APPLICATION OF COUPLED FINITE/INFINITE ELEMENT METHOD TO MODELING OF ABSORPTION AND SCATTERING OF EM WAVES

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ABSTRACT

The paper reviews the construction of the Mie series solution for the problem of scattering a plane wave on a dielectric, absorption sphere; and the technology of coupled $hp$ finite/infinite element discretization for a general class of exterior Maxwell problems. The performed numerical experiments indicate that $G^1$-continuous geometry representation is sufficient for modeling propagation and absorption of EM waves in curvilinear geometries.

Keywords: finite/infinite elements, Mie theory, electromagnetic scattering, error estimation

1. INTRODUCTION

Scattering and absorption are the widest and most important classes of problems in computational electromagnetics. Nowadays, there exist a variety of new applications, ranging from high-speed communications and computing up to medical applications [1]. The calculation and the measurement of electromagnetic energy absorption in humans for near-field exposure conditions are particularly important because of the very high strengths of leakage fields from industrial Radio Frequency (RF) equipment. The proposed work is motivated by studying EM waves transfer properties of the human head. One particular interest for the proposed work is the level and distribution of Specific Absorption Rates (SAR) in the human head.

In the case of geometrically complex media and discontinuous materials, the Finite Element/Infinite Element (FE/IE) approach seems to be a natural exact way to deal with the unbounded region problem [2, 3, 4]. The approach used in the presented work bypasses certain difficulties common to hybrid finite element-boundary integral type of methods by using $H$ (curl)-compatible curvilinear finite and infinite elements. The idea of the discretization by the coupled FE/IE is standard [4, 5]. The scatterer is surrounded with a truncating sphere, and the truncated domain is discretized using $hp$ finite elements. The mesh is then matched with the infinite element discretization outside the truncating sphere with a strong enforcement of the continuity conditions across the finite/infinite elements boundary. From the algorithmical point of view, the infinite elements are treated as standard finite elements, except for the definition of the approximation in the radial direction. The method is supported with stability analysis and convergence proof [6] for a spherical truncating surface.

In this paper, two major contributions are presented:

- Verification of the FE/IE code using the Mie series solution.
A complete mathematical solution to the problem of scattering of electromagnetic plane wave by a sphere first was obtained by Gustav Mie in 1908 [7], the well known Mie infinite series solution. The presented FE/IE code is verified by comparing the magnitude and phase of both electric field $E$ and magnetic field $H$ with the exact Mie solution for the benchmark problem.

- Assessment of geometry induced error, a proper definition of the FE error for curvilinear geometries and $H(\text{curl})$-conforming elements.

Following the general framework of handling curvilinear geometries in high accuracy FE simulations [8], $H(\text{curl})$-discretization errors are computed for Maxwell problems. The accuracy of FE/IE approximation is assessed using a precise definition of solution error incorporating the effects of geometry approximation. Both geometry and solution error for curvilinear geometries are computed and discussed.

A broad outline of the paper follows. Section 2 describes the classical formulation for the scattering and absorption EM problems. Section 3 discusses the implementation details of Mie-Scattering and Mie-Absorption solutions. Section 4 presents a couple of examples to verify the FE code and outline the essentials of the hybrid FE/IE approach. The computation of the error using a precise definition of $H(\text{curl})$ norm is discussed next. Numerical, highly accurate, and fairly converged solutions are presented. Section 5 summarizes the work and sketches future developments.

2. SCATTERING AND ABSORPTION EM PROBLEMS

2.1 Time harmonic Maxwell equations

We shall discuss the simplest (linear, isotropic) version of Maxwell’s equations. Given a domain $\Omega \subset \mathbb{R}^3$, we wish to determine electric field $E(x)$ and magnetic field $H(x)$ that satisfy:

- Faraday’s law
  \[ \nabla \times E = -\frac{\partial}{\partial t}(\mu H), \]

- Ampere’s law
  \[ \nabla \times H = J^{\text{imp}} + \sigma E + \frac{\partial}{\partial t}(\epsilon E), \]

- Gauss law for the magnetic flux
  \[ \nabla \cdot (\mu H) = 0, \]

- Gauss law for the electric flux
  \[ \nabla \cdot (\epsilon E) = \rho. \]

Here $\mu, \sigma, \epsilon$ denote the material data: permeability, conductivity, and permittivity, assumed to be piecewise constant, $J^{\text{imp}}$ denotes a prescribed, given impressed current, and $\rho$ is the corresponding free charge density.

In order to reduce the number of unknowns, the first order Maxwell system is usually reduced to a single vector-valued wave equation, expressed either in terms of $E$, or $H$. Assuming the “positive” anzatz $E(x, t) = \Re(E(x) e^{i\omega t})$, the Maxwell equations reduced into,

\[ \nabla \times (\frac{1}{\mu} \nabla \times E) - (\omega^2 \epsilon - i \omega \sigma)E = -i\omega J^{\text{imp}}. \]  

The sign in the anzatz time factor $e^{i\omega t}$ affects the phase of the EM fields. Substituting $i$ instead of $-i$, we can easily switch in between the two formulations. Our FE/IE code is using the positive anzatz, however, in most of the EE literature, the negative sign is assumed. Once the electric field has been determined, the Faraday equation can be integrated to find the corresponding magnetic field.
2.2 Scattering of a plane wave on a dielectric sphere

Let $\Omega$ denote a sphere with radius $a$ occupied by a dielectric. We seek the solution in terms of electric field $E(x)$ defined in the internal domain $\Omega$ with material constants $\mu, \sigma, \epsilon$ and external free space domain $\Omega_0^+$ with material constants $\mu_0, \epsilon_0$, see Fig. 1. In FE/IE simulations, the free space domain $\Omega$ is partitioned into a near field domain $\Omega_0^-$ between the boundary of the scatterer $\Gamma$ and a spherical truncating boundary $\Gamma_0$; and a far field domain $\Omega_0^+$ outside of the truncating boundary. Finite elements are used then to discretize the Maxwell equations within the truncating boundary, i.e., $\epsilon, \in \Omega \cup \Omega_0^-$, with the infinite element discretization applied to $\Omega_0^+$. The construction is motivated with large variations of the solution in the near field domain (scattering from edges, vertices, rough surfaces), which requires good local resolution capabilities provided by finite elements. Typically, the truncating boundary is placed one wavelength away from the scatterer.

The excitation is given in terms of an incident linearly polarized plane wave of amplitude $E_0$ and free space wave number $k_0$. We shall assume an x-polarized wave, propagating in z direction,

$$E_{x}^{inc} = E_0 e^{-i k_0 z}. \quad (2)$$

The total electric field $E^{tot}$ can be decomposed as,

$$E^{tot} = E^{inc} + E^{sca}, \quad (3)$$

where $E^{sca}$ is the unknown scattered field to be determined. The total field $E^{tot}$ must satisfy the reduced wave equation (1), and the incident field $E^{inc}$ satisfies the same equation in the free space, i.e., with $\mu = \mu_0, \epsilon = \epsilon_0, \sigma_0 = 0$. With $\mu = \mu_0$ in the whole space, this implies that the impressed current $J^{imp}$ can be expressed as,

$$J^{imp} = \frac{\omega (\epsilon - \epsilon_0) + i \omega \sigma}{-i \omega} E^{inc}. \quad (4)$$

By the same reasoning, $E^{sca}$ also satisfies the following boundary conditions:

- the interface conditions along $\Gamma$,

$$n \times [E^{sca}] = -n \times [E^{inc}] = 0 \quad (5)$$

$$n \times [\frac{1}{\mu} \nabla \times E^{sca}] = -n \times [\frac{1}{\mu} \nabla \times E^{inc}] = 0$$

Here [ ] denotes a jump of function $f$ across $\Gamma$, 

- Silver - Müller radiation condition at $\infty$,

$$\lim_{r \to \infty} [n \times E^{sca} - j k_0 e_r \times (e_r \times E^{sca})] = 0 \quad (6)$$
where $r$ is a spherical coordinate with corresponding unit vector $e_r$ and origin at an arbitrary point (taken usually within the scatterer). This implies that, asymptotically, the scattered field is an outgoing spherical wave with leading factor $e^{-ij_0r}$. Note that the incident field needs not satisfy the radiation condition. The interface and radiation conditions ensure uniqueness of the solution \[9\].

### 2.3 Nondimensionalization of Maxwell equations

Dimensional analysis is performed for any numerical simulations which require equations to be independent of units of measurement. For reduced wave equation (1), the following scales are introduced,

- **Material scale** - $\epsilon_0, \mu_0$
  For free space, $\epsilon_0 \approx \frac{1}{36\pi}10^{-9}$ [C$^2$/Nm$^2 = $F/m], $\mu_0 = 4\pi10^{-7}$ [N/A$^2 = $h/m]. Material constants can be expressed in terms of the relative permittivity $\epsilon_r$, and relative permeability $\mu_r$
  $$\epsilon_r = \frac{\epsilon}{\epsilon_0}, \quad \mu_r = \frac{\mu}{\mu_0},$$

- **Spatial scale** - $a$
  We shall use the radius of the sphere $a$ as a spatial scale for defining nondimensional coordinates corresponding to the physical coordinates $x$
  $$x' = \frac{x}{a},$$

- **Solution scale** - $E_0$
  The total $E^{tot}$ and scattered electric fields $E^{sca}$ are scaled by the amplitude of incident wave $E_0$
  $$E^{tot'} = \frac{E^{tot}}{E_0},$$
  $$E^{sca'} = \frac{E^{sca}}{E_0},$$

As a result, the corresponding nondimensional quantities are introduced,

- **Nondimensional wave number (frequency)** - $\Omega$
  $$\Omega = \omega\sqrt{\epsilon_0\mu_0}a,$$

- **Nondimensional conductivity** - $\Sigma$
  $$\Sigma = \sigma\sqrt{\frac{\mu_0}{\epsilon_0}a},$$

- **Nondimensional impressed current** - $J^{imp'}$
  $$J^{imp'} = \frac{J^{imp}\sqrt{\frac{\mu_0}{\epsilon_0}}a}{E_0}.$$

The dimensionless form of reduced wave equation is identical with (1),

$$\nabla \times \left( \frac{1}{\mu_r} \nabla \times E^{sca'} \right) - \left( \Omega^2 \epsilon_r - i\Sigma \right) E^{sca'} = -i\Omega J^{imp'}.$$

(8)

In what follows, we shall drop the primes in the notation and return to the original notation $x, \mu, \epsilon, \omega, \sigma, J^{imp}$, with the understanding that the original variables have been replaced with their non-dimensional counterparts.
Table 1: Nondimensional material constants for a homogeneous sphere.

<table>
<thead>
<tr>
<th>εr</th>
<th>µr</th>
<th>Ω</th>
<th>Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>π</td>
<td>0</td>
</tr>
</tbody>
</table>

3. MIE-SERIES SOLUTIONS FOR THE SCATTERING AND ABSORPTION OF EM WAVES ON A SPHERE

The Mie series solution is a robust and efficient algorithm used to compute electromagnetic plane wave scattering by a dielectric sphere of arbitrary radius and refractive index. In this study, we implement the Mie-theory for a homogeneous sphere with radius \( a_0 = 1 \) (nondimensional), and material properties and angular frequency specified in Table 1. Following the common approach, we position the truncating sphere one wavelength away from the scatterer, the radius of the truncating sphere, \( c = a + 2a_0 = 3 \). For the human head, \( a = 0.1 \text{m} \), and the nondimensional frequency \( n = \frac{f}{2\pi a_0} \) compares to the physical frequency \( f = 2a_0 \omega_0 \) [9].

We select the origin of a Cartesian coordinate system at the center of the sphere, with the positive \( z \) axis along the direction of propagation of the incident wave. The incident electric vector is polarized in the direction of the \( x \) axis. The incident field \( \mathbf{E}^{\text{inc}} \) is defined in the whole space. The Mie solution will be represented in terms of the scattered field \( \mathbf{E}^{\text{sca}} \) outside of the spherical domain \( \Omega_0 \), and the total field \( \mathbf{E}^{\text{tot}} \) within scatterer \( \Omega_0 \).

The boundary conditions require that the transverse components of the total electric and magnetic fields are continuous across the material interface,

\[
\begin{align*}
\mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{sca}} &= \mathbf{E}^{\text{tot}} \times \hat{e}_r, \\
\mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{sca}} &= \mathbf{H}^{\text{tot}} \times \hat{e}_r,
\end{align*}
\]

where \( \hat{e}_r \) is the unit outward normal vector of the sphere.

The FE/IE code is implemented for accurate and reliable numerical modeling of the Mie theory in modern way [10, 11, 12]. The incident plane wave \( \mathbf{E}^{\text{inc}} \) as well as the scattering field \( \mathbf{E}^{\text{sca}} \) is expanded into radiating spherical vector wave functions. The internal field \( \mathbf{E}^{\text{tot}} \) is expanded into regular spherical vector wave functions. By enforcing the boundary condition (10) on the spherical surface, the expansion coefficients of the scattered field can be computed.

3.1 Solutions to vector wave equation

Taking gradient of (1), we learn that the electric field must be divergence free in a homogeneous region,

\[-(\omega^2 \epsilon - i\omega\sigma)\nabla \cdot \mathbf{E} = -i\omega \nabla \cdot \mathbf{J}^{\text{imp}} = 0.\]

Consequently, within any homogeneous region, solution of (1) reduces to the vector Helmholtz equation,

\[\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0,\]

with \( k^2 = \omega^2 \epsilon \mu - i\omega \sigma \). Additionally, solution \( \mathbf{E} \) must be divergence-free,

\[\nabla \cdot \mathbf{E} = 0.\]

Solution of (12) and (13) can be expanded in terms of vector harmonics \( \mathbf{M} \) and \( \mathbf{N} \),

\[\mathbf{M} = \nabla \times (\mathbf{r} \psi), \quad \mathbf{N} = \frac{\nabla \times \mathbf{M}}{k},\]

where \( \mathbf{r} \) is the pilot vector, and \( \psi \) is a solution to the scalar Helmholtz equation,

\[\nabla^2 \psi + k^2 \psi = 0.\]

Thus, finding a solution of (1) in a homogeneous domain, has been reduced to the solution of the scalar Helmholtz equation. Separation of variables \( \psi = f(r)g(\theta)h(\phi) \) leads to three separated equations,
• the Bessel equation in \( r \),

\[
\frac{d}{dr} \left( r^2 \frac{df}{dr}(r) \right) + [(kr)^2 - n(n+1)]f(r) = 0, \tag{17}
\]

• Legendre equation in \( \theta \)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dg}{d\theta}(\theta) \right) + \left[ n(n+1) - \frac{m^2}{\sin^2 \theta} \right] g(\theta) = 0 \tag{18}
\]

• second-order linear ordinary differential equation in \( \phi \)

\[
\frac{d^2 h}{d\phi^2}(\phi) + m^2 h(\phi) = 0 \tag{19}
\]

where the separation constants \( m^2 \) and \( n(n+1) \) have been determined from conditions that \( g \) and \( h \) must satisfy. As a result, we generate two families of scalar spherical harmonics,

\[
\psi_{emn} = \cos m \phi P^m_n(\cos \theta) z_n(kr),
\]

\[
\psi_{omn} = \sin m \phi P^m_n(\cos \theta) z_n(kr), \quad m = 1, \ldots, n
\]

where subscripts \( e \) and \( o \) denote even and odd. Here \( P^m_n(\cos \theta) \) are associated Legendre functions of the first kind, of degree \( n \) and order \( m \); \( z_n \) is any of the four functions: spherical Bessel functions of the first and second kind, \( j_n \) and \( y_n \), spherical Hankel functions of the first and second kind, \( h_n^{(1)} \) and \( h_n^{(2)} \) [13]. Because of the completeness of functions \( \cos m \phi \) and \( \sin m \phi \), solution to (16) may be expanded into an infinite series.

![Figure 2: Reproducing a plane wave. Contour plots of the exact solution of (a) \( E_x \) and (b) \( H_y \) on the internal domain \( \Omega \) of the dielectric sphere and near field domain \( \Omega_o \) ](image)

3.2 Incident field

In order to enable modal decoupling, it is necessary to expand the plane wave in vector spherical harmonics,

\[
E^{inc} = \sum_{n=0}^{\infty} \sum_{m=-l}^{l} (B_{emn} M_{emn} + B_{omn} M_{omn} + A_{emn} N_{emn} + A_{omn} N_{omn}). \tag{21}
\]
For incident wave, where

\[ E_{\text{inc}} = \sum_{n=1}^{\infty} E_n(M_{01n} - iN_{01n}) \]

and

\[ H_{\text{inc}} = -\frac{k}{\omega\mu} \sum_{n=1}^{\infty} E_n(M_{11n} + iN_{11n}). \]

The spherical harmonics \( M_{01n} \) and \( N_{01n} \) are given as,

\[ M_{01n} = \begin{bmatrix} \cos \phi \tau_n(\cos \theta) z_n(\rho) \\ -\sin \phi \tau_n(\cos \theta) z_n(\rho) \end{bmatrix} \]

\[ N_{01n} = \begin{bmatrix} n(n+1) \cos \phi \tau_n(\cos \theta) \frac{z_n(\rho)}{\rho} \\ \cos \phi \tau_n(\cos \theta) \frac{[\rho j_n(\rho)]'}{\rho} \\ -\sin \phi \tau_n(\cos \theta) \frac{[\rho j_n(\rho)]'}{\rho} \end{bmatrix} \]

where \( \tau_n = P_n^{(1)} / \sin \theta \) and \( \tau_n = dP_n^{(1)} / d\theta \) describe the angular scattering patterns of the spherical harmonics.

For incident wave, \( \rho = \Omega \tau \) in (27); \( z_n \) denotes \( j_n \) since \( y_n(\rho) \) possesses a singularity at \( \rho = 0 \). Note that the derivatives of the spherical Bessel functions can be calculated from,

\[ [\rho j_n(\rho)]' = \rho j_n(\rho) - n j_n(\rho). \]

The other two harmonic vectors can be obtained as,

\[ M_{11n} = \frac{\partial M_{01n}}{\partial \phi} \]

\[ N_{01n} = -\frac{\partial N_{01n}}{\partial \phi}. \]

### 3.3 Near fields

Near fields include two fields: the scattered wave outside the sphere, and the internal wave produced inside the sphere itself. Both are similar to the expansion of the incident wave in form, but the scattered wave outside the sphere involves spherical Hankel functions of the first kind, while that inside involves spherical Bessel functions of the first kind.

The boundary conditions (10) , the orthogonality of the vector harmonics and the form of the expansion of the incident field (23) (24) dictate the form of the expansions of the scattered field and the field inside the sphere: the coefficients in these expansions vanish for all \( m \neq 1 \). The series of coefficients obtained are denoted \( a_n, b_n, c_n \) and \( d_n \) [14],

\[ a_n = \frac{I^2 j_n(\Omega)[\Omega j_n(\Omega)]'}{I^2 j_n(\Omega)[\Omega h_n^{(1)}(\Omega)]'} - \mu_1 j_n(\Omega)[\Omega j_n(\Omega)]' \]

\[ b_n = \frac{\mu_1 j_n(\Omega)[\Omega j_n(\Omega)]'}{\mu_1 j_n(\Omega)[\Omega h_n^{(1)}(\Omega)]'} - j_n(\Omega)[\Omega j_n(\Omega)]' \]
The incident fields $E_i$

$\begin{align*}
\text{Order of approximation} & \quad \text{Relative Error in energy norm} \\
1 & \quad 10^{-5} \\
2 & \quad 10^{-4} \\
3 & \quad 10^{-3} \\
4 & \quad 10^{-2} \\
5 & \quad 10^{-1} \\
6 & \quad 0 
\end{align*}$

Figure 3: Reproducing the plane wave. The FE and geometry errors vs the order of approximation

Here $l = \sqrt{\mu_r \rho}$ denotes the refractive index of the host medium. Often $\mu_r = 1$, then the parameters used in radiative transfer $E_{\text{ sca}}$ depend on $a_n$ and $b_n$, but not on $c_n$ and $d_n$. However in our case, the latter coefficients are needed when the electric field inside the sphere is of interest, e.g. to test the field penetration in the sphere, and to compute the distribution of absorption.

The total internal fields $E_t$ and $H_t$ for the internal fields are given by

$$E^{\text{tot}} = \sum_{n=1}^{\infty} E_n (c_n M_{\text{el}n} - id_n N_{\text{el}n})$$

$$H^{\text{tot}} = \frac{-k}{\omega \mu} \sum_{n=1}^{\infty} E_n (d_n M_{\text{el}n} + i c_n N_{\text{el}n})$$

where $\rho = r \Omega$ and $z_n = j_n$ for the two spherical harmonics $M_{\text{el}n}$ and $N_{\text{el}n}$ in (27).

Finally, the expansion of the scattered field is therefore,

$$E_s = \sum_{n=1}^{\infty} E_n (i a_n N_{\text{el}n} - b_n M_{\text{el}n})$$

$$H_s = \frac{k}{\omega \mu} \sum_{n=1}^{\infty} E_n (i b_n N_{\text{el}n} + a_n M_{\text{el}n}).$$

In the region outside the sphere, both $j_n$ and $y_n$ are well behaved; and, the expansion of the scattered field involves a linear combinations of these two functions. One of the two types of Hankel function $h_n$ is required by considering the asymptotic expansions of $h_n$ [15],

$$h_n^{(1)}(kr) \approx \frac{(-i)^n e^{ikr}}{ikr}$$

$$h_n^{(2)}(kr) \approx -\frac{(i)^n e^{-ikr}}{ikr}.$$
If the positive Anzatz is used, (34) represents an outgoing spherical wave and (35) corresponds to an incoming spherical wave. Therefore,

\[ z_n = h_n^{(2)} = j_n(z) - iy_n(z) \]  

is used in the two spherical harmonics \( M_{\alpha n} \) and \( N_{\epsilon n} \) in (30). On the other hand, if the anzatz time factor has a negative sign, we use \( z_n = h_n^{(1)} \) instead. All infinite series can be truncated after \( n_{\text{max}} \) terms. For this number Bohren and Huffman (1983) proposed the value

\[ n_{\text{max}} = \Omega + 4\Omega^{1/3} + 2 \]  

and this value is used here as well.

4. VERIFICATION OF THE FE/IE CODE

We use the Mie theory to verify the \( hp \)-adaptive FE method described in [16, 17]. The method is based on hexahedral elements, with locally variable order of approximation \( p \) and element size \( h \). The \( hp \) method is ideal for modeling singular solutions such as those resulting from scattering of EM waves on irregular geometries or discontinuous materials. In general, the \( hp \) methods can model highly irregular solutions, unaccessible with standard finite element or finite difference approximations. The used element generalizes Nedelec’s elements of the first kind for hexahedra [18], and the approximation is based on H(curl)-conforming shape functions.

![Figure 4](image)

**Figure 4**: Reproducing the internal field. Contour plots of the exact solution (a) \( E_x \) and (b) \( H_y \) of the internal fields domain \( \Omega \) of the sphere.

4.1 Reproducing a plane wave

As a first test for the Finite Element code alone (no infinite elements yet), we attempt to reproduce the plane wave solution within the domain including the sphere and the near field exterior domain \( \Omega^- \). Free space material constants are assumed throughout the domain, and the problem is driven with non-homogeneous Dirichlet boundary conditions corresponding to the plane wave. Since the incident electric vector of plane wave is polarized in the direction of the \( x \) axis, all field components vanish except for \( E_x \) and \( H_y \), see Fig.2.
Seven curvilinear hexahedra are used to represent the geometry of the dielectric sphere, and another six curvilinear hexahedra are used for the near field domain \( \Omega_0 \).

The FE error is evaluated in relative \( H(\text{curl}) \) norm incorporating the \( L^2 \)-norm of the electric field and the \( L^2 \)-norm of the curl of the electric field, proportional to the magnetic field \( \nabla \times E = i\omega \mu H \) \[8\],

\[
\mathcal{E}_{c,H(\text{curl})} = \frac{\|E(x) - E_{hp}^{c}(x)\|_{H(\text{curl})}}{\|E(x)\|_{H(\text{curl})}}.
\]

Here the \( H(\text{curl}) \) norm is defined as,

\[
\|E\|_{H(\text{curl})} = \left( \int_{\Omega} |E|^2 + |

\nabla \times E|^2 \, dx \right)^{1/2}.
\]

The FE error is defined and evaluated on the exact geometrical manifold (the sphere), with the approximate solution \( E_{hp} \) defined as,

\[
E_{hp}(x) = E_{hp}(x_{hp}(x_{exact}(x))) = (E_{hp} \circ x_{hp} \circ x_{exact}^{-1})(x)
\]

where

- \( x_{exact} \) is the exact parametrization for a particular curvilinear hexahedron, defined on the reference hexahedron;
- \( x_{hp} \) is the approximate parametrization using the geometry degree of freedom (d.o.f);
- \( E_{hp} \) is the FE solution defined on the approximate manifold.

The definition \[8\] includes thus the effects of approximating the geometry of the spherical domain as well.

In the preasymptotic range, for linear and quadratic elements, the error equals 82% and 15% as the mesh does not reproduce the wave form yet. The FE error drops dramatically with \( p = 3 \) to an acceptable level of 4% and continuous to decrease linearly on the linear-log scale, which indicates the expected exponential convergence. Fig.3 demonstrates the exponential convergence plot according to Table 2, along with the corresponding convergence curve for the geometry error (See \[8\] for a precise definition).

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>Relative Error in ( H(\text{curl}) ) norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>p=1</td>
<td>8.2114E-001</td>
</tr>
<tr>
<td>p=2</td>
<td>1.5463E-001</td>
</tr>
<tr>
<td>p=3</td>
<td>4.3978E-002</td>
</tr>
<tr>
<td>p=4</td>
<td>1.2905E-002</td>
</tr>
<tr>
<td>p=5</td>
<td>3.7920E-003</td>
</tr>
<tr>
<td>p=6</td>
<td>1.0750E-003</td>
</tr>
<tr>
<td>p=7</td>
<td>2.9892E-004</td>
</tr>
</tbody>
</table>

Table 2: Reproducing a plane wave. Relative Error in \( H(\text{curl}) \) norm vs the order of approximation

We mention that the plane wave expansion into spherical harmonics has been verified by comparing it numerically with the standard representation in Cartesian coordinates.

### 4.2 Reproducing the Mie series solution inside of the dielectric sphere

As a second test, we use the Finite Element code to reproduce the Mie series solution within the dielectric sphere only. Material constants are summarized in Table 1. The sphere is modeled with the same seven curvilinear hexahedra, each of which is divided uniformly (in the reference domain) into \( 2 \times 2 \times 2 \) elements \( (h = 1/2) \). Note that the radius of the sphere is equal to one wavelength \( a = \lambda \). Therefore, again, a minimum order \( p = 3 \) to reproduce the wave is expected.

For the internal field \( E^{\text{tot}} \) and \( H^{\text{tot}} \), one propagating wave is clearly discernible in Fig.4. The accuracy of the simulation is reflected by the relative FE error evaluated over the exact physical domain, see Table 3. Fig.5 demonstrates the exponential convergence of both geometry and the FE errors.
4.3 Scattering of a plane wave on a dielectric sphere

Following the discussion in the Introduction, we recall now the main idea of coupled FE/IE discretization. Finite elements are used only within scatterer domain $\Omega$ and the near field external domain $\Omega_0^+$, herefore called the finite element domain $\Omega \cup \Omega_0^-$. The remaining far-field exterior domain $\Omega_0^+$ is discretized using infinite elements (IE). The IE discretisation of $\Omega_0^+$ is fully compatible with the $hp$-FE discretisation of the near-field domain $\Omega_0^-$. For $H(\text{curl})$-conforming elements, this means that the tangential traces of the IE shape functions must coincide with the tangential traces of the FE shape functions for adjacent infinite and finite elements. For parametric elements, this implies that the parametrization of the truncating sphere(or its isoparametric approximation) for the adjacent finite and infinite elements is identical, and the tangential trace space of the IE space of shape functions must be identical with the corresponding FE trace space in the master coordinates.

The major components of the standard spherical formulation are reproduced below. Infinite elements are images of a master infinite element

$$\hat{D} = \{(\xi, \eta, r) \in \mathbb{R}^3 : (\xi, \eta) \in (0, 1)^2, r \in (0, \infty)\},$$

(41)
through the transformation

\[ \mathbf{x}(\xi, \eta, r) = \frac{r}{c} \hat{\mathbf{x}}(\xi, \eta). \]  

(42)

where \( c \) is the radius of the truncating sphere; \( \hat{\mathbf{x}}(\xi, \eta) \) is a parametrization of the face of a selected finite element which is adjacent to the truncating sphere \( \Gamma_0 \). For exact geometry elements, parameterizations of this type can be constructed by utilizing our Geometrical Modeling Package (GMP) [19]. In this way every infinite element shares its base with exactly one finite element. For isoparametric elements used in our code, \( \hat{\mathbf{x}}(\xi, \eta) \) denotes the isoparametric FE approximation of the truncating sphere obtained in the mesh generation process by interpolating the exact parametrization. With each infinite element associated is a set of shape functions which allow for enforcing the continuity of the tangential component of the \( \mathbf{E} \)-field across the truncating sphere, and which guarantee the \( O(1/r) \) decay of solution as \( r \to \infty \). Unfortunately, this last condition and the requirement of integrability of the appropriate expressions in the weak formulation require selecting the test functions with a decay rate \( O(1/r^3) \). That is, the trial functions \( \mathbf{E} \) and the test functions \( \mathbf{F} \) must be different and, therefore, the discrete formulation becomes unsymmetric.

![Figure 6: Scattering of a plane wave on a dielectric sphere. Contour plots on cross section \( z = 0 \) of the exact solution (a) \( E_x \) and (b) \( H_y \) for the scattered fields in domain \( \Omega \) and \( \Omega_0 \).](image)

**Variational Formulation**

We select test functions \( \mathbf{F} \) that have continuous tangential components across any material interface \( \Gamma \). In the unbounded exterior domain \( \Omega_0^+ \) we assume that the test function decays fast enough at infinity in order to guarantee that all the involved integrals become Lebesgue integrable. Multiplying the reduced wave

\footnote{We use the projection-based interpolation}
equation (1) by the appropriate test function $F$, integrating over the domain, and then integrating by parts, we obtain the following weak formulation.

Find the scattered field $E$ such that,

$$b(E, F) = l(F)$$

(43)

where the sesquilinear and anti-linear forms $b$ and $l$ are defined as follows

$$b(E, F) = \int_{\Omega} \left\{ \frac{1}{\mu} \nabla \times E \cdot \nabla \times F - (\omega^2 \varepsilon - j\omega \sigma) E \cdot F \right\} \, dx$$

(44)

$$l(F) = -i\omega \int_{\Omega} J^{imp} \cdot F \, dx$$

where $J^{imp}$ is the impressed current in (4). Since $\mu_r$ is one in our case, there is no contribution from the surface integral over material discontinuities. The continuity across the FE/IE boundaries is enforced in exactly the same way as between any two finite elements, i.e. the tangential component of the electric field is continuous, whereas the normal component may be discontinuous.

The method has been implemented in its most straightforward version, with the trial shape functions in the radial direction derived from the asymptotic form of the solution in the spherical coordinates (33), and the test shape functions selected in such a way that all the integrals exist in the Lebesgue sense. The trial and test functions on a master element are defined as follows [2].

$$E = \sum_{n=0}^{N} \sum_{m=1}^{K} T_{mn} \psi_n(r)[[\hat{e}_n(\xi, \eta)]_1, [\hat{e}_m(\xi, \eta)]_2, 0] + \sum_{n=0}^{N} \sum_{m=1}^{L} R_{mn} \alpha_n(r)[0, 0, \hat{g}_m(\xi, \eta)],$$

$$F = \sum_{i=0}^{N} \sum_{j=1}^{K} U_{ij} \phi_i(r)[[\hat{e}_j(\xi, \eta)]_1, [\hat{e}_j(\xi, \eta)]_2, 0] + \sum_{i=0}^{N} \sum_{j=1}^{L} S_{ij} \beta_i(r)[0, 0, \hat{g}_i(\xi, \eta)].$$

(45)
Figure 8: Scattering of a plane wave on a dielectric sphere. Magnitude and phase of Mie and FE/IE solution of magnetic field $H_y$ in terms of $z$ coordinate.

where $\hat{e}_j$ are the two-dimensional vector-valued edge shape functions defined for the common face, $\hat{g}_i(\xi, \eta)$ are the scalar-valued shape functions defined for a square master element, $T_{mn}, R_{mn}, U_{mn}, S_{mn}$ are arbitrary coefficients. The functions $\psi_n, \alpha_n, \phi_i, \beta_i$ define the behavior of the approximation as $r \to \infty$. The choice of trial shape functions is dictated by the form of the exact solution outside of the truncating sphere,

$$\psi_n(r) = \begin{cases} e^{-jk_0(r-c)} & n = 0 \\ \left(\frac{r}{c}\right)^n - 1 e^{-jk_0(r-c)} & n > 0 \end{cases}$$  \quad (46)$$

$$\alpha_n(r) = \left(\frac{c}{r}\right)^{n+2} e^{-jk_0(r-c)}.$$  

The choice of the test shape functions is dictated by the Lebesgue integrability condition, and the condition on reproducibility of gradients (exact sequence property, see [2] for a detailed discussion),

$$\phi_n(r) = \frac{1}{r} \gamma_n(r)$$  \quad (47)

$$\beta_n(r) = \gamma'_n(r)$$

where

$$\gamma_n(r) = \begin{cases} \left(\frac{r}{c}\right)^2 e^{-jk_0(r-c)} , & n = 0 \\ \left(\frac{r}{c}\right)^{n+2} - \left(\frac{r}{c}\right)^2 e^{-jk_0(r-c)} , & n > 0 \end{cases}$$  \quad (48)$$

In computations, infinite elements are processed in a way similar to $hp$-finite elements. Their shape functions are logically associated with appropriate edge, face and central nodes. Their spectral orders are fully specified.
by the orders of approximation on a common face of the adjacent finite elements, and by the number of terms in the radial direction N. In evaluation of stiffness matrices, integration in the radial direction is performed exactly while in the tangential directions via the standard Gaussian quadratures. The IE stiffness matrices are assembled into the global stiffness matrix in a standard way.

**Figure 9:** Scattering of a plane wave on a dielectric sphere. The geometry and FE/IE solution errors vs order of approximation.

The sphere with the same geometry, and material constants as in last section are used here to verify the FE/IE code. (33) provides the exact solution for scattered field outside the sphere in domain $\Omega_{0}^{-}$. According to (3), the internal scattered field is then calculated by removing incident field $E^{\text{inc}}$ (23) from total internal field $E^{\text{tot}}$ (32). Contour plots of selected components of scattered fields $E^{\text{sca}}$ and $H^{\text{sca}}$ evaluated using the Mie series, are presented in Fig. 6.

The quality of the FE/IE discretization is illustrated in Fig. 7 and Fig. 8, showing the variation of $E_{x}$ and $H_{y}$ components (magnitude and phase) along the z-axis. The solid blue line indicates the (exact) Mie solution. Recall that the order of elements refers here to the order of the scalar-valued approximation in the de Rham diagram. For the scalar master hexahedral element of order $p \times p \times p$, the order of approximation for the $E_{x}$ component is $p-1 \times p \times p$, and the order of approximation for the $H_{y}$ component is $p-1 \times p \times p-1$. For affine meshes, the variation of $E_{x}$ and $H_{y}$ components along the z-axis would have been a piecewise polynomial of order $p$ and $p-1$, respectively. For curvilinear meshes, however, the definition of the shape functions involves use of the Piola transformations and the interpretation of the FE results is less straightforward.

The green line in Fig. 7 and Fig. 8 corresponds to the FE/IE solution obtained with quadratic elements yielding a global (relative) $H(\text{curl})$ error of 46%. The quality of the approximation considerably improves with cubic elements that deliver a global error of 10%. For quartic elements ($p = 4$), the global error drops to 2%, and thus the FE/IE and Mie solutions become visually indistinguishable. The total fields $E^{\text{tot}} = E^{\text{inc}} + E^{\text{sca}}$ for internal domain $\Omega$ and near field domain $\Omega_{0}^{-}$ are also illustrated in Fig.10.

The obtained results make us confident that the FE/IE code is bug-free and can be used now to solve the actual human head problem.
Table 4: Scattering of a plane wave on a dielectric sphere. Relative Error in $H(\text{curl})$ norm in terms of order of approximation

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>Relative Error in $H(\text{curl})$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>p=2</td>
<td>4.6364E-001</td>
</tr>
<tr>
<td>p=3</td>
<td>1.0219E-001</td>
</tr>
<tr>
<td>p=4</td>
<td>2.2091E-002</td>
</tr>
<tr>
<td>p=5</td>
<td>7.4803E-003</td>
</tr>
<tr>
<td>p=6</td>
<td>1.9685E-003</td>
</tr>
</tbody>
</table>

5. COMPUTATION OF SAR’S FOR A HUMAN HEAD MODEL

We conclude our numerical experiments with the dielectric sphere problem by computing Specific Absorption Rates (SAR’s) for material data corresponding to the Eureka Project [20]. The experimental setup reported in [20] was supposed to resemble the real like situation with the human head. The absorbing human tissue is simulated by adequately designed water solution of sugar, salt, hydroxyethylcellulose, and bactericide filling a spherical flask made of glass. Its radius is around 100 mm. The material properties of the media are described in Table 5,

<table>
<thead>
<tr>
<th>Frequency $f$</th>
<th>Relative permittivity $\epsilon_r$</th>
<th>Conductivity $\sigma$</th>
<th>Density $\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>900MHZ</td>
<td>41.5</td>
<td>0.86</td>
<td>$1 \times 10^6$ kg/m$^3$</td>
</tr>
</tbody>
</table>

Table 5: Material properties for the head model.

The corresponding input nondimensional variables for the FE/IE code are as follows,

<table>
<thead>
<tr>
<th>$\epsilon_r$</th>
<th>$\Sigma$</th>
<th>$\mu_r$</th>
<th>$\Omega$</th>
<th>$a$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.5</td>
<td>32.42</td>
<td>1</td>
<td>0.6$\pi$</td>
<td>1</td>
<td>4.333</td>
</tr>
</tbody>
</table>

Table 6: Nondimensional data for the head model.

While in the experimental study, the sphere was opened at its northern hemisphere, and filled with the solution only to about 3/4 of its height, our simulation is performed for the whole sphere to simplify the mesh generation. Note that $a = 0.3\lambda_0 \approx 2\lambda$, the head model consists of seven curvilinear hexahedral blocks which are covered by $3 \times 3 \times 3$ meshes starting with elements of order $p = 2$. The near field of the head is modeled with six curvilinear hexahedra with the same subdivision and element order as for the head model. Fig.11 presents the distribution of $E_x$-field and $H_y$-field in the FE domain. Note the refractive index

$$ I = \sqrt{\left(\epsilon_r - \frac{\Sigma}{\Omega}\right) \mu_r} = \sqrt{(41.5 - 17.2i) \cdot 1} = 6.5736 - 1.3083i. \tag{49} $$

Figures 12, 13 and 14 summarize the convergence experiments reported in the same fashion as for the previous problem.

Pointwise and average of SAR’s

In order to study EM waves transfer properties of the human head, the focus is now on the level and distribution of SAR inside the head model. Once the electric field in the head model is determined, the corresponding SAR is defined as,

$$ SAR = \frac{|E^{\text{tot}}|^2}{\rho} \sigma [\text{mW/g}] \tag{50} $$

averaged either over 1 or 10 g of the tissue mass. The value of SAR can be calculated pointwise according to FE/IE solution $|E|$ in terms of physical coordinates $x$. From SAR distribution in the finite element domain of the head model, see Fig.15(a), we observe relatively strong attenuation of the waves due to the absorbing properties of the liquid. The location of high value SAR in the head model can be easily discernible as the red area in Fig.15(b).
Figure 10: Scattering of a plane wave on a dielectric sphere. Contour plots of cross section $z = 0$ of the exact solution (a) $E_x$ and (b) $H_y$ of the total fields.

The average SAR can be computed directly from the Mie series solution [12]. This allows to test again the computational accuracy. The orthogonality of spherical vector harmonics yields the formula for the integral of the total field intensity,

$$
\int_{\Omega} |E^{tot}|^2 \, dx = \pi \sum_{n=1}^{\infty} \int_{-1}^{1} d(cos\theta) \int_{0}^{\pi} r^2 \, dr \left( |c_n|^2 (m_\theta + m_\phi) + |d_n|^2 (n_r + n_\theta + n_\phi) \right)
$$

The integration over azimuth $\phi$ has already been performed, leading to the $\pi$ factor. Here $m_\theta, m_\phi$ and $n_r, n_\theta, n_\phi$ are absolute-square values of the series terms of the components of the vector-waves in (27),

$$
m_\theta = g_n \pi_r^2 \pi_\theta^2 (cos\theta) |j_n(\rho)|^2; \quad m_\phi = g_n \pi_r^2 \pi_\phi^2 (cos\theta) |j_n(\rho)|^2;
$$

$$
n_r = g_n sin^2 \theta \pi_r^2 (cos\theta) \left| j_n(\rho) \right|^2; \quad n_\theta = g_n \pi_\theta^2 \pi_\theta^2 (cos\theta) \left| (\rho j_n(\rho))' \right|^2; \quad n_\phi = g_n \pi_\phi^2 (cos\theta) \left| (\rho j_n(\rho))' \right|^2.
$$

Here $g_n$ stands for

$$
g_n = \frac{2n+1}{n(n+1)}
$$
Integrals in $\cos \theta$ can be evaluated analytically,

\begin{align}
    m_n &= \int_{-1}^{1} (m_\theta + m_\phi) d(\cos \theta) = 2(2n + 1)|j_n(\rho)|^2 \\
n_n &= \int_{-1}^{1} (n_r + n_\theta + n_\phi) d(\cos \theta) = 2n(2n + 1) \left\{ (n + 1) \left| \frac{j_n(\rho)}{\rho} \right|^2 + \left| \frac{p j_n(\rho)'}{\rho} \right|^2 \right\}^2
\end{align}

The final formula for the average SAR’s becomes,

\[ SAR = \frac{3\sigma}{4\rho a^3} \sum_{n=1}^{n_{max}} \int_{0}^{a} (m_n |e_n|^2 + n_n |d_n|^2) r^2 dr. \]  

Substituting the head model data, we obtain the (head) average $SAR = 2.581 [mW/g]$. A direct numerical evaluation using the FE/IE yields $SAR = 2.607 [mW/g]$.

6. CONCLUSIONS AND FUTURE WORK

In this paper, the Mie series solution is used to verify the FE/IE code. The presented numerical results are certainly encouraging from the practical point of view. The coupled finite/infinite elements provide a reliable
approximation not only within the truncated domain but outside it as well. The convergence tests done for the presented benchmark problems, confirm fully the predicted exponential convergence with respect to order of approximation $p$.

Future work will focus on a human head model with a $G^1$-continuous reconstructed geometry model [21]. As the parameterizations used to represent both the dielectric and truncating spheres are only $G^1$-continuous (for the exact geometry manifold).

References


Figure 13: Scattering of a plane wave on a head model. Magnitude and phase of Mie and FE/IE solution of magnetic field $H_y$ in terms of $z$ coordinate.


Figure 14: Scattering of a plane wave on a head model. Contour plots of the exact solution of (a) $E_x$ and (b) $H_y$.


[20] “Eureka Project SarSys,Benchmark Phantom Protokol.” courtesy of Niels Kuster, ETH Laboratory for EMF and Microwave Electronic

Figure 15: Scattering of a plane wave on a head model. (a) SAR distribution in both the head model and its near field domain. (b) SAR distribution within the head model.