetic field is uniformly bounded and moreover, the electric current converges uniformly to zero i.e. on every compact interval $z \in I = [a,b]$

$$\lim_{\sigma \to 0} |\sigma \boldsymbol{E}(z,t)| \leq \lim_{\sigma \to 0} \sigma |E^+| = 0\,. \tag{60}$$

In lossless media uniform plane waves propagate harmonically

$$\boldsymbol{E}(z,t) = E^+ e^{j(-\beta z+\theta^+)} e^{j\omega t} \boldsymbol{e}_x \tag{61}$$

$$\boldsymbol{H}(z,t) = \frac{E^+}{\eta} e^{j(-\beta z+\theta^+)} e^{j\omega t} \boldsymbol{e}_y\,, \tag{62}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and $\theta_\eta = 0$. The latter means that electric and magnetic wave are in time phase.

The same reasoning applies to the backward-traveling waves.

A.2 Perfect conductor

In this section we concentrate on a uniform forward traveling plane wave

$$\begin{aligned} \boldsymbol{E}(z,t) &= E^{+}e^{-\alpha z}e^{j(-\beta z+\theta^{+})}e^{j\omega t}\boldsymbol{e}_x \\ \boldsymbol{H}(z,t) &= \frac{E^{+}}{\eta}e^{-\alpha z}e^{j(-\beta z+\theta^{+}-\theta_\eta)}e^{j\omega t}\boldsymbol{e}_y \end{aligned}$$

and analyze its behavior for $\sigma \to \infty$. In lossy media, the propagation constant γ has a nonzero real part:

$$\alpha = \left(-\frac{1}{2}\left(\omega^2\mu\varepsilon - \omega\mu\sqrt{\omega^2\varepsilon^2+\sigma^2}\right)\right)^{\frac{1}{2}} > 0\,.$$

This results always in an exponential decrease of the phasors.

The second difference between lossless and lossy media concerns the intrinsic impedance of the medium. $\hat{\eta}$ is complex and the phase angle of the intrinsic impedance θ_η results in the electric and magnetic fields of time phase by the phase angle θ_η.

We would like to consider now the limiting behavior of an electromagnetic plane wave for $\sigma \to \infty$. The limiting case is a model for a perfectly conducting material i.e. a material whose conductivity is so hight that free charge moves instantly, without time delay. We recall that α depends continuously from σ so that

$$\lim_{\sigma\to\infty} \alpha(\sigma) = \infty\,. \tag{63}$$

For a forward traveling plane wave we gain the following estimates for $z \in I = [a,b], 0 < a < b < \infty$:

$$\begin{aligned} \lim_{\sigma\to\infty} |\boldsymbol{E}(z,t)| &\le |E^{+}| \lim_{\sigma\to\infty} e^{-\alpha z} = 0 \\ \lim_{\sigma\to\infty} \left|\frac{d\boldsymbol{E}(z,t)}{dz}\right| &\le |E^{+}| \lim_{\sigma\to\infty} \alpha e^{-\alpha a} = 0 \\ \lim_{\sigma\to\infty} |\sigma\boldsymbol{E}(z,t)| &\le |E^{+}| \lim_{\sigma\to\infty} \sigma e^{-\frac{1}{2}\sqrt{\sigma}a} = 0\,. \end{aligned}$$

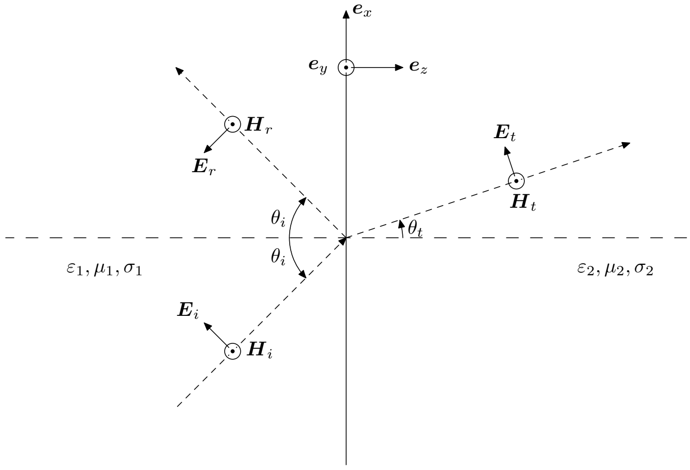
Due to the first two limits the electric field component as well as its curl converges uniformly to zero. Due to Faraday's law the magnetic field component converges uniformly to zero. Finally, the last equation says that within a perfect electric conductor the electric current vanishes uniformly. Therefore, no electromagnetic wave exists within a perfect conductor.

A.3 Scattering of an oblique incident plane wave

We would like to exploit the analytical knowledge of the uniform plane waves to get a feeling for scattering problems. We consider two different materials with data $\varepsilon_1, \mu_1, \sigma_1$ and $\varepsilon_2, \mu_2, \sigma_2$. The boundary between these two medias is assumed to be plane. The plane of incidence is the plane containing the propagation vector of the incident wave and the normal to the boundary. In our case it is the xz plane. The polarization of the plane wave, i.e. the angle of incidence, is arbitrary. It is however always possible to represent the plane wave as a superposition of a so-called parallel polarized and a perpendicular polarized plane wave. For perpendicular polarization, the incident electric field vector is perpendicular to the plane of incidence, as shown in the picture below. For parallel polarization, the incident electric field vector is parallel to or in the plane of incidence. Let us have a closer look at the scattering of parallel polarized and perpendicular polarized plane waves. The representation of the solutions in either case is taken from [8].

1. **Parallel Polarization.**

Figure 1: Parallel polarization



For the case of parallel polarization the phasors of the incident $\boldsymbol{E}_i, \boldsymbol{H}_i$, the reflected $\boldsymbol{E}_r, \boldsymbol{H}_r$ and the transmitted electromagnetic field $\boldsymbol{E}_s, \boldsymbol{H}_s$ read

$$\begin{aligned}
\boldsymbol{E}_i &= E_i \left(\cos(\theta_i)\boldsymbol{e}_x - \sin(\theta_i)\boldsymbol{e}_z\right) e^{-\gamma_1(\sin(\theta_i)x+\cos(\theta_i)z)} \\
\boldsymbol{H}_i &= \frac{E_i}{\hat{\eta}_1} e^{-\gamma_1(\sin(\theta_i)x+\cos(\theta_i)z)} \boldsymbol{e}_y \\
\boldsymbol{E}_r &= -E_r \left(\cos(\theta_i)\boldsymbol{e}_x - \sin(\theta_i)\boldsymbol{e}_z\right) e^{\gamma_1(-\sin(\theta_i)x+\cos(\theta_i)z)} \\
\boldsymbol{H}_r &= \frac{E_r}{\hat{\eta}_1} e^{\gamma_1(-\sin(\theta_i)x+\cos(\theta_i)z)} \boldsymbol{e}_y \\
\boldsymbol{E}_t &= E_t \left(\cos(\theta_t)\boldsymbol{e}_x - \sin(\theta_t)\boldsymbol{e}_z\right) e^{-\gamma_2(\sin(\theta_t)x+\cos(\theta_t)z)} \\
\boldsymbol{H}_t &= \frac{E_t}{\hat{\eta}_1} e^{-\gamma_2(\sin(\theta_t)x+\cos(\theta_t)z)} \boldsymbol{e}_y \,.
\end{aligned}$$

Due to the boundary conditions

$$\begin{aligned}
\gamma_D \boldsymbol{E}_i + \gamma_D \boldsymbol{E}_r &= \gamma_D \boldsymbol{E}_t \\
\gamma_D \boldsymbol{H}_i + \gamma_D \boldsymbol{H}_r &= \gamma_D \boldsymbol{H}_t
\end{aligned}$$

one deduces the following conditions upon the coefficients

$$\begin{aligned}
\frac{\sin\theta_i}{\sin\theta_t} &= \frac{\gamma_2}{\gamma_1} \\
\frac{E_i + E + r}{\hat{\eta}_1} e^{-\gamma_1 \sin\theta_i x} &= \frac{E_t}{\hat{\eta}_2} e^{-\gamma_2 \sin\theta_t x} \,.
\end{aligned}$$

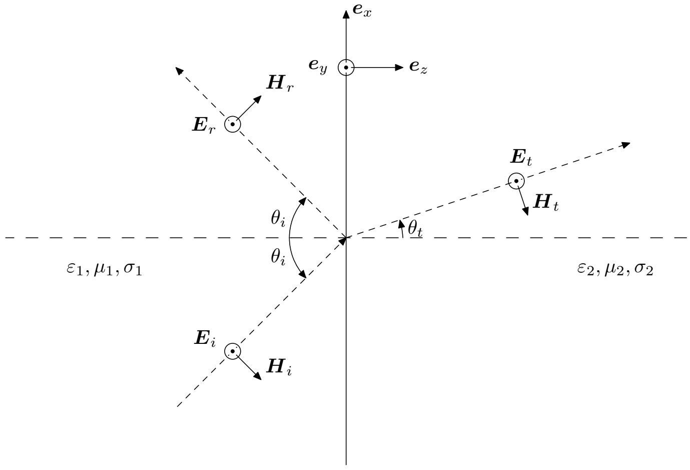
Now, we check the jumps in the normal component of the vector fields $\mu\boldsymbol{H}$, $\sigma\boldsymbol{E}$ and $\varepsilon\boldsymbol{E}$ respectively.

Normal traces of the **magnetic flux**: $[\boldsymbol{n}\cdot\mu\boldsymbol{H}] = 0$ for all $0 \leq \sigma_1, \sigma_2 < \infty$ because this polarization has no normal component.

Normal traces of the **electric flux** $[\boldsymbol{n} \cdot \varepsilon\boldsymbol{E}]$ and the **electric current** $[\boldsymbol{n} \cdot \sigma\boldsymbol{E}]$: ,

$$\begin{aligned}
\boldsymbol{n} \cdot (\boldsymbol{E}_i + \boldsymbol{E}_r)\,|_{z=0} &= -\left(E_i + E_r\right) \sin\theta_i e^{-\gamma_1 \sin\theta_i x} \\
&= -\hat{\eta}_1 \sin\theta_i \frac{E_i + E_r}{\hat{\eta}_1} e^{-\gamma_1 \sin\theta_i x} = -\frac{\hat{\eta}_1}{\hat{\eta}_2}\frac{\gamma_2}{\gamma_1} \sin\theta_t E_t e^{-\gamma_2 \sin\theta_t x} \\
&= \frac{\hat{\eta}_1 \gamma_2}{\hat{\eta}_2 \gamma_1} \left(\boldsymbol{n} \cdot E_t|_{z=0}\right) \,.
\end{aligned}$$

Figure 2: Perpenticular polarization



A closer look at the coefficient reveals

$$\begin{aligned}\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} &= \left(\frac{j\omega\mu_1}{\sigma_1+j\omega\varepsilon_1}\frac{\sigma_2+j\omega\varepsilon_2}{j\omega\mu_2}\frac{j\omega\mu_2(\sigma_2+j\omega\varepsilon)}{j\omega\mu_1(\sigma_1+j\omega\varepsilon)}\right)^{\frac{1}{2}} \\ &= \frac{\sigma_2+j\omega\varepsilon_2}{\sigma_1+j\omega\varepsilon_1}\,.\end{aligned}$$

For $\omega > 0$, we conclude

- $[\boldsymbol{n}\cdot\mu\boldsymbol{H}] = 0$ for every media.
- If $\frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_1}{\varepsilon_2}$, $[\boldsymbol{n}\cdot\varepsilon\boldsymbol{E}] = 0$ for lossy media, because $\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} = \frac{\varepsilon_2}{\varepsilon_1}$.
- If $\frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_1}{\varepsilon_2}$, $[\boldsymbol{n}\cdot\sigma\boldsymbol{E}] = 0$ for lossy media, because $\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} = \frac{\sigma_2}{\sigma_1}$.
- If $\frac{\varepsilon_1}{\varepsilon_2} \neq \frac{\varepsilon_1}{\varepsilon_2}$, $[\boldsymbol{n}\cdot\varepsilon\boldsymbol{E}] \neq 0$ for lossy media, because $\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} \neq \frac{\varepsilon_2}{\varepsilon_1}$.
- If $\frac{\varepsilon_1}{\varepsilon_2} \neq \frac{\varepsilon_1}{\varepsilon_2}$, $[\boldsymbol{n}\cdot\sigma\boldsymbol{E}] \neq 0$ for lossy media, because $\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} \neq \frac{\sigma_2}{\sigma_1}$.
- $[\boldsymbol{n}\cdot\varepsilon\boldsymbol{E}] = 0$ for lossless media, because $\frac{\hat{\eta}_1\gamma_2}{\hat{\eta}_2\gamma_1} = \frac{\varepsilon_2}{\varepsilon_1}$.

2. **Perpendicular Polarization.**

   For the case of perpendicular polarization the phasors of the incident $\boldsymbol{E}_i, \boldsymbol{H}_i$, the reflected $\boldsymbol{E}_r, \boldsymbol{H}_r$ and the transmitted electromagnetic field $\boldsymbol{E}_s, \boldsymbol{H}_s$ read

$$\begin{aligned}\boldsymbol{E}_i &= E_i e^{-\gamma_1(\sin(\theta_i)x+\cos(\theta_i)z)}\boldsymbol{e}_y \\ \boldsymbol{H}_i &= \frac{E_i}{\hat{\eta}_1}\left(\cos(\theta_i)\boldsymbol{e}_x - \sin(\theta_i)\boldsymbol{e}_z\right)e^{-\gamma_1(\sin(\theta_i)x+\cos(\theta_i)z)} \\ \boldsymbol{E}_r &= E_r e^{\gamma_1(-\sin(\theta_i)x+\cos(\theta_i)z)}\boldsymbol{e}_y \\ \boldsymbol{H}_r &= \frac{E_r}{\hat{\eta}_1}\left(\cos(\theta_i)\boldsymbol{e}_x - \sin(\theta_i)\boldsymbol{e}_z\right)e^{\gamma_1(-\sin(\theta_i)x+\cos(\theta_i)z)} \\ \boldsymbol{E}_t &= E_t e^{-\gamma_2(\sin(\theta_t)x+\cos(\theta_t)z)}\boldsymbol{e}_y \\ \boldsymbol{H}_t &= \frac{E_t}{\hat{\eta}_1}\left(\cos(\theta_t)\boldsymbol{e}_x - \sin(\theta_t)\boldsymbol{e}_z\right)e^{-\gamma_2(\sin(\theta_t)x+\cos(\theta_t)z)}\,.\end{aligned}$$

   Due to the boundary conditions

$$\begin{aligned}\gamma_D\boldsymbol{E}_i + \gamma_D\boldsymbol{E}_r &= \gamma_D\boldsymbol{E}_t \\ \gamma_D\boldsymbol{H}_i + \gamma_D\boldsymbol{H}_r &= \gamma_D\boldsymbol{H}_t\end{aligned}$$

one deduces the following conditions upon the coefficients

$$\begin{aligned}\frac{\sin\theta_i}{\sin\theta_t} &= \frac{\gamma_2}{\gamma_1}\\ \frac{E_i+E_r}{\hat{\eta}_1}e^{-\gamma_1\sin\theta_i x} &= \frac{E_t}{\hat{\eta}_2}e^{-\gamma_2\sin\theta_t x}\,.\end{aligned}$$

Now, we check the jumps in the normal component of the vector fields $\mu\boldsymbol{H}$, $\sigma\boldsymbol{E}$ and $\varepsilon\boldsymbol{E}$ respectively.

- $[\boldsymbol{n}\cdot\mu\boldsymbol{H}]=0$ for every media, because

$$\begin{aligned}\boldsymbol{n}\cdot(\boldsymbol{H}_i+\boldsymbol{H}_r)\,|_{z=0} &= -\frac{E_i+E_r}{\hat{\eta}_1}\sin\theta_i e^{-\gamma_1\sin\theta_i x}\\ &= -\frac{\hat{\eta}_2}{\hat{\eta}_1}\frac{\gamma_2}{\gamma_1}\sin\theta_t\frac{E_t}{\hat{\eta}_2}e^{-\gamma_2\sin\theta_t x}\\ &= \frac{\hat{\eta}_2\gamma_2}{\hat{\eta}_1\gamma_1}\left(\boldsymbol{n}\cdot H_t|_{z=0}\right)\,.\end{aligned}$$

  A closer look at the coefficient reveals

$$\frac{\hat{\eta}_2\gamma_2}{\hat{\eta}_1\gamma_1} = \frac{\mu_2}{\mu_1}\,.$$

- $[\boldsymbol{n}\cdot\varepsilon\boldsymbol{E}]=[\boldsymbol{n}\cdot\sigma\boldsymbol{E}]=0$, because the normal component of the electric field vanishes.

B Survey: Interface conditions

Transmission problem between two conductors

$0 < \sigma_1 < \infty,\ \varepsilon_1,\ \mu_1 \quad \Omega_1 = \text{ conductor 1}$

$0 < \sigma_2 < \infty,\ \varepsilon_2,\ \mu_2 \quad \Omega_2 = \text{ conductor 2}$

$$\begin{array}{rcllll}
\nabla \times \boldsymbol{E} & = & -j\omega \boldsymbol{B} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{E}] = \boldsymbol{0} \\
\nabla \times \boldsymbol{H} & = & j\omega \boldsymbol{D} + \boldsymbol{J} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{H}] = \boldsymbol{0} \\
\nabla \cdot \boldsymbol{D} & = & \rho & + & \rho_\Gamma \delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{D}] = \rho_\Gamma \\
\nabla \cdot \boldsymbol{B} & = & 0 & + & 0\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{B}] = 0 \\
\\
\nabla \cdot \boldsymbol{J} & = & -j\omega\rho & - & j\omega\rho_\Gamma\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{J}] = -j\omega\rho_\Gamma
\end{array}$$

Transmission problem between two perfect dielectrics

$\sigma_1 = 0,\ \varepsilon_1,\ \mu_1 \quad \Omega_1 = \text{ perfect dielectric 1}$

$\sigma_2 = 0,\ \varepsilon_2,\ \mu_2 \quad \Omega_2 = \text{ perfect dielectric 2}$

$$\begin{array}{rcllllll}
\nabla \times \boldsymbol{E} & = & -j\omega \boldsymbol{B} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{E}] & = & \boldsymbol{0} \\
\nabla \times \boldsymbol{H} & = & j\omega \boldsymbol{D} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{H}] & = & \boldsymbol{0} \\
\nabla \cdot \boldsymbol{D} & = & 0 & + & 0\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{D}] & = & 0 \\
\nabla \cdot \boldsymbol{B} & = & 0 & + & 0\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{B}] & = & 0
\end{array}$$

$- - -$

Transmission problem between a conductor and a perfect dielectric

$0 < \sigma_1 < \infty,\ \varepsilon_1,\ \mu_1 \quad \Omega_1 = \text{ conductor}$

$\sigma_2 = 0,\ \varepsilon_2,\ \mu_2 \quad \Omega_2 = \text{ perfect dielectric}$

$$\begin{array}{rcllllll}
\nabla \times \boldsymbol{E} & = & -j\omega \boldsymbol{B} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{E}] & = & \boldsymbol{0} \\
\nabla \times \boldsymbol{H} & = & j\omega \boldsymbol{D} + \boldsymbol{J}_1\chi_{\Omega_1} & + & \boldsymbol{0}\delta_\Gamma & [\boldsymbol{n} \times \boldsymbol{H}] & = & \boldsymbol{0} \\
\nabla \cdot \boldsymbol{D} & = & \rho_1\chi_{\Omega_1} & + & \rho_\Gamma\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{D}] & = & \rho_\Gamma \\
\nabla \cdot \boldsymbol{B} & = & 0 & + & 0\delta_\Gamma & [\boldsymbol{n} \cdot \boldsymbol{B}] & = & 0 \\
\\
\nabla \cdot \boldsymbol{J}_1 & = & -j\omega\rho\chi_{\Omega_1} & - & j\omega\rho_\Gamma\delta_\gamma & \boldsymbol{n} \cdot \boldsymbol{J}_1 & = & -j\omega\rho_\Gamma
\end{array}$$

Transmission problem between a perfect conductor and a conductor

$\sigma_1 = \infty,\ \varepsilon_1,\ \mu_1 \quad \Omega_1 = \text{ perfect conductor}$

$0 < \sigma_2 < \infty,\ \varepsilon_2,\ \mu_2 \quad \Omega_2 = \text{ conductor}$

$$\begin{array}{rcllllll}
\nabla \times \boldsymbol{E}_2 & = & -j\omega \boldsymbol{B}_2 & + & \boldsymbol{0}\delta_\Gamma & \boldsymbol{n} \times \boldsymbol{E}_2 & = & \boldsymbol{0} \\
\nabla \times \boldsymbol{H}_2 & = & j\omega \boldsymbol{D}_2 + \boldsymbol{J}_2 & + & \boldsymbol{J}_\Gamma\delta_\Gamma & \boldsymbol{n} \times \boldsymbol{H}_2 & = & \boldsymbol{J}_\Gamma \\
\nabla \cdot \boldsymbol{D}_2 & = & \rho_2 & + & \rho_\Gamma\delta_\Gamma & \boldsymbol{n} \cdot \boldsymbol{D}_2 & = & \rho_\Gamma \\
\nabla \cdot \boldsymbol{B}_2 & = & 0 & + & 0\delta_\Gamma & \boldsymbol{n} \cdot \boldsymbol{B}_2 & = & 0 \\
\\
\nabla \cdot \boldsymbol{J}_2 & = & -j\omega\rho_2 & - & (j\omega\rho_\Gamma + \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma)\,\delta_\Gamma & \boldsymbol{n} \cdot \boldsymbol{J}_2 & = & -j\omega\rho_\Gamma - \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma
\end{array}$$

Transmission problem between a perfect conductor and a perfect dielectric

$$\sigma_1 = \infty,\, \varepsilon_1,\, \mu_1 \quad \Omega_1 = \text{ perfect conductor}$$

$$\sigma_2 = 0,\, \varepsilon_2,\, \mu_2 \quad \Omega_2 = \text{ perfect dielectric}$$

$$\begin{array}{rclcl} \nabla \times \boldsymbol{E}_2 & = & -j\omega \boldsymbol{B}_2 & + & \boldsymbol{0}\delta_\Gamma \\ \nabla \times \boldsymbol{H}_2 & = & j\omega \boldsymbol{D}_2 & + & \boldsymbol{J}_\Gamma \delta_\Gamma \\ \nabla \cdot \boldsymbol{D}_2 & = & 0 & + & \rho_\Gamma^\infty \delta_\Gamma \\ \nabla \cdot \boldsymbol{B}_2 & = & 0 & + & 0\delta_\Gamma \\ \\ 0 & = & 0 & - & (j\omega\rho_\Gamma + \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma)\,\delta_\Gamma \end{array} \qquad \begin{array}{rcl} \boldsymbol{n} \times \boldsymbol{E}_2 & = & \boldsymbol{0} \\ \boldsymbol{n} \times \boldsymbol{H}_2 & = & \boldsymbol{J}_\Gamma \\ \boldsymbol{n} \cdot \boldsymbol{D}_2 & = & \rho_\Gamma \\ \boldsymbol{n} \cdot \boldsymbol{B}_2 & = & 0 \\ \\ \nabla_\Gamma \cdot \boldsymbol{J}_\Gamma & = & -j\omega\rho_\Gamma \end{array}$$

References

[1] S. S. Antman and J. E. Osborn. The principle of virtual work and integral laws of motion. *Arch. Rational Mech. Anal.*, 69(3):231–262, 1979.

[2] A. C. Balanis. *Electromagnetics*. Wiley, 1989.

[3] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95(3):459–485, 2003.

[4] L. Demkowicz and L. Vardapetyan. Modeling of electromagnetic absorption/scattering problems using hp-adaptive finite elements. *Comput. Methods Appl. Mech. Engrg.*, 152(1-2):103–124, 1998. Symposium on Advances in Computational Mechanics, Vol. 5 (Austin, TX, 1997).

[5] J. D. Jackson. *Classical electrodynamics*. John Wiley & Sons Inc., New York, second edition, 1975.

[6] Demkowicz L., Kurtz J., Pardo D., Paszynski M., Rachowicz W., and Zdunek A. *Computing with hp-adaptive finite elements. Vol. 2*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2008. Frontiers: three dimensional elliptic and Maxwell problems with applications.

[7] J.-C. Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Integral representations for harmonic problems.

[8] C.R. Paul and S.A. Nasar. *Introduction to Electromagnetic Fields*. McGraw-Hill, Inc., 1987.