CONSTRUCTING PACKINGS IN PROJECTIVE SPACES AND GRASSMANNIAN SPACES VIA ALTERNATING PROJECTION

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Abstract. This report presents a numerical method for finding good packings on spheres, in projective spaces, and in Grassmannian manifolds equipped with various metrics. In each case, producing a good packing is equivalent to constructing a matrix that has certain structural and spectral properties. By alternately enforcing the structural condition and then the spectral condition, it is frequently possible to reach a matrix that satisfies both. One may then extract a packing from this matrix.

This approach is both powerful and versatile. In cases where experiments have been performed, the alternating projection method yields packings that compete with the best packings recorded. It also extends to problems that have not been studied numerically. For example, it can be used to produce packings of subspaces in real and complex Grassmannian spaces equipped with the Fubini–Study distance. Some of these novel configurations are essentially optimal.

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1. Introduction

Let us begin with the standard facetious example. Imagine that twelve mutually inimical nations build their capital cities on the surface of a featureless globe. Being concerned about missile strikes, they wish to locate the closest pair of cities as far apart as possible. This is a prototypical packing problem, and the solution places one city at each vertex of a regular icosahedron.

1.1. **Abstract Packing Problems.** We shall study packing problems set in a compact metric space $\mathbb{M}$ with distance function $\mathrm{dist}_{\mathbb{M}}$. The *packing radius* of a finite set $\mathscr{X}$ is the minimum distance between some pair of distinct points drawn from $\mathscr{X}$. That is,

$$\mathrm{pack}_{\mathbb{M}}(\mathscr{X}) \quad \stackrel{\mathrm{def}}{=} \quad \min_{m \neq n} \quad \mathrm{dist}_{\mathbb{M}}(x_m, x_n).$$

In other words, the packing radius of a set is the largest open ball that can be centered at one point of the set without encompassing any other point. An *optimal packing* of N points is an ensemble $\mathscr{X}$ that solves the mathematical program

$$\max_{|\mathscr{X}|=N} \quad \mathrm{pack}_{\mathbb{M}}(\mathscr{X})$$

where $|\cdot|$ returns the cardinality of a finite set. The optimal packing problem is guaranteed to have a solution because the metric space is compact and the objective is a continuous function of the ensemble $\mathscr{X}$.

In this paper, we shall consider a *feasibility problem* closely connected with optimal packing. Given a number ρ, the goal is to produce a set of N points for which

$$\mathrm{pack}_{\mathbb{M}}(\mathscr{X}) \quad \geq \quad \rho. \tag{1.1}$$

It is notoriously difficult to solve this problem, and it is even more difficult to determine the maximum value of ρ for which the feasibility problem is soluble. This maximum value of ρ corresponds with the radius of an optimal packing.

1.2. **Our Approach.** In this report, we shall develop an elegant method for solving packing problems on Euclidean spheres, in projective spaces, and in Grassmannian spaces equipped with various metrics. Here is a brief outline of the technique.

(1) Each configuration of points is associated with a matrix whose entries are related to the inter-point distances.
(2) One proves that a configuration solves the feasibility problem (1.1) if and only if its matrix possesses both a structural property and a spectral property.
(3) Randomly choose an initial configuration, and construct its matrix.
(4) Alternately enforce the structural condition and the spectral condition in hope of reaching a matrix that satisfies both.
(5) Extract a configuration of points from the output matrix.

As a first illustration of our method, we shall construct packings of points on the surface of a Euclidean sphere. This problem has been studied for about 75 years, and hundreds of putatively optimal packings have been tabulated by N. J. A. Sloane and his colleagues [Slo04b]. A simple implementation of our algorithm was able to reproduce many of these configurations.

Afterward, we show how to produce packings in real and complex projective spaces. A projective space is the collection of all lines through the origin of a Euclidean space, and the distance between two lines is the acute angle between them. Some applications that require good projective packings include sparse approximation [Tro03, Tro04] and communications [SH03].

The natural generalization of line packing is subspace packing. This problem is set in a *Grassmannian space*, which is the collection of all subspaces of fixed dimension in a Euclidean space. The distance between two subspaces is a function of the principal angles between them. Our algorithm can match the best packings of real subspaces recorded by Sloane and his colleagues [Slo04a]. We

have also extended our technique to produce packings of complex subspaces and to produce packings with respect to several new metrics. In most cases, the algorithm provides very satisfactory results. Our success at solving so many different packing problems underscores the power and versatility of the alternating projection approach.

1.3. **A Historical Interlude.** The problem of constructing optimal packings in various metric spaces has a long and lovely history. The most famous example, perhaps, is Kepler's Conjecture that an optimal packing of spheres in three-dimensional Euclidean space[1] locates them at the points of a face-centered cubic lattice. For millennia, greengrocers have applied this theorem when stacking oranges, but it has only been established rigorously within the last few years [Hal04]. Packing problems play a major role in modern communications because error-correcting codes may be interpreted as packings in the Hamming space of binary strings [CT91]. The standard reference on packing is the *magnum opus* of Conway and Sloane [CS98]. Classical monographs on the subject were written by L. Fejes Tóth [Tót64] and C. A. Rogers [Rog64].

2. Packing on Spheres

The example that heads this report requests an optimal packing of points on the surface of a two-dimensional sphere. This is often referred to as *Tammes' Problem* in honor of a Dutch botanist who raised the question in 1930 [Tam30]. We return to it here because it provides the most transparent illustration of our *modus operandi*.

2.1. **The Sphere.** Let $\mathbb{R}^d$ denote the d-dimensional real inner-product space. The usual symmetric inner product will be written as $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^* \boldsymbol{x}$, where * denotes the conjugate transpose operator. The Euclidean norm falls from the inner product: $\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$.

The $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ is defined as the set of all unit vectors in $\mathbb{R}^d$.

$$\mathbb{S}^{d-1} \quad \stackrel{\text{def}}{=} \quad \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}.$$

We measure the distance between two points on the sphere as the Euclidean distance of the chord joining them.

$$\operatorname{dist}_{\mathbb{S}}(\boldsymbol{x}, \boldsymbol{y}) \quad \stackrel{\text{def}}{=} \quad \|\boldsymbol{x} - \boldsymbol{y}\|_2 .$$

Equipped with this distance, the sphere becomes a compact metric space.

2.2. **Packings and Matrices.** Suppose that we wish to produce a configuration of N points in $\mathbb{S}^{d-1}$ with packing radius ρ. We may represent each configuration of points with a collection $\mathscr{X}$ of unit vectors drawn from $\mathbb{R}^d$.

$$\mathscr{X} \quad = \quad \{\boldsymbol{x}_n\}_{n=1}^N.$$

The packing radius of $\mathscr{X}$ is defined as

$$\operatorname{pack}_{\mathbb{S}}(\mathscr{X}) \quad \stackrel{\text{def}}{=} \quad \min_{m \neq n} \quad \operatorname{dist}_{\mathbb{S}}(\boldsymbol{x}_n, \boldsymbol{x}_m) \quad = \quad \min_{m \neq n} \quad \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2 ,$$

and the feasibility problem requests a configuration $\mathscr{X}$ for which

$$\min_{m \neq n} \quad \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2 \quad \geq \quad \rho.$$

It is better to reorganize this condition so that it depends only on the inner products between pairs of vectors. Therefore, we seek a collection $\mathscr{X}$ for which

$$\max_{m \neq n} \quad \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \leq \quad \mu \tag{2.1}$$

where the parameter μ satisfies the relationship $\mu = 1 - \frac{1}{2}\rho^2$.

[1]The infinite extent of a Euclidean space necessitates a more subtle definition of an optimal packing.

Form the elements of $\mathscr{X}$ into a $d \times N$ matrix:

$$\mathsf{X} \quad \stackrel{\text{def}}{=} \quad \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_N \end{bmatrix}.$$

In the sequel, we shall not distinguish between the matrix X and the collection of its columns. To detect whether X solves the feasibility problem (2.1), one must compute the inner products between its columns. We prefer to work with a matrix that registers these inner products explicitly. The obvious candidate is the *Gram matrix* of X,

$$\mathsf{G} \quad \stackrel{\text{def}}{=} \quad \mathsf{X}^*\mathsf{X}.$$

The (m, n) entry of the Gram matrix is precisely the inner product between $\boldsymbol{x}_n$ and $\boldsymbol{x}_m$.

We may reformulate the feasibility problem purely in terms of the Gram matrix. Suppose that the configuration X satisfies (2.1) with parameter μ. Then its Gram matrix G must have the following six properties:

(1) G is real symmetric.
(2) G has a unit diagonal.
(3) $g_{mn} \le \mu$ whenever $m \ne n$.
(4) G is positive semi-definite.
(5) G has rank d.
(6) G has trace N.

Some of these properties are redundant, but we have listed them separately for reasons soon to become apparent.

Conversely, suppose that a matrix G satisfies Properties (1)–(6). Then it is always possible to extract a configuration of N points that solves (2.1). More precisely, there exists a real $d \times N$ matrix X with unit-norm columns so that $\mathsf{G} = \mathsf{X}^*\mathsf{X}$. The off-diagonal entries of G do not exceed μ, so the inner products between distinct columns of X do not exceed μ. We conclude that Properties (1)–(6) completely characterize solutions of the feasibility problem with parameter μ.

For reference, a positive semi-definite (PSD) matrix is defined to have real, nonnegative eigenvalues. It can be shown that every PSD matrix is (conjugate) symmetric. To indicate that A is PSD, we write $\mathsf{A} \succcurlyeq \mathsf{0}$.

2.3. **Alternating Projection.** Observe that Properties (1)–(3) are *structural* properties. By this, we mean that they constrain the entries of the matrix. Properties (4)–(6), on the other hand, are *spectral* properties. That is, they control the eigenvalues of the matrix. It is not easy to enforce structural and spectral properties simultaneously, so we must resort to half measures. Starting from an initial matrix, our algorithm will alternately enforce (1)–(3) and then (4)–(6) in hope of reaching a matrix that satisfies all six properties at once.

To be more rigorous, let us define the structural constraint set

$$\mathscr{H} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{H} \in \mathbb{R}^{N \times N} : \mathsf{H} = \mathsf{H}^*, \quad \operatorname{diag} \mathsf{H} = \mathbf{e}, \quad \text{and} \quad h_{mn} \le \mu \text{ for } m \ne n\}. \tag{2.2}$$

The symbol $\mathbf{e}$ represents a conformal vector of ones. We also define the spectral constraint set

$$\mathscr{G} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{G} \in \mathbb{R}^{N \times N} : \mathsf{G} \succcurlyeq \mathsf{0}, \quad \operatorname{rank} \mathsf{G} \le d, \quad \text{and} \quad \operatorname{trace} \mathsf{G} = N\}. \tag{2.3}$$

Both constraint sets are compact. The structural constraint set $\mathscr{H}$ is convex, but the spectral constraint set is not.

To solve the feasibility problem (2.1), we must find a matrix that lies in the intersection of $\mathscr{G}$ and $\mathscr{H}$. This section will present a high-level statement of our approach. The next two sections will provide implementation details. We remind the reader that the Frobenius norm of a matrix is defined as

$$\|\mathsf{A}\|_{\mathrm{F}} \quad \stackrel{\text{def}}{=} \quad \left[\sum\nolimits_{m,n} |a_{mn}|^2\right]^{1/2}$$

Algorithm 2.1 (Alternating Projection)**.**
INPUTS:

- An $N \times N$ (conjugate) symmetric matrix G_0.
- The maximum number of iterations T.

OUTPUTS:

- G_{out} is an $N \times N$ matrix that belongs to $\mathscr{G}$ and that has a unit diagonal.

PROCEDURE:

(1) Initialize $t = 0$.
(2) Determine a matrix H_t that solves

$$\min_{\mathsf{H} \in \mathscr{H}} \quad \|\mathsf{H} - \mathsf{G}_t\|_{\mathrm{F}} .$$

(3) Determine a matrix G_{t+1} that solves

$$\min_{\mathsf{G} \in \mathscr{G}} \quad \|\mathsf{G} - \mathsf{H}_t\|_{\mathrm{F}} .$$

(4) Increment t.
(5) If $t < T$, return to Step (2).
(6) Define the diagonal matrix $\mathsf{D} = \operatorname{diag} \mathsf{G}_T$.
(7) Return the matrix

$$\mathsf{G}_{\text{out}} = \mathsf{D}^{-1/2}\, \mathsf{G}_T\, \mathsf{D}^{-1/2}.$$

The iterates generated by this algorithm need not converge. Therefore, we have chosen to halt the algorithm after a fixed number of steps instead of checking the behavior of the sequence of iterates.

The scaling in the the last step normalizes the diagonal of the matrix but preserves its inertia (i.e., numbers of negative, zero, and positive eigenvalues). It follows that the output matrix always admits a factorization $\mathsf{G}_{\text{out}} = \mathsf{X}^*\mathsf{X}$ where X is a $d \times N$ matrix with unit-norm columns.

The idea of applying alternating projection to feasibility problems first appeared in the work of von Neumann [vN50]. He proved that an alternating projection between two closed subspaces of a Hilbert space converges to the orthogonal projection of the initial iterate onto the intersection of the two subspaces. Cheney and Goldstein subsequently showed that an alternating projection between two closed, convex subsets of a Hilbert space always converges to a point in their intersection (provided that the intersection is nonempty) [CG59]. Unfortunately, these results do not apply to our problem because the spectral constraint set $\mathscr{G}$ is not convex.

2.4. **The Matrix Nearness Problems.** To implement Algorithm 2.1, we must solve the matrix nearness problems in Steps (2) and (3). The first one is straightforward.

Proposition 2.2. *Let G be a real, symmetric matrix. With respect to Frobenius norm, the unique matrix in $\mathscr{H}$ closest to G has a unit diagonal and off-diagonal entries that satisfy*

$$h_{mn} \quad = \quad \begin{cases} g_{mn} & \text{if } g_{mn} \le \mu, \text{ and} \\ \mu & \text{otherwise.} \end{cases}$$

It is rather more difficult to find a nearest matrix in the spectral constraint set. To state the result, we define the plus operator $(x)_+ = \max\{0, x\}$.

Proposition 2.3. *Let H be a conjugate symmetric matrix whose eigenvalue decomposition is $\sum_{n=1}^{N} \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^*$ with the eigenvalues decreasingly ordered: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$. With respect to Frobenius norm, a matrix in $\mathscr{G}$ closest to H is given by*

$$\sum\nolimits_{n=1}^{d} (\lambda_n - \gamma)_+ \boldsymbol{u}_n \boldsymbol{u}_n^*$$

where the scalar γ is chosen so that the matrix has trace N. This best approximation is unique provided that $\lambda_d > \lambda_{d+1}$.

The nearest matrix described by this theorem can be computed efficiently with standard tools of numerical linear algebra [GVL96].

Proof. Given an Hermitian matrix $\boldsymbol{A}$, denote by $\boldsymbol{\lambda}(\boldsymbol{A})$ the vector of decreasingly ordered eigenvalues. Then we may decompose $\boldsymbol{A} = \boldsymbol{U}\,\operatorname{diag}\boldsymbol{\lambda}(\boldsymbol{A})\,\boldsymbol{U}^*$ for some unitary matrix $\boldsymbol{U}$.

We must solve the optimization problem

$$\begin{aligned} \min_{\boldsymbol{A}} \ \tfrac{1}{2}\,\|\boldsymbol{A}-\boldsymbol{H}\|_{\mathrm{F}}^2 \qquad \text{subject to} \qquad & \lambda_n(\boldsymbol{A}) \geq 0 \qquad \text{for } n = 1, \dots, d, \\ & \lambda_n(\boldsymbol{A}) = 0 \qquad \text{for } n = d+1, \dots, N, \text{ and} \\ & \mathbf{e}^*\,\boldsymbol{\lambda}(\boldsymbol{A}) = N. \end{aligned}$$

First, we fix the eigenvalues of $\boldsymbol{A}$ and minimize with respect to the unitary part of its decomposition. In consequence of the Wielandt–Hoffman Theorem [HJ85], the objective function is bounded below:

$$\tfrac{1}{2}\,\|\boldsymbol{A}-\boldsymbol{H}\|_{\mathrm{F}}^2 \quad \geq \quad \tfrac{1}{2}\,\|\boldsymbol{\lambda}(\boldsymbol{A}) - \boldsymbol{\lambda}(\boldsymbol{H})\|_2^2\,.$$

Equality holds if and only $\boldsymbol{A}$ and $\boldsymbol{H}$ are simultaneously diagonalizable by a unitary matrix. Therefore, if we decompose $\boldsymbol{H} = \boldsymbol{U}\,\operatorname{diag}\boldsymbol{\lambda}(\boldsymbol{H})\,\boldsymbol{U}^*$, the objective function attains its minimal value whenever $\boldsymbol{A} = \boldsymbol{U}\,\operatorname{diag}\boldsymbol{\lambda}(\boldsymbol{A})\,\boldsymbol{U}^*$. Note that the matrix $\boldsymbol{U}$ may not be uniquely determined.

We find the optimal set of eigenvalues $\boldsymbol{\xi} = \boldsymbol{\lambda}(\boldsymbol{A})$ by solving the (strictly) convex program

$$\begin{aligned} \min_{\boldsymbol{\xi}} \ \tfrac{1}{2}\,\|\boldsymbol{\xi} - \boldsymbol{\lambda}(\boldsymbol{H})\|_2^2 \qquad \text{subject to} \qquad & \xi_n \geq 0, \qquad \text{for } n = 1, \dots, d, \\ & \xi_n = 0, \qquad \text{for } n = d+1, \dots, N, \text{ and} \\ & \mathbf{e}^*\,\boldsymbol{\xi} = N. \end{aligned}$$

This minimization is accomplished with an application of Karush–Kuhn–Tucker theory [Roc70]. In short, the top d eigenvalues of $\boldsymbol{H}$ are translated by an equal amount, and those that become negative are set to zero. The size of the translation is chosen to fulfill the trace condition. The entries of the optimal $\boldsymbol{\xi}$ are nonincreasing on account of the ordering of $\boldsymbol{\lambda}(\boldsymbol{H})$.

Finally, the uniqueness claim follows from the fact that the eigenspace associated with the top d eigenvectors of $\boldsymbol{H}$ is uniquely determined if and only if $\lambda_d(\boldsymbol{H}) > \lambda_{d+1}(\boldsymbol{H})$. □

2.5. **The Initial Matrix.** The success of the algorithm sometimes depends on adequate selection of the input matrix $\boldsymbol{G}_0$. We have found the following strategy is reasonably efficient and effective.

Algorithm 2.4 (Initial Matrix)**.**
INPUTS:

- The dimension d.
- The number of vectors N.
- An upper bound τ on the inner product between vectors.

OUTPUTS:

- An $N \times N$ matrix $\boldsymbol{G}$ with rank d, with a unit diagonal, and with off-diagonal entries that do not exceed τ.

PROCEDURE:

(1) Initialize $n = 1$.
(2) Choose a vector $\boldsymbol{x}_n$ uniformly at random from $\mathbb{S}^{d-1}$.
(3) If $\langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle \leq \tau$ for each $m = 0, \dots, n-1$, then increment n.
(4) If $n \leq N$, return to Step (2).
(5) Form the matrix $\boldsymbol{X} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \dots \quad \boldsymbol{x}_N]$.
(6) Return the Gram matrix $\boldsymbol{G} = \boldsymbol{X}^*\boldsymbol{X}$.

To implement Step (2), we choose a d-dimensional vector whose entries are iid standard normal. This vector is scaled to have unit norm [Ste80].

It is essential to be aware that this procedure will not halt if τ is too small (or if we are unlucky in our random choices). The selection of τ may require a little experimentation, but it is usually not very important. Indeed, we have often set $\tau = 1$ without any ill effect.

We have also investigated several other methods for constructing starting points. In particular, one might pick a large number of random unit vectors and then greedily remove the vectors that have the largest inner products with the others. Some additional ideas appear in [TDHS03].

2.6. **Convergence of the Algorithm.** Although alternating projection need not generate a convergent sequence of iterates, it is possible to say a little about the behavior of the algorithm. For reference, the distance between a matrix and a collection of matrices is defined as

$$\operatorname{dist}(M, \mathscr{C}) \quad \stackrel{\text{def}}{=} \quad \inf_{C \in \mathscr{C}} \|C - M\|_{\mathrm{F}} .$$

Theorem 2.5 (Global Convergence)**.** *Suppose that Algorithm 2.1 generates an (infinite) sequence of iterates $\{(G_t, H_t)\}$. This sequence has at least one accumulation point.*

- *Every accumulation point lies in $\mathscr{G} \times \mathscr{H}$.*
- *Every accumulation point $(\overline{G}, \overline{H})$ satisfies*

$$\left\|\overline{G} - \overline{H}\right\|_{\mathrm{F}} = \lim_{t \to \infty} \|G_t - H_t\|_{\mathrm{F}} .$$

- *Every accumulation point $(\overline{G}, \overline{H})$ satisfies*

$$\left\|\overline{G} - \overline{H}\right\|_{\mathrm{F}} = \operatorname{dist}(\overline{G}, \mathscr{H}) = \operatorname{dist}(\overline{H}, \mathscr{G}).$$

For a proof of this theorem, please see the report [TDHS03].

Even when an alternating algorithm converges in the usual sense, it may converges quite slowly. This is the greatest weakness of our approach.

2.7. **Numerical Experiments.** At this point, we must ask the inevitable question of whether the algorithm actually works. In principle, this question is difficult to resolve because the optimal packing radius is unknown for most combinations of d and N. Therefore, we have attempted to reproduce the best packings tabulated by N. J. A. Sloane and his colleagues [Slo04b]. It is highly likely that their packings are optimal or nearly optimal.

Our experiments were performed on a desktop computer with a simple Matlab implementation of the algorithm. For various d and N, we used Sloane's tables to determine a value for the parameter μ of the feasibility problem (2.1). Our results are summarized in Table 1, which may be found in Appendix B. Following Sloane, we have reported the degrees of arc subtended by the closest pair of points in lieu of the Euclidean distance between them. In most cases, we obtained packing radii within a tenth of a degree of the best known. Although we did not time the individual calculations, the implementation and experiments were completed within the space of a single afternoon.

It should also be disclosed that the alternating projection algorithm performs somewhat worse (but still respectably) when d or N becomes large. We do not know whether the starting points must be chosen more intelligently, whether additional random trials are necessary, or whether the algorithm suffers some intrinsic limitation.

3. Packing in Projective Spaces

Imagine that we wish to destroy a tumor by firing laser beams at it from several directions. The beams should coincide at the tumor, but the acute angle between each pair should remain as large as possible to avoid damaging the surrounding tissue [CHS96]. In other words, we seek a packing in projective space, which is the collection of all lines through the origin of a Euclidean space.

Emboldened by our success with packing on spheres, we shall adapt our algorithm to address this new problem.

3.1. **Projective Spaces.** Let $\mathbb{C}^d$ denote the d-dimensional, complex inner-product space. The usual Hermitian inner product will be written as $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^* \boldsymbol{x}$, where * denotes the complex conjugate transpose. As before, the Euclidean norm derives from the inner product: $\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$.

Define an equivalence relation on d-dimensional complex vectors:

$$\boldsymbol{y} \equiv \boldsymbol{x} \qquad \Longleftrightarrow \qquad \boldsymbol{y} = \zeta \, \boldsymbol{x} \quad \text{for a nonzero complex number } \zeta.$$

Under this equivalence relation, every nonzero vector is identified with the one-dimensional subspace spanned by that vector. The zero vector falls in a class by itself. The $(d-1)$-dimensional *complex projective space* is the collection of nonzero, d-dimensional complex vectors, modulo this equivalence relation:

$$\mathbb{P}^{d-1}(\mathbb{C}) \quad \stackrel{\text{def}}{=} \quad \frac{\mathbb{C}^d \setminus \{\mathbf{0}\}}{\mathbb{C}^\times}.$$

(The symbol $\mathbb{C}^\times$ refers to the nonzero complex numbers.) In words, $\mathbb{P}^{d-1}(\mathbb{C})$ is the set of one-dimensional subspaces of $\mathbb{C}^d$. The real projective space $\mathbb{P}^{d-1}(\mathbb{R})$ is defined in much the same way:

$$\mathbb{P}^{d-1}(\mathbb{R}) \quad \stackrel{\text{def}}{=} \quad \frac{\mathbb{R}^d \setminus \{\mathbf{0}\}}{\mathbb{R}^\times}.$$

It may be viewed as the collection of all lines through the origin of $\mathbb{R}^d$. On analogy, we shall refer to the elements of a complex projective space as *lines*. We concentrate on the complex case because the real case follows from a transparent adaptation.

The natural metric for $\mathbb{P}^{d-1}(\mathbb{C})$ is the acute angle between two lines or—what is equivalent—the sine of the acute angle. Therefore, the projective distance between two d-dimensional vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ will be calculated as

$$\operatorname{dist}_{\mathbb{P}}(\boldsymbol{x}, \boldsymbol{y}) \quad \stackrel{\text{def}}{=} \quad \left[1 \; - \; \left(\frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|}{\|\boldsymbol{x}\|_2 \, \|\boldsymbol{y}\|_2} \right)^2 \right]^{1/2}. \tag{3.1}$$

In particular, if both vectors have unit norm,

$$\operatorname{dist}_{\mathbb{P}}(\boldsymbol{x}, \boldsymbol{y}) \quad = \quad \sqrt{1 \; - \; |\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2}.$$

Evidently, the distance between two lines ranges between zero and one. Equipped with this distance, $\mathbb{P}^{d-1}(\mathbb{C})$ forms a compact metric space [CHS96].

3.2. **Packings and Matrices.** Suppose that we wish to construct a configuration of N lines in $\mathbb{P}^{d-1}(\mathbb{C})$ with a packing radius no less than ρ. We shall represent each configuration of lines in $\mathbb{P}^{d-1}(\mathbb{C})$ by a collection $\mathscr{X}$ of unit vectors in $\mathbb{C}^d$.

$$\mathscr{X} \quad = \quad \{\boldsymbol{x}_n\}_{n=1}^N.$$

The packing radius of the configuration $\mathscr{X}$ in projective space is defined as

$$\operatorname{pack}_{\mathbb{P}}(\mathscr{X}) \quad \stackrel{\text{def}}{=} \quad \min_{m \neq n} \operatorname{dist}_{\mathbb{P}}(\boldsymbol{x}_m, \boldsymbol{x}_n) \quad = \quad \min_{m \neq n} \sqrt{1 - |\langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle|^2},$$

and the feasibility problem requests a configuration $\mathscr{X}$ for which

$$\min_{m \neq n} \sqrt{1 - |\langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle|^2} \quad \geq \quad \rho.$$

As before, we clear the debris from this inequality to obtain an equivalent condition:

$$\max_{m \neq n} |\langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle| \quad \leq \quad \mu \tag{3.2}$$

where $\mu = \sqrt{1 - \rho^2}$.

Form the elements of $\mathscr{X}$ into a $d \times N$ matrix:

$$\mathsf{X} \quad \stackrel{\text{def}}{=} \quad \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_N \end{bmatrix}.$$

As usual, the Gram matrix is $\mathsf{G} = \mathsf{X}^*\mathsf{X}$. It follows that the configuration X solves the feasibility problem (3.2) if and only if its Gram matrix has the following properties:

(1) G is Hermitian.
(2) G has a unit diagonal.
(3) $|g_{mn}| \le \mu$ whenever $m \neq n$.
(4) G is positive semi-definite.
(5) G has rank d.
(6) G has trace N.

These conditions are almost identical with the ones we developed for Tammes' Problem. We expect, therefore, that the alternating projection algorithm will apply.

3.3. **Implementation Details.** We define the convex structural constraint set

$$\mathscr{H} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{H} \in \mathbb{C}^{N\times N} : \mathsf{H} = \mathsf{H}^*, \quad \operatorname{diag}\mathsf{H} = \mathbf{e}, \quad \text{and} \quad |h_{mn}| \le \mu \text{ for } m \neq n \,\}. \tag{3.3}$$

The only difference between (2.2) and (3.3) is the absolute value in the bound on the off-diagonal entries. As before, the spectral constraint set is

$$\mathscr{G} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{G} \in \mathbb{C}^{N\times N} : \mathsf{G} \succcurlyeq \mathsf{0}, \quad \operatorname{rank}\mathsf{G} \le d, \quad \text{and} \quad \operatorname{trace}\mathsf{G} = N\}.$$

It is clear that we may solve the feasibility problem (3.2) by producing a matrix in the intersection of $\mathscr{G}$ and $\mathscr{H}$.

First, we use Algorithm 2.4 to construct a starting matrix. Some minor changes are necessary. In Step (2), we wish to construct a uniformly random line in $\mathbb{P}^{d-1}(\mathbb{C})$. To do so, one selects a complex vector whose real and imaginary parts are chosen from independent standard normal distributions. This vector is rescaled to have unit Euclidean norm [Ste80]. In Step (3), one tests whether $|\langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle| \le \tau$ for each $m < n$.

We then apply Algorithm 2.1 to this initial matrix. Proposition 2.3 allows us to determine a nearest matrix from the spectral constraint set $\mathscr{G}$. To compute the nearest matrix from the structural constraint set $\mathscr{H}$, we use the following result.

Proposition 3.1. *Let G be an Hermitian matrix. With respect to Frobenius norm, the unique matrix in $\mathscr{H}$ closest to G has a unit diagonal and off-diagonal entries that satisfy*

$$h_{mn} \quad = \quad \begin{cases} g_{mn} & \text{if } |g_{mn}| \le \mu, \text{ and} \\ \mu\, g_{mn} / |g_{mn}| & \text{otherwise.} \end{cases}$$

The proof is immediate.

3.4. **Numerical Experiments.** To test how well the algorithm performs for the feasibility problem (3.2), we first attempted to reproduce the real projective packings that N. J. A. Sloane and his colleagues have tabulated [Slo04a]. Table 2 summarizes our calculations. We have reported the acute angle between the closest pair of lines instead of its sine. In the range of parameters we explored, our packing radii usually fall within a tenth of a degree of the best known. The algorithm was less successful when d or N grew large.

To our knowledge, the paper [ARU01] contains the only numerical work on packing in complex projective spaces. Unfortunately, the authors have provided very few examples of good complex packings. We have attempted to produce a more extensive table. Our results appear in Table 3. Rankin's Bound, equation (A.1) of the sequel, describes an upper limit on how far our configurations decline from the best possible. In many cases, our packings very nearly meet the bound. In other cases the gap is significant, so it may be possible to improve these results somewhat. Figure 1

compares the quality of the best real projective packings from [Slo04a] with the best complex projective packings we obtained.

3.5. **Equiangular Tight Frames.** Rankin's Bound shows that the packing radius of N lines in a $(d-1)$-dimensional projective space must satisfy the inequality

$$\operatorname{pack}_{\mathbb{P}}(\mathscr{X})^2 \quad \leq \quad \frac{(d-1)\,N}{d\,(N-1)}.$$

This result holds in both the real and complex setting, and Appendix A provides a derivation. When this bound is met, something special happens:

(1) X is *equiangular*. That is, the acute angle between each pair of lines is identical.
(2) X is a *tight frame*, which is equivalent to the statement that $\mathsf{X}\mathsf{X}^* = (N/d)\,\mathsf{I}$.

This second property places a very strict spectral constraint on the Gram matrix of X. By enforcing this stronger condition, we can improve the ability of the algorithm to locate configurations that meet the Rankin Bound. The work described in this section has previously been reported in [TDHS03]. Equiangular tight frames were defined and studied in [SH03] under the cognomen *optimal Grassmannian frames*. The literature contains several necessary conditions on d and N for the existence of equiangular tight frames [HP04, STDH04].

The definition of a tight frame, $\mathsf{X}\mathsf{X}^* = (N/d)\,\mathsf{I}$, is exactly equivalent with the condition that the matrix $(d/N)\,\mathsf{X}^*\mathsf{X}$ be an orthogonal projector. So we define the spectral constraint set

$$\mathscr{G}_{\mathrm{TF}} \quad \stackrel{\mathrm{def}}{=} \quad \{\mathsf{G} \in \mathbb{C}^{N\times N} : \mathsf{G} = \mathsf{G}^*, \quad \operatorname{rank}\mathsf{G} = d, \quad \text{and} \quad \mathsf{G}^2 = (N/d)\,\mathsf{G}\}.$$

To construct equiangular tight frames, we select an initial matrix at random and perform an alternating projection between $\mathscr{G}_{\mathrm{TF}}$ and the structural constraint set $\mathscr{H}$ defined in (3.3). The following proposition shows how to calculate the member of $\mathscr{G}_{\mathrm{TF}}$ nearest to a given matrix.

Proposition 3.2. *Let H be an Hermitian matrix whose eigenvalue decomposition is $\sum_{n=1}^{N} \lambda_n\,\boldsymbol{u}_n\,\boldsymbol{u}_n^*$ with the eigenvalues decreasingly ordered: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. With respect to Frobenius norm, a matrix in $\mathscr{G}_{\mathrm{TF}}$ closest to H is given by*

$$(N/d)\,\sum\nolimits_{n=1}^{d} \boldsymbol{u}_n\,\boldsymbol{u}_n^*.$$

This best approximation is unique provided that $\lambda_d > \lambda_{d+1}$.

The proof is essentially the same as that of Proposition 2.3. A more detailed proof appears in [TDHS03].

Where the necessary conditions permit, we have attempted to construct real and complex equiangular tight frames. Table 4, which is reproduced from [TDHS03], lists the results of these experiments. In dimensions d from two through six, the algorithm was able to construct every real equiangular tight frame that exists. We were also able to construct many complex equiangular tight frames in dimensions two through six. We have fewer theorems that delineate when complex equiangular tight frames can exist, so it is not currently possible to exclude the existence of others. Once again, the algorithm was less successful in higher dimensions [TDHS03].

3.6. **Other Directions.** Adaptability is the major advantage of the alternating projection algorithm. Let us mention several other projective packing problems to which it could be applied.

Although equiangular tight frames do not exist for all combinations of d and N, one may construct tight frames with large packing radii. The alternating projection algorithm naturally applies to this feasibility problem. This idea was developed in the conference paper [HTDS04]. It is also possible to enforce other types of spectral conditions, such as lower and upper bounds on eigenvalues. To implement these algorithms, one just needs to solve the matrix nearness problem induced by the spectral constraint set. The report [TDHS03] considers other types of structured tight frames that arise in electrical engineering applications.

Another possibility is to construct weighted line packings by assigning different norms to the vectors that represent the lines. This constraint will be reflected in the diagonal entries of the Gram matrix. The off-diagonal entries then represent weighted cosines of the angles between lines. As a result, vectors with large norms must be more orthogonal to the other vectors in the configuration. This type of constraint arises when constructing optimal signature sequences for CDMA systems. See [TDH04] for a discussion of this problem from a matrix-theoretic point of view.

We are also intrigued by the possibility of packing lines in quaternionic projective spaces. To our knowledge, no one has developed numerical algorithms that can approach this problem. In principle, the alternating projection method requires no adjustment. Of course, it may take a serious effort to implement quaternionic arithmetic and linear algebra.

4. Packing in Grassmannian Spaces

A line is just a one-dimensional subspace. The obvious generalization of line packing, therefore, is subspace packing. The alternating algorithm also applies in this setting, but we must address some new challenges. The problem of subspace packing was initially raised in the inspiring paper [CHS96].

4.1. **Grassmannian Spaces.** The complex Grassmannian space $\mathbb{G}(K, \mathbb{C}^d)$ is the collection of all K-dimensional subspaces of $\mathbb{C}^d$. This space is isomorphic to a quotient of unitary groups:

$$\mathbb{G}(K, \mathbb{C}^d) \quad \cong \quad \frac{U(d)}{U(K) \times U(d-K)}.$$

To understand the equivalence, note that each orthonormal basis for $\mathbb{C}^d$ can be split into K vectors, which span a K-dimensional subspace, and $(d-K)$ vectors, which span the orthogonal complement of the subspace. To obtain a unique representation of the subspace, one must modulate by rotations that fix the subspace and rotations that fix its complement. Similarly, the real Grassmannian space $\mathbb{G}(K, \mathbb{R}^d)$ is the collection all K-dimensional subspaces of $\mathbb{R}^d$. It is isomorphic to a quotient of orthogonal groups:

$$\mathbb{G}(K, \mathbb{R}^d) \quad \cong \quad \frac{O(d)}{O(K) \times O(d-K)}.$$

We shall concentrate on complex Grassmannians since the real case follows from a transparent adaptation.

4.2. **Metrics on Grassmannian Spaces.** Spheres and projective spaces each admit an essentially unique method for measuring distance. Moving up to a Grassmannian space, the situation becomes more complicated. Many interesting metrics exist, and each one yields a different packing problem.

Suppose that $\mathscr{S}$ and $\mathscr{T}$ are two subspaces in $\mathbb{G}(K, \mathbb{C}^d)$. These subspaces are inclined against each other by K different *principal angles*. The smallest principal angle θ_1 is the minimum angle formed by any pair of vectors drawn from the two subspaces. The second principal angle θ_2 is defined as the smallest angle attained between a pair of vectors orthogonal to the first set of vectors. The remaining principal angles are defined recursively. The principal angles are increasing, and each one lies in the range $[0, \pi/2]$. We shall only consider metrics that are functions of the principal angles between two subspaces.

Let us present a more computational definition of the principal angles [GVL96]. Suppose that the columns of S and T form orthonormal bases for the subspaces $\mathscr{S}$ and $\mathscr{T}$. Formally, S is a $d \times K$ matrix that satisfies $\mathsf{S}\mathsf{S}^* = \mathsf{I}$ and $\operatorname{colspan} \mathsf{S} = \mathscr{S}$. Analogously, the matrix T. Next, we compute a singular value decomposition of the product $\mathsf{S}^*\mathsf{T}$:

$$\mathsf{S}^*\mathsf{T} \quad = \quad \mathsf{U}\mathsf{C}\mathsf{V}^*,$$

where U and V are $K \times K$ unitary matrices and C is a nonnegative, diagonal matrix with nonincreasing entries. The matrix C is uniquely determined, and its entries list the cosines of the principal angles between $\mathscr{S}$ and $\mathscr{T}$:

$$c_{kk} \quad = \quad \cos\theta_k.$$

This definition of the principal angles is most convenient numerically because singular value decompositions can be computed quickly with standard software.

We are now in a position to detail some metrics on the Grassmannian space.

(1) The *chordal distance* between $\mathscr{S}$ and $\mathscr{T}$ is given by

$$\operatorname{dist}_{\text{chord}}(\mathscr{S}, \mathscr{T}) \quad \stackrel{\text{def}}{=} \quad \sqrt{\sin^2\theta_1 + \cdots + \sin^2\theta_K} \quad = \quad \sqrt{K - \|\mathsf{S}^*\mathsf{T}\|_{\text{F}}^2}. \tag{4.1}$$

The values of this metric range between zero and $\sqrt{K}$. The chordal distance is the easiest to work with, and it is also yields the most symmetrical packings [CHS96].

(2) The *spectral distance* is

$$\operatorname{dist}_{\text{spec}}(\mathscr{S}, \mathscr{T}) \quad \stackrel{\text{def}}{=} \quad \sin\theta_1 \quad = \quad \min_k \, \sin\theta_k \quad = \quad \sqrt{1 - \|\mathsf{S}^*\mathsf{T}\|_{2,2}^2}. \tag{4.2}$$

We use $\|\cdot\|_{2,2}$ to denote the spectral norm, which returns the largest singular value of a matrix. The spectral distance takes values between zero and one.

(3) The *Fubini–Study distance* is defined by

$$\operatorname{dist}_{\text{FS}}(\mathscr{S}, \mathscr{T}) \quad \stackrel{\text{def}}{=} \quad \arccos\left(\prod\nolimits_k \cos\theta_k\right) \quad = \quad \arccos\left|\det \mathsf{S}^*\mathsf{T}\right|. \tag{4.3}$$

This metric takes values between zero and $\pi/2$. From a group-theoretic point of view, the Fubini–Study distance is the most natural because it is the unique Riemannian metric that is invariant under actions of the unitary group on the Grassmannian space.

If the subspaces are one-dimensional, observe that each of these metrics reduces to (the sine of) the acute angle between the two subspaces, which is just the distance we defined on the projective space. The Grassmannian space admits other interesting metrics, some of which are listed in [BN02].

4.3. **Configurations and Matrices.** Next, we must discuss how to represent a configuration of N subspaces in the Grassmannian space $\mathbb{G}(K, \mathbb{C}^d)$. Let $\mathscr{X} = \{\mathsf{X}_n\}$ be a collection of N complex matrices with dimensions $d \times K$. Each of these matrices will provide a basis for one of the N subspaces, so we require that the columns of X_n form an orthonormal set for each n. We collate these matrices into a $d \times KN$ matrix

$$\mathsf{X} \quad \stackrel{\text{def}}{=} \quad \begin{bmatrix} \mathsf{X}_1 & \mathsf{X}_2 & \dots & \mathsf{X}_N \end{bmatrix}.$$

As always, the Gram matrix of X is defined as $\mathsf{G} = \mathsf{X}^*\mathsf{X}$. It is best to regard the Gram matrix as an $N \times N$ block matrix comprised of $K \times K$ blocks, and we shall index it as such. Observe that each block satisfies

$$\mathsf{G}_{mn} \quad = \quad \mathsf{X}_m^* \, \mathsf{X}_n$$

In particular, each diagonal block G_{nn} is an identity matrix. Meanwhile, the singular values of the off-diagonal block G_{mn} equal the cosines of the principal angles between the two subspaces $\operatorname{colspan}\mathsf{X}_m$ and $\operatorname{colspan}\mathsf{X}_n$.

As we shall see, each metric on the Grassmannian space leads to a measure of magnitude for the off-diagonal blocks of the Gram matrix. The Gram matrix solves the feasibility problem if and only if each off-diagonal block has sufficiently small magnitude.

4.4. **Packings with Chordal Distance.** Suppose that we seek a packing of N subspaces in $\mathbb{G}(K, \mathbb{C}^d)$ equipped with the chordal distance. If X is a configuration of N subspaces, its packing radius is

$$\operatorname{pack}_{\text{chord}}(\mathsf{X}) \quad \stackrel{\text{def}}{=} \quad \min_{m \neq n} \operatorname{dist}_{\text{chord}}(\mathsf{X}_m, \mathsf{X}_n) \quad = \quad \min_{m \neq n} \sqrt{K - \|\mathsf{X}_m^* \mathsf{X}_n\|_{\text{F}}^2}.$$

Given a parameter ρ, the feasibility problem requests a configuration X that satisfies

$$\min_{m \neq n} \sqrt{K - \|\mathsf{X}_m^* \mathsf{X}_n\|_{\text{F}}^2} \quad \geq \quad \rho.$$

As usual, we rearrange to obtain a simpler condition:

$$\max_{m \neq n} \|\mathsf{X}_m^* \mathsf{X}_n\|_{\text{F}} \quad \leq \quad \mu \tag{4.4}$$

where $\mu = \sqrt{K - \rho^2}$. It is immediately clear that the configuration X solves the feasibility problem (4.4) if and only if its Gram matrix G has the following properties:

(1) G is Hermitian.
(2) Each diagonal block of G is an identity matrix.
(3) $\|\mathsf{G}_{mn}\|_{\text{F}} \leq \mu$ whenever $m \neq n$.
(4) G is positive semi-definite.
(5) G has rank d.
(6) G has trace N.

This enumeration leads directly to an algorithm.

The structural constraint is the convex set

$$\mathscr{H} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{H} \in \mathbb{C}^{KN \times KN} : \mathsf{H} = \mathsf{H}^*, \quad \mathsf{H}_{nn} = \mathsf{I} \text{ for all } n, \quad \text{and} \quad \|\mathsf{H}_{mn}\|_{\text{F}} \leq \mu \text{ for all } m \neq n\}.$$

The spectral constraint set remains

$$\mathscr{G} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{G} \in \mathbb{C}^{KN \times KN} : \mathsf{G} \succcurlyeq \mathsf{0}, \quad \operatorname{rank} \mathsf{G} \leq d, \quad \text{and} \quad \operatorname{trace} \mathsf{G} = N\}.$$

Solving the feasibility problem (4.4) with parameter μ is equivalent to exhibiting a matrix in the intersection of $\mathscr{G}$ and $\mathscr{H}$.

Algorithm 2.4 allows us to build a starting matrix. To construct a subspace uniformly at random with respect to the left-invariant Haar measure on $\mathbb{G}(K, \mathbb{C}^d)$, we use the striking method developed in [Ste80]. Choose a $d \times K$ matrix whose (complex) entries are iid standard normal, and perform a QR decomposition. The first K columns of the unitary part of the decomposition form an orthonormal basis for a random subspace. In Step (3), the test requires that $\|\mathsf{X}_m^* \mathsf{X}_n\|_{\text{F}} \leq \tau$ for each $m < n$.

Next, we apply alternating projection to this initial matrix. To determine the nearest matrix from the structural constaint set, we use the following result.

Proposition 4.1. *Let G be an Hermitian matrix. With respect to the Frobenius norm, the unique matrix in $\mathscr{H}$ nearest to G has a block-identity diagonal and off-diagonal blocks that satisfy*

$$\mathsf{H}_{mn} \quad = \quad \begin{cases} \mathsf{G}_{mn} & \text{if } \|\mathsf{G}_{mn}\|_{\text{F}} \leq \mu, \text{ and} \\ \mu \, \mathsf{G}_{mn} / \|\mathsf{G}_{mn}\|_{\text{F}} & \text{otherwise.} \end{cases}$$

It is nice to see how this result generalizes Proposition 3.1. We leave the proof for the reader.

In Step (6) of the alternating projection algorithm, we extract the diagonal blocks of G_T. It follows that Step (7) scales each diagonal block to equal the identity matrix without changing the inertia of the matrix. Therefore, we may factor the output matrix to obtain a $d \times KN$ configuration matrix X. The N blocks of this matrix represent K-dimensional subspaces of $\mathbb{C}^d$.

Sloane and his colleagues have tabulated packings of subspaces in $\mathbb{G}(K, \mathbb{R}^d)$ equipped with the chordal distance [Slo04a]. We have attempted to reproduce some of these packings. Table 5

compares the results. In most cases, the packing radii are identical in the first two decimal places. We have also computed a table of good packings in complex Grassmannian spaces equipped with the chordal distance. See Table 6 for the numbers. Most of the complex packings improve on the corresponding real packings. Comparing the complex packings against the Rankin Bound (A.1), we discover that many of the complex packings are essentially optimal. Figure 2 provides a direct comparison of some real and complex Grassmannian packings with respect to the chordal distance.

4.5. **Packings with Spectral Distance.** To construct packings with respect to the spectral distance, we tread a familiar path. Suppose that we wish to produce a configuration of N subspaces in $\mathbb{G}(K, \mathbb{C}^d)$ with a packing radius ρ. The feasibility problem requires that

$$\max_{m \neq n} \|\mathsf{X}_m^* \mathsf{X}_n\|_{2,2} \quad \leq \quad \mu \tag{4.5}$$

where $\mu = \sqrt{1 - \rho^2}$. This leads to the convex structural constraint set

$$\mathscr{H} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{H} \in \mathbb{C}^{KN \times KN} : \mathsf{H} = \mathsf{H}^*, \quad \mathsf{H}_{nn} = \mathsf{I} \text{ for all } n, \quad \text{and} \quad \|\mathsf{H}_{mn}\|_{2,2} \leq \mu \text{ for all } m \neq n\}.$$

The spectral constraint set is the same as usual. The next proposition shows how to find the matrix in $\mathscr{H}$ closest to an initial matrix. In preparation, define the truncation operator $[x]_\mu = \min\{x, \mu\}$ for nonnegative numbers, and extend it to nonnegative matrices by applying it to each component.

Proposition 4.2. *Let G be an Hermitian matrix. With respect to the Frobenius norm, the unique matrix in $\mathscr{H}$ nearest to G has a block identity diagonal. If the off-diagonal block G_{mn} has singular value decomposition $\mathsf{U}_{mn} \mathsf{C}_{mn} \mathsf{V}_{mn}^*$, then*

$$\mathsf{H}_{mn} \quad = \quad \begin{cases} \mathsf{G}_{mn} & \text{if } \|\mathsf{G}_{mn}\|_{2,2} \leq \mu, \text{ and} \\ \mathsf{U}_{mn} \, [\mathsf{C}_{mn}]_\mu \, \mathsf{V}_{mn}^* & \text{otherwise.} \end{cases}$$

Proof. For each block H_{mn}, we must solve the optimization problem

$$\min_{\mathsf{A}} \ \tfrac{1}{2} \|\mathsf{A} - \mathsf{H}_{mn}\|_{\mathrm{F}}^2 \qquad \text{subject to} \qquad \|\mathsf{A}\|_{2,2} \leq \mu.$$

The Frobenius norm is strictly convex and the spectral norm is convex, so this problem has a unique solution.

Let $\boldsymbol{\sigma}(\cdot)$ return the vector of decreasingly ordered singular values of a matrix. Suppose that H_{mn} has the singular value decomposition $\mathsf{H}_{mn} = \mathsf{U} \operatorname{diag} \boldsymbol{\sigma}(\mathsf{H}_{mn}) \, \mathsf{V}^*$. The constraint in the optimization problem depends only on the singular values of A, and so the Wielandt–Hoffman Theorem for singular values [HJ85] allows us to check that the solution has the form $\mathsf{A} = \mathsf{U} \operatorname{diag} \boldsymbol{\sigma}(\mathsf{A}) \, \mathsf{V}^*$.

To determine the singular values $\boldsymbol{\xi} = \boldsymbol{\sigma}(\mathsf{A})$ of the solution, we must solve the (strictly) convex program

$$\min_{\boldsymbol{\xi}} \ \tfrac{1}{2} \|\boldsymbol{\xi} - \boldsymbol{\sigma}(\mathsf{H}_{mn})\|_2^2 \qquad \text{subject to} \qquad \xi_k \leq \mu.$$

An easy application of Karush–Kuhn–Tucker theory [Roc70] proves that the solution is obtained by truncating the singular values of H_{mn} that exceed μ. □

We have performed some limited experiments on packing subspaces with respect to the spectral distance, and we have compared our results against the upper bound (A.2). Turn to Table 7 for the calculations. Meanwhile, Figure 3 illustrates the differences between the real packings and the complex packings. Many of our configurations very nearly attained the upper bound, which implies that the subspaces are not only equidistant but also equi-isoclinic. That is, all principal angles between all pairs of subspaces are identical. We should mention that the algorithm failed to produce reasonable packings of 8 and 9 subspaces in $\mathbb{G}(3, \mathbb{R}^6)$.

4.6. **Packings with Fubini–Study Distance.** Packing with respect to the Fubini–Study distance is much harder. Suppose that we wish to construct a configuration of N subspaces whose Fubini–Study packing radius exceeds ρ. The feasibility condition is

$$\max_{m \neq n} \ |\det \mathsf{X}_m^* \, \mathsf{X}_n| \quad \leq \quad \mu \tag{4.6}$$

where $\mu = \cos \rho$. This leads to the structural constraint set

$$\mathscr{H} \quad \stackrel{\text{def}}{=} \quad \{\mathsf{H} \in \mathbb{C}^{KN \times KN} : \mathsf{H} = \mathsf{H}^*, \quad \mathsf{H}_{nn} = \mathsf{I} \text{ for all } n, \quad \text{and} \quad |\det \mathsf{H}_{mn}| \leq \mu \text{ for all } m \neq n\}.$$

Unhappily, this set is no longer convex. To produce a nearest matrix in $\mathscr{H}$, we must solve a nonlinear programming problem. The following proposition describes a numerically favorable formulation.

Proposition 4.3. *Let G be an Hermitian matrix. Suppose that the off-diagonal block G_{mn} has singular value decomposition $\mathsf{U}_{mn} \mathsf{C}_{mn} \mathsf{V}_{mn}^*$. Let $\boldsymbol{c}_{mn} = \operatorname{diag} \mathsf{C}_{mn}$, and find a (real) vector $\boldsymbol{x}_{mn}$ that solves the optimization problem*

$$\min_{\boldsymbol{x}} \quad \tfrac{1}{2} \|\exp(\boldsymbol{x}) - \boldsymbol{c}_{mn}\|_2^2 \qquad \textit{subject to} \qquad \mathbf{e}^* \, \boldsymbol{x} \leq \log \mu.$$

In Frobenius norm, a matrix H from $\mathscr{H}$ that is closest to G has a block-identity diagonal and off-diagonal blocks

$$\mathsf{H}_{mn} \quad = \quad \begin{cases} \mathsf{G}_{mn} & \textit{if } |\det \mathsf{G}_{mn}| \leq \mu, \textit{ and} \\ \mathsf{U}_{mn} \ \operatorname{diag}(\exp \boldsymbol{x}_{mn}) \ \mathsf{V}_{mn}^* & \textit{otherwise.} \end{cases}$$

We use $\exp(\cdot)$ to denote the componentwise exponential of a vector. It can be shown that the objective function in the optimization problem is quasi-convex but not convex.

Proof. For each off-diagonal block H_{mn}, we must solve the optimization problem

$$\min\nolimits_{\mathsf{A}} \ \tfrac{1}{2} \ \|\mathsf{A} - \mathsf{H}_{mn}\|_{\mathrm{F}}^2 \qquad \text{subject to} \qquad |\det \mathsf{A}| \leq \mu.$$

We may reformulate this problem as

$$\min\nolimits_{\mathsf{A}} \ \tfrac{1}{2} \ \|\mathsf{A} - \mathsf{H}_{mn}\|_{\mathrm{F}}^2 \qquad \text{subject to} \qquad \sum\nolimits_{k=1}^{K} \log \sigma_k(\mathsf{A}) \leq \log \mu.$$

A familiar argument proves that the solution matrix has the same left and right singular vectors as H_{mn}. To obtain the singular values $\boldsymbol{\xi} = \boldsymbol{\sigma}(\mathsf{A})$ of the solution, we consider the mathematical program

$$\min\nolimits_{\boldsymbol{\xi}} \ \tfrac{1}{2} \ \|\boldsymbol{\xi} - \boldsymbol{\sigma}(\mathsf{H}_{mn})\|_2^2 \qquad \text{subject to} \qquad \sum\nolimits_{k=1}^{K} \log \xi_k \leq \log \mu.$$

Change variables to complete the argument. □

At present, we use general-purpose nonlinear programming software to solve the optimization problems required by the proposition. With this imperfect implementation, we have performed a few experiments on packing with respect to the Fubini–Study distance, which are described in Table 8; see also Figure 4. The alternating projection algorithm appears to succeed reasonably well. For small N, in fact, the packings have the maximum possible radius. Unfortunately, we do not know a general upper bound that allows us to gauge the quality of the remaining packings. Moreover, our current software is very slow. To perform large-scale experiments, it will probably be necessary to tailor an algorithm that can solve the nonlinear programming problems more quickly. A detailed study of packing with respect to the Fubini–Study distance must remain a topic for future research.

5. Conclusions

We have shown that the alternating projection algorithm can be used to solve many different packing problems. The method is easy to understand and to implement, even while it is versatile and powerful. In cases where experiments have been performed, we have often been able to match the best packings known. Moreover, we have extended the method to solve problems that have not been studied numerically. Using the bounds in the appendix, we have been able to show that many of our packings are essentially optimal. It seems clear that alternating projection is an effective numerical algorithm for packing. We hope that it will allow researchers to explore *terra incognita*.

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Appendix A. Bounds on Packing Radii

To assay the quality of our packings, it helps to have some upper bounds on the packing radius. Researchers have developed very sophisticated machinery for computing these bounds, but we shall content ourselves with the simplest means. In spite of our willful naïveté, the results we present here suffice to establish that many of our packings are essentially optimal. We begin with the Rankin Bound on the minimum distance among a set of points on a sphere.

Theorem A.1 (Rankin [Ran47]). *Suppose that $\{\boldsymbol{x}_n\}$ is a collection of N points on a sphere of radius r centered at the origin of a real Euclidean space. Then*

$$\max_{m \neq n} \langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle \quad \geq \quad \frac{r^2}{1 - N}.$$

It follows that

$$\min_{m \neq n} \|\boldsymbol{x}_m - \boldsymbol{x}_n\|_2^2 \quad \leq \quad \frac{2\, r^2\, N}{N - 1}.$$

Equality holds if only if the points are equidistant. This event requires that $N \leq d + 1$.

Proof. Let G be the Gram matrix of the ensemble. Since the Gram matrix is positive semi-definite,

$$\sum\nolimits_{m,n} \langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle \quad = \quad \mathbf{e}^*\, \mathsf{G}\, \mathbf{e} \quad \geq \quad 0.$$

On the other hand, if we define $\mu = \max_{m \neq n} \langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle$, then

$$\sum\nolimits_{m,n} \langle \boldsymbol{x}_m, \boldsymbol{x}_n \rangle \quad \leq \quad r^2\, N \quad + \quad N\,(N-1)\,\mu.$$

Combine the inequalities and rearrange to develop the bound on the maximum inner product. The distance bound follows after a little more algebra.

Equality requires that each inner product equal μ, which implies that each pair of points is equidistant. It follows that the points lie at vertices of a regular simplex. In d dimensions, a simplex contains exactly $(d + 1)$ vertices. Conversely, a calculation shows the vertices of a regular simplex yield equality in the bounds. □

We require upper bounds on the radii of subspace packings. Conway, Hardin, and Sloane have developed a wonderful approach to this problem. They first embed the chordal Grassmannian space isometrically into a Euclidean sphere, and then they apply Rankin's Bound.

Theorem A.2 (Conway–Hardin–Sloane [CHS96]). *The chordal Grassmannian space $\mathbb{G}(K, \mathbb{F}^d)$ may be embedded isometrically into a real Euclidean sphere whose squared radius is $\frac{1}{2}\, K\,(d - K)/d$. When $\mathbb{F} = \mathbb{R}$, the dimension of the embedding space is $\binom{d+1}{2} - 1$. When $\mathbb{F} = \mathbb{C}$, the dimension is $d^2 - 1$.*

Proof. Suppose that the columns of S and T form orthonormal bases for two K-dimensional subspaces of $\mathbb{F}^d$. Then we may calculate that

$$\begin{aligned} 2\,\mathrm{dist}_{\mathrm{chord}}(\mathsf{S},\mathsf{T})^2 &= 2\,(K - \|\mathsf{S}^*\,\mathsf{T}\|_{\mathrm{F}}^2) \\ &= \|\mathsf{S}\mathsf{S}^*\|_{\mathrm{F}}^2 + \|\mathsf{T}\mathsf{T}^*\|_{\mathrm{F}}^2 - 2\,\mathrm{Re}\,\mathrm{trace}\,\mathsf{S}\mathsf{S}^*\,\mathsf{T}\mathsf{T}^* \\ &= \|\mathsf{S}\mathsf{S}^* - \mathsf{T}\mathsf{T}^*\|_{\mathrm{F}}^2\,. \end{aligned}$$

That is, the squared chordal distance between two subspaces is equal to half the squared Frobenius distance between the orthogonal projectors onto the two subspaces.

Suppose that $\mathscr{S}$ is a K-dimensional subspace of $\mathbb{F}^d$, and let P be the unique orthogonal projector onto $\mathscr{S}$. The projector has trace K, so we may subtract a multiple of the identity to zero the trace: $\widehat{\mathsf{P}} = \mathsf{P} - (K/d)\,\mathsf{I}$. The new matrix satisfies $\|\widehat{\mathsf{P}}\|_{\mathrm{F}}^2 = K\,(d-K)/d$, so the de-traced projectors all lie on a sphere. Moreover, since we translate each rank-K projector by the same amount, this operation does not change the Frobenius distance between them. Therefore, the map $\mathscr{S} \mapsto \frac{1}{\sqrt{2}}\,\widehat{\mathsf{P}}$ is an isometric embedding of the chordal Grassmannian space into a Euclidean sphere with squared radius $\frac{1}{2}\,K\,(d-K)/d$.

To determine the dimension of the embedding space, we count the degrees of freedom in the de-traced projectors. In the real case, a d-dimensional symmetric matrix contains $\frac{1}{2}\,d\,(d+1)$ free real-valued entries, from which we subtract one to account for the fixed trace. In the complex case, a d-dimensional Hermitian matrix contains d real entries on the diagonal and $\frac{1}{2}\,d\,(d-1)$ complex entries in the strict lower triangle. Accounting for the trace, we conclude that the embedding dimension is $d^2 - 1$. □

Combining Theorem A.1 with Theorem A.2, we obtain an upper bound on the packing radius.

Corollary A.3 (Conway–Hardin–Sloane [CHS96])**.** *An upper bound on the packing radius of N subspaces in the chordal Grassmannian space $\mathbb{G}(K,\mathbb{F}^d)$ is*

$$\mathrm{pack}_{\mathrm{chord}}(\mathscr{X})^2 \quad\leq\quad \frac{K\,(d-K)}{d}\,\frac{N}{N-1}. \tag{A.1}$$

If the bound is met, all pairs of subspaces are equidistant. When $\mathbb{F} = \mathbb{R}$, the bound is attainable only if $N \leq \frac{1}{2}\,d\,(d+1)$. When $\mathbb{F} = \mathbb{C}$, the bound is attainable only if $N \leq d^2$.

We shall refer to the inequality (A.1) as the Rankin Bound for packings with respect to the chordal distance. When $K = 1$, the corollary applies to packings in projective space. Finally, we draw a new corollary that gives a bound for the spectral distance.

Corollary A.4. *We have the following bound on the packing radius of N subspaces in the Grassmannian space $\mathbb{G}(K,\mathbb{F}^d)$ equipped with the spectral distance.*

$$\mathrm{pack}_{\mathrm{spec}}(\mathscr{X})^2 \quad\leq\quad \frac{d-K}{d}\,\frac{N}{N-1}. \tag{A.2}$$

If the bound is met, all pairs of subspaces are equi-isoclinic. When $\mathbb{F} = \mathbb{R}$, the bound is attainable only if $N \leq \frac{1}{2}\,d\,(d+1)$. When $\mathbb{F} = \mathbb{C}$, the bound is attainable only if $N \leq d^2$.

Proof. The monotonicity of power means [HLP52] yields the inequality

$$\min_k\ \sin\theta_k \quad\leq\quad \left[K^{-1}\sum\nolimits_{k=1}^{K}\sin^2\theta_k\right]^{1/2}.$$

Equality holds if and only if $\theta_1 = \cdots = \theta_K$. It follows that

$$\mathrm{pack}_{\mathrm{spec}}(\mathscr{X})^2 \quad\leq\quad K^{-1}\,\mathrm{pack}_{\mathrm{chord}}(\mathscr{X})^2 \quad\leq\quad \frac{d-K}{d}\,\frac{N}{N-1}.$$

If the second inequality is met, then all pairs of subspaces are equidistant with respect to the chordal metric. Moreover, if the first inequality is met, then the principal angles between each pair of subspaces are constant. That is, the subspaces are equi-isoclinic. □

In fact, the previous corollary overestimates the maximum possible number of equi-isoclinic subspaces. The following result is better, although its authors still do not believe it is sharp.

Theorem A.5 (Lemmens–Seidel [LS73]). *The maximum number of equi-isoclinic K-dimensional subspaces of $\mathbb{R}^d$ is no greater than*

$$\tfrac{1}{2}\, d\,(d+1) - \tfrac{1}{2}\, K\,(K+1) + 1.$$

Similarly, the maximum number of equi-isoclinic K-dimensional subspaces of $\mathbb{C}^d$ does not exceed

$$d^2 - K^2 + 1.$$

We do not know any bounds for packings with respect to the Fubini–Study metric.

Appendix B. Tables and Figures

These tables and figures are intended only to describe the results of our experiments. It is likely that many of the packing radii could be improved with additional effort. In all cases, we present the results of calculations for the stated problem, even if we obtained a better packing by solving a different problem. For example, a complex packing should always improve on the corresponding real packing. If the numbers indicate otherwise, it just means that the complex experiment yielded an inferior result. As a second example, the optimal packing radius should increase monotonically with the number of points. When the numbers indicate otherwise, it means that running the algorithm with more points yielded a better result than running it with fewer. These failures may reflect the difficulty of various packing problems.

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		Packing Radii (Degrees)		
	N	JAT	NJAS	Difference
$\mathbb{S}^2$	4	109.47	109.47	0.00
	5	90.00	90.00	0.00
	6	90.00	90.00	0.00
	7	77.87	77.87	0.00
	8	74.86	74.86	0.00
	9	70.53	70.53	0.00
	10	66.15	66.15	0.00
	11	63.43	63.43	0.00
	12	63.43	63.43	0.00
	13	56.67	57.14	0.47
	14	55.33	55.67	0.34
	15	53.59	53.66	0.07
	16	52.16	52.24	0.09
	17	51.04	51.09	0.05
	18	49.53	49.56	0.03
	19	47.49	47.69	0.20
	20	47.42	47.43	0.01
$\mathbb{S}^3$	5	104.47	104.48	0.00
	6	89.98	90.00	0.02
	7	90.00	90.00	0.00
	8	90.00	90.00	0.00
	9	80.60	80.68	0.08
	10	80.40	80.41	0.00
	11	76.68	76.68	0.00
	12	75.52	75.52	0.00
	13	71.82	72.10	0.28
	14	71.21	71.37	0.16
	15	69.43	69.45	0.02
	16	67.16	67.19	0.04
	17	65.26	65.65	0.40
	18	64.14	64.99	0.85
	19	63.90	64.26	0.36
	20	64.26	64.26	0.00
$\mathbb{S}^4$	6	101.54	101.54	0.00
	7	90.00	90.00	0.00
	8	90.00	90.00	0.00
	9	90.00	90.00	0.00
	10	90.00	90.00	0.00
	11	82.24	82.37	0.12
	12	81.14	81.15	0.00
	13	79.10	79.21	0.11
	14	78.46	78.46	0.00
	15	78.46	78.46	0.00
	16	78.46	78.46	0.00
	17	74.31	74.31	0.00
	18	74.01	74.01	0.00
	19	72.42	73.03	0.62
	20	72.58	72.58	0.00

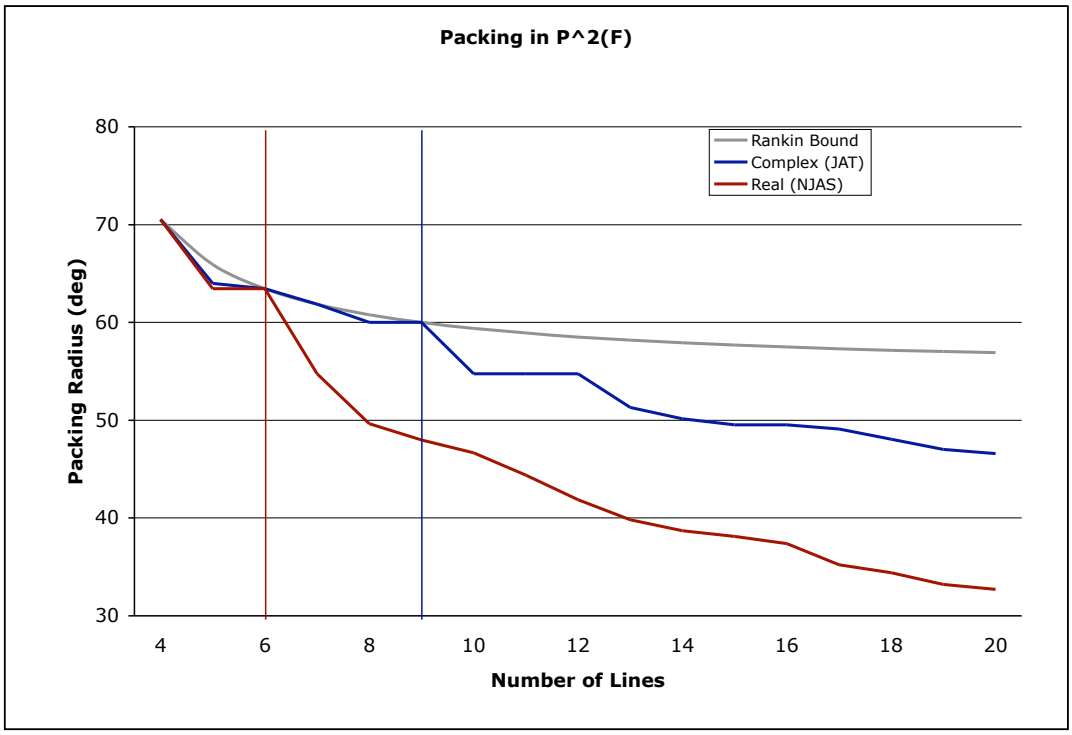
Table 1. Packing on spheres: We compare our calculations (JAT) against the best packings (NJAS) reported in [Slo04b]. The number of points is denoted by N. The packing radius of a configuration is measured as the number of degrees of arc subtended by the two closest points. The final column records the difference between the third and second columns.

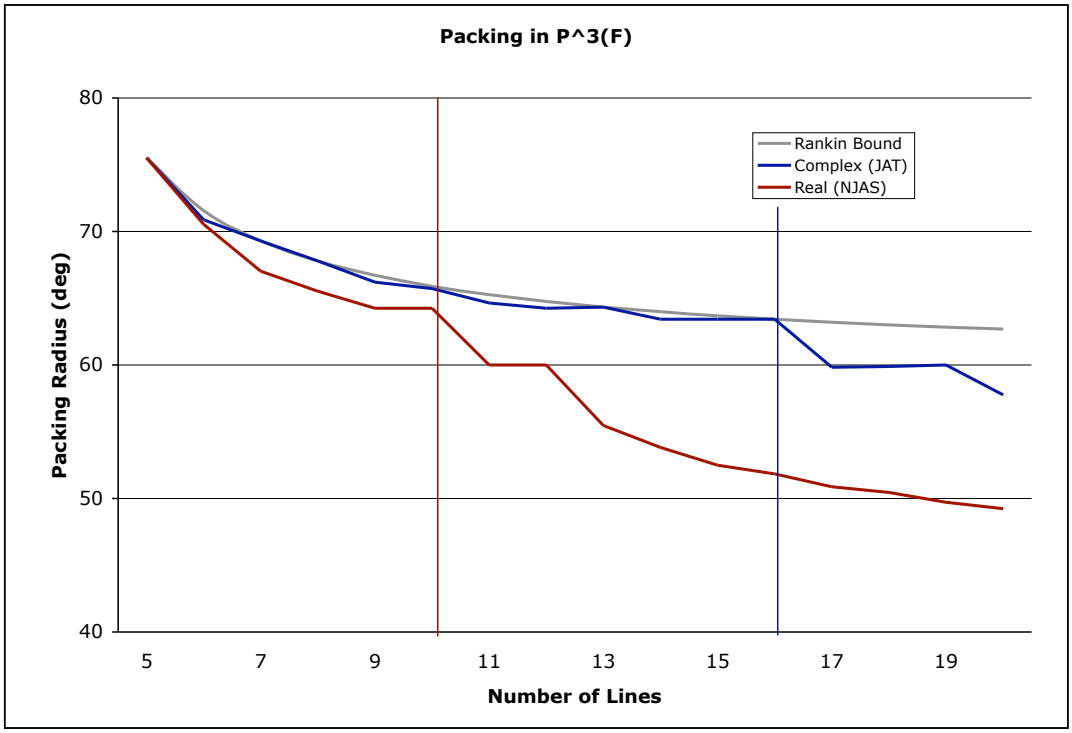
		Packing Radii (Degrees)		
	N	JAT	NJAS	Difference
$\mathbb{P}^2(\mathbb{R})$	4	70.53	70.53	0.00
	5	63.43	63.43	0.00
	6	63.43	63.43	0.00
	7	54.74	54.74	0.00
	8	49.64	49.64	0.00
	9	47.98	47.98	0.00
	10	46.67	46.67	0.00
	11	44.40	44.40	0.00
	12	41.88	41.88	0.00
	13	39.81	39.81	0.00
	14	38.52	38.68	0.17
	15	37.93	38.13	0.20
	16	37.36	37.38	0.02
	17	35.00	35.24	0.23
	18	34.22	34.41	0.19
	19	32.93	33.21	0.28
	20	32.48	32.71	0.23
$\mathbb{P}^3(\mathbb{R})$	5	75.52	75.52	0.00
	6	70.53	70.53	0.00
	7	67.02	67.02	0.00
	8	65.53	65.53	0.00
	9	64.26	64.26	0.00
	10	64.26	64.26	0.00
	11	60.00	60.00	0.00
	12	60.00	60.00	0.00
	13	55.46	55.46	0.00
	14	53.63	53.84	0.21
	15	52.07	52.50	0.43
	16	50.97	51.83	0.85
	17	50.66	50.89	0.23
	18	50.28	50.46	0.18
	19	49.65	49.71	0.06
	20	49.11	49.23	0.12
$\mathbb{P}^4(\mathbb{R})$	6	78.46	78.46	0.00
	7	73.37	73.37	0.00
	8	70.79	70.80	0.01
	9	70.53	70.53	0.00
	10	70.53	70.53	0.00
	11	67.25	67.25	0.00
	12	67.02	67.02	0.00
	13	65.57	65.73	0.17
	14	65.18	65.72	0.54
	15	65.50	65.53	0.03
	16	63.43	63.43	0.00
	17	61.21	61.26	0.04
	18	61.05	61.05	0.01
	19	58.41	60.00	1.59
	20	55.97	60.00	4.03

Table 2. Packing in real projective spaces: We compare our calculations (JAT) with the best packings (NJAS) reported in [Slo04a]. The number of points is denoted by N. The packing radius of a configuration is measured as the acute angle between the closest pair of lines. The final column lists the differences between the fourth and third columns.

		Packing Radii (Degrees)		
	N	JAT	Rankin	Difference
$\mathbb{P}^1(\mathbb{C})$	3	60.00	60.00	0.00
	4	54.74	54.74	0.00
	5	45.00	52.24	7.24
	6	45.00	50.77	5.77
$\mathbb{P}^2(\mathbb{C})$	4	70.53	70.53	0.00
	5	64.00	65.91	1.90
	6	63.44	63.43	0.00
	7	61.87	61.87	0.00
	8	60.00	60.79	0.79
	9	60.00	60.00	0.00
	10	54.73	59.39	4.66
	11	54.73	58.91	4.18
	12	54.73	58.52	3.79
	13	51.32	58.19	6.88
	14	50.13	57.92	7.79
	15	49.53	57.69	8.15
$\mathbb{P}^3(\mathbb{C})$	5	75.52	75.52	0.00
	6	70.88	71.57	0.68
	7	69.29	69.30	0.01
	8	67.78	67.79	0.01
	9	66.21	66.72	0.51
	10	65.71	65.91	0.19
	11	64.64	65.27	0.63
	12	64.24	64.76	0.52
	13	64.34	64.34	0.00
	14	63.43	63.99	0.56
	15	63.43	63.69	0.26
	16	63.43	63.43	0.00
	17	59.84	63.21	3.37
	18	59.89	63.02	3.12
	19	60.00	62.84	2.84
	20	57.76	62.69	4.93
$\mathbb{P}^4(\mathbb{C})$	6	78.46	78.46	0.00
	7	74.52	75.04	0.51
	8	72.81	72.98	0.16
	9	71.24	71.57	0.33
	10	70.51	70.53	0.02
	11	69.71	69.73	0.02
	12	68.89	69.10	0.21
	13	68.19	68.58	0.39
	14	67.66	68.15	0.50
	15	67.37	67.79	0.43
	16	66.68	67.48	0.80
	17	66.53	67.21	0.68
	18	65.87	66.98	1.11
	19	65.75	66.77	1.02
	20	65.77	66.59	0.82
	21	65.83	66.42	0.60
	22	65.87	66.27	0.40
	23	65.90	66.14	0.23
	24	65.91	66.02	0.11
	25	65.91	65.91	0.00

Table 3. Packing in complex projective spaces: We compare our calculations (JAT) against the Rankin Bound, equation (A.1). The number of points is denoted by N. The packing radius of a configuration is measured as the acute angle between the closest pair of lines. The final column shows how far the calculated ensemble falls short of the bound.


Packing in P^2(F)
Packing Radius (deg)
80
70
60
50
40
30
4
6
8
10
12
14
16
18
20
Number of Lines
Rankin Bound
Complex (JAT)
Real (NJAS)


Packing in P^3(F)
Packing Radius (deg)
80
70
60
50
40
5
7
9
11
13
15
17
19
Number of Lines
Rankin Bound
Complex (JAT)
Real (NJAS)

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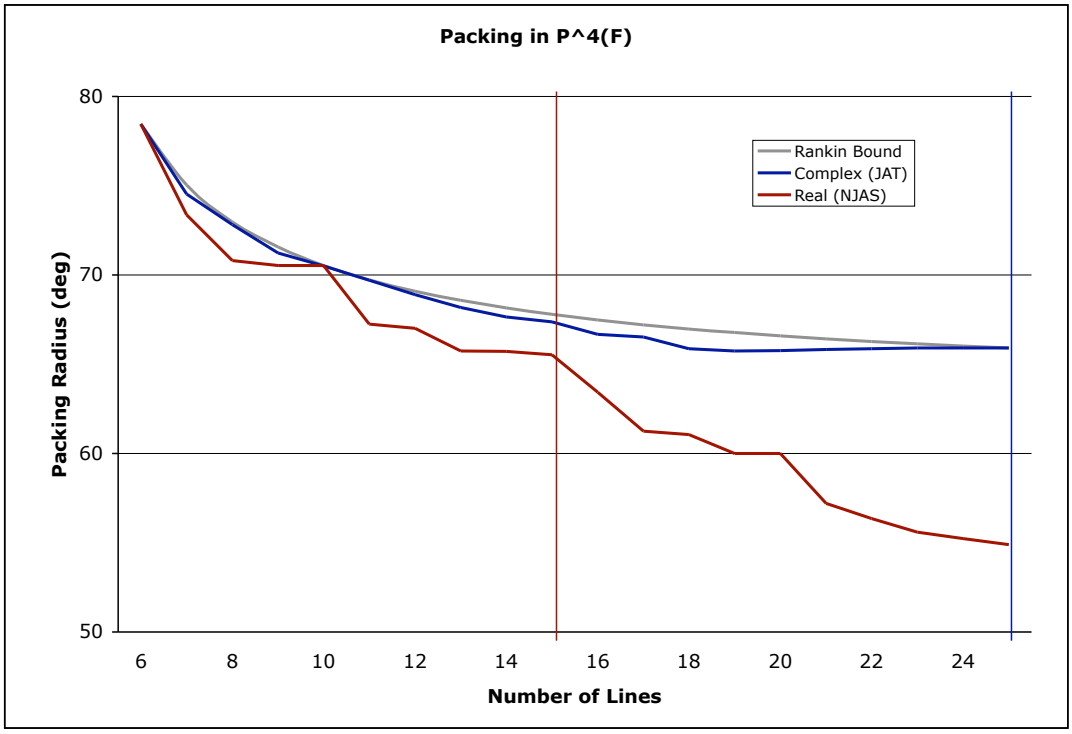

FIGURE 1. REAL AND COMPLEX PROJECTIVE PACKINGS: These three graphs compare the packing radii attained by configurations in real and complex projective spaces. The red line indicates the best real packings obtained by Sloane and his colleagues [Slo04a]. The blue line indicates the best complex packings produced by the author. Rankin's upper bound (A.1) is depicted in gray. The vertical red line marks the largest number of real lines for which the Rankin Bound is attainable, while the blue line marks the maximum number of complex lines for which the Rankin Bound is attainable.

	d				
N	2	3	4	5	6
3	$\mathbb{R}$	$\mathbb{R}$	..	..	..
4	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	..	..
5	..	.	$\mathbb{R}$	$\mathbb{R}$	..
6	..	$\mathbb{R}$	.	$\mathbb{R}$	$\mathbb{R}$
7	..	$\mathbb{C}$	$\mathbb{C}$	.	$\mathbb{R}$
8	..	.	$\mathbb{C}$	.	.
9	..	$\mathbb{C}$	.	.	$\mathbb{C}$
10	..	..	.	$\mathbb{R}$	.
11	..	..	.	$\mathbb{C}$	$\mathbb{C}$
12	..	..	.	.	$\mathbb{C}$
13	..	..	$\mathbb{C}$	.	.
14	..	..	.	.	.
15	..	..	.	.	.
16	..	..	$\mathbb{C}$	.	$\mathbb{R}$
17	..	..	..	.	.
18	..	..	..	.	.
19	..	..	..	.	.

	d				
N	2	3	4	5	6
20	..	..	..	.	.
21	..	..	..	$\mathbb{C}$	.
22	..	..	..	.	.
23	..	..	..	.	.
24	..	..	..	.	.
25	..	..	..	$\mathbb{C}$	.
26	..	..	..	..	.
27	..	..	..	..	.
28	..	..	..	..	.
29	..	..	..	..	.
30	..	..	..	..	.
31	..	..	..	..	$\mathbb{C}$
32	..	..	..	..	.
33	..	..	..	..	.
34	..	..	..	..	.
35	..	..	..	..	.
36	..	..	..	..	$\mathbb{C}$

TABLE 4. EQUIANGULAR TIGHT FRAMES: The symbols $\mathbb{R}$ and $\mathbb{C}$ respectively indicate that alternating projection was able to compute a real or complex equiangular tight frame of N vectors in a d-dimensional Euclidean space. Every real equiangular tight frame is automatically a complex equiangular tight frame. One period (.) means that no real equiangular tight frame exists, and two periods (..) mean that no equiangular tight frame exists at all. An equiangular tight frame is characterized as a projective packing that meets the Rankin Bound, equation (A.1). This table is reproduced from [TDHS03].

		Squared Packing Radii		
	N	JAT	NJAS	Difference
$\mathbb{G}(2,\mathbb{R}^4)$ Chordal Distance	3	1.5000	1.5000	0.0000
	4	1.3333	1.3333	0.0000
	5	1.2500	1.2500	0.0000
	6	1.2000	1.2000	0.0000
	7	1.1656	1.1667	0.0011
	8	1.1423	1.1429	0.0005
	9	1.1226	1.1231	0.0004
	10	1.1111	1.1111	0.0000
	11	0.9981	1.0000	0.0019
	12	0.9990	1.0000	0.0010
	13	0.9996	1.0000	0.0004
	14	1.0000	1.0000	0.0000
	15	1.0000	1.0000	0.0000
	16	0.9999	1.0000	0.0001
	17	1.0000	1.0000	0.0000
	18	0.9992	1.0000	0.0008
	19	0.8873	0.9091	0.0218
	20	0.8225	0.9091	0.0866
$\mathbb{G}(2,\mathbb{R}^5)$ Chordal Distance	3	1.7500	1.7500	0.0000
	4	1.6000	1.6000	0.0000
	5	1.5000	1.5000	0.0000
	6	1.4400	1.4400	0.0000
	7	1.4000	1.4000	0.0000
	8	1.3712	1.3714	0.0002
	9	1.3464	1.3500	0.0036
	10	1.3307	1.3333	0.0026
	11	1.3069	1.3200	0.0131
	12	1.2973	1.3064	0.0091
	13	1.2850	1.2942	0.0092
	14	1.2734	1.2790	0.0056
	15	1.2632	1.2707	0.0075
	16	1.1838	1.2000	0.0162
	17	1.1620	1.2000	0.0380
	18	1.1589	1.1909	0.0319
	19	1.1290	1.1761	0.0472
	20	1.0845	1.1619	0.0775

Table 5. Packing in real Grassmannians with chordal distance: We compare our calculations (JAT) against the best packings (NJAS) reported in [Slo04a]. The number of points is denoted by N. The squared packing radius is the squared chordal distance (4.1) between the closest pair of subspaces. The last column lists the differences between the fourth and third columns.

		Squared Packing Radii		
	N	JAT	Rankin	Difference
$\mathbb{G}(2,\mathbb{C}^4)$ Chordal Distance	3	1.5000	1.5000	0.0000
	4	1.3333	1.3333	0.0000
	5	1.2500	1.2500	0.0000
	6	1.2000	1.2000	0.0000
	7	1.1667	1.1667	0.0000
	8	1.1429	1.1429	0.0000
	9	1.1250	1.1250	0.0000
	10	1.1111	1.1111	0.0000
	11	1.0999	1.1000	0.0001
	12	1.0906	1.0909	0.0003
	13	1.0758	1.0833	0.0076
	14	1.0741	1.0769	0.0029
	15	1.0698	1.0714	0.0016
	16	1.0658	1.0667	0.0009
	17	0.9975	1.0625	0.0650
	18	0.9934	1.0588	0.0654
	19	0.9868	1.0556	0.0688
	20	0.9956	1.0526	0.0571
$\mathbb{G}(2,\mathbb{C}^5)$ Chordal Distance	3	1.7500	1.8000	0.0500
	4	1.6000	1.6000	0.0000
	5	1.5000	1.5000	0.0000
	6	1.4400	1.4400	0.0000
	7	1.4000	1.4000	0.0000
	8	1.3714	1.3714	0.0000
	9	1.3500	1.3500	0.0000
	10	1.3333	1.3333	0.0000
	11	1.3200	1.3200	0.0000
	12	1.3090	1.3091	0.0001
	13	1.3000	1.3000	0.0000
	14	1.2923	1.2923	0.0000
	15	1.2857	1.2857	0.0000
	16	1.2799	1.2800	0.0001
	17	1.2744	1.2750	0.0006
	18	1.2686	1.2706	0.0020
	19	1.2630	1.2667	0.0037
	20	1.2576	1.2632	0.0056
$\mathbb{G}(2,\mathbb{C}^6)$ Chordal Distance	4	1.7778	1.7778	0.0000
	5	1.6667	1.6667	0.0000
	6	1.6000	1.6000	0.0000
	7	1.5556	1.5556	0.0000
	8	1.5238	1.5238	0.0000
	9	1.5000	1.5000	0.0000
	10	1.4815	1.4815	0.0000
	11	1.4667	1.4667	0.0000
	12	1.4545	1.4545	0.0000
	13	1.4444	1.4444	0.0000
	14	1.4359	1.4359	0.0000
	15	1.4286	1.4286	0.0000
	16	1.4221	1.4222	0.0001
	17	1.4166	1.4167	0.0000
	18	1.4118	1.4118	0.0000
	19	1.4074	1.4074	0.0000
	20	1.4034	1.4035	0.0001
	21	1.3999	1.4000	0.0001
	22	1.3968	1.3968	0.0001
	23	1.3923	1.3939	0.0017
	24	1.3886	1.3913	0.0028
	25	1.3862	1.3889	0.0027

Continued...

		Squared Packing Radii		
	N	JAT	Rankin	Difference
$\mathbb{G}(3, \mathbb{C}^6)$ Chordal Distance	3	2.2500	2.2500	0.0000
	4	2.0000	2.0000	0.0000
	5	1.8750	1.8750	0.0000
	6	1.8000	1.8000	0.0000
	7	1.7500	1.7500	0.0000
	8	1.7143	1.7143	0.0000
	9	1.6875	1.6875	0.0000
	10	1.6667	1.6667	0.0000
	11	1.6500	1.6500	0.0000
	12	1.6363	1.6364	0.0001
	13	1.6249	1.6250	0.0001
	14	1.6153	1.6154	0.0000
	15	1.6071	1.6071	0.0000
	16	1.5999	1.6000	0.0001
	17	1.5936	1.5938	0.0001
	18	1.5879	1.5882	0.0003
	19	1.5829	1.5833	0.0004
	20	1.5786	1.5789	0.0004
	21	1.5738	1.5750	0.0012
	22	1.5687	1.5714	0.0028
	23	1.5611	1.5682	0.0070
	24	1.5599	1.5652	0.0053
	25	1.5558	1.5625	0.0067
	26	1.5542	1.5600	0.0058
	27	1.5507	1.5577	0.0070
	28	1.5502	1.5556	0.0054
	29	1.5443	1.5536	0.0092
	30	1.5316	1.5517	0.0201
	31	1.5283	1.5500	0.0217
	32	1.5247	1.5484	0.0237
	33	1.5162	1.5469	0.0307
	34	1.5180	1.5455	0.0274
	35	1.5141	1.5441	0.0300
	36	1.5091	1.5429	0.0338

Table 6. Packing in complex Grassmannians with chordal distance: We compare our calculations (JAT) against the Rankin Bound, equation (A.1). The number of points is denoted by N. The squared packing radius is calculated as the squared chordal distance (4.1) between the closest pair of subspaces. The final column shows how much the computed ensemble declines from the Rankin Bound. When the bound is met, all pairs of subspaces are equidistant.

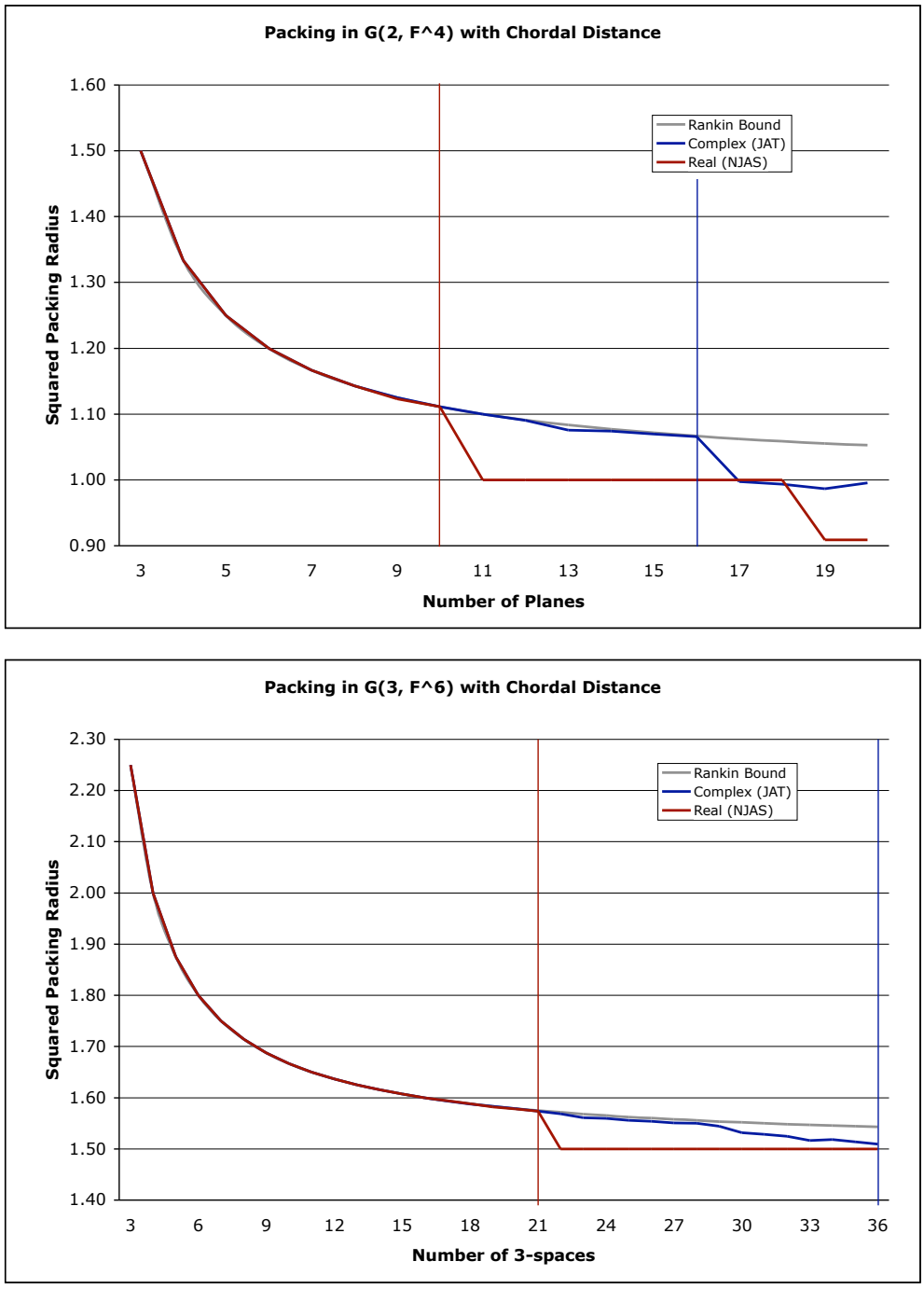

FIGURE 2. PACKING IN GRASSMANNIANS WITH CHORDAL DISTANCE: The red line indicates the best real packings obtained by Sloane and his colleagues [Slo04a]. The blue line indicates the best complex packings produced by the author. Rankin's upper bound (A.1) appears in gray. The vertical red line marks the largest number of real subspaces for which the Rankin Bound is attainable, while the blue line marks the maximum number of complex subspaces for which the Rankin Bound is attainable.

		Squared Packing Radii		
	N	$\mathbb{F} = \mathbb{R}$	$\mathbb{F} = \mathbb{C}$	Rankin
$\mathbb{G}(2, \mathbb{F}^4)$ Spectral Distance	3	0.7500	0.7500	0.7500
	4	0.6667	0.6667	0.6667
	5	0.5000	0.6250	0.6250
	6	0.4286	0.6000	0.6000
	7	0.3122	0.5000	0.5833
	8	0.2851	0.4374	0.5714
	9	0.2544	0.4363	0.5625
	10	0.2606	0.4375	0.5556
$\mathbb{G}(2, \mathbb{F}^5)$ Spectral Distance	3	0.7500	0.7500	0.9000
	4	0.7500	0.7500	0.8000
	5	0.6700	0.7497	0.7500
	6	0.6014	0.6637	0.7200
	7	0.5596	0.6667	0.7000
	8	0.4991	0.6060	0.6857
	9	0.4590	0.5821	0.6750
	10	0.4615	0.5196	0.6667
$\mathbb{G}(2, \mathbb{F}^6)$ Spectral Distance	4	0.8889	0.8889	0.8889
	5	0.7999	0.8333	0.8333
	6	0.8000	0.8000	0.8000
	7	0.7500	0.7778	0.7778
	8	0.7191	0.7597	0.7619
	9	0.6399	0.7500	0.7500
	10	0.6344	0.7407	0.7407
	11	0.6376	0.7333	0.7333
	12	0.6214	0.7273	0.7273
$\mathbb{G}(3, \mathbb{F}^6)$ Spectral Distance	3	0.7500	0.7500	0.7500
	4	0.5000	0.6667	0.6667
	5	0.4618	0.4999	0.6250
	6	0.4238	0.5000	0.6000
	7	0.3590	0.4408	0.5833
	8	—	0.4413	0.5714
	9	—	0.3258	0.5625

Table 7. Packing in Grassmannians with spectral distance: We compare our real ($\mathbb{F} = \mathbb{R}$) and complex ($\mathbb{F} = \mathbb{C}$) packings against the Rankin Bound, equation (A.2). The number of points is denoted by N. The squared packing radius of a configuration is the squared spectral distance (4.2) between the closest pair of subspaces. When the Rankin Bound is met, all pairs of subspaces are equi-isoclinic. The algorithm refused to produce configurations of 8 and 9 subspaces in $\mathbb{G}(3, \mathbb{R}^6)$ with reasonable packing radii.

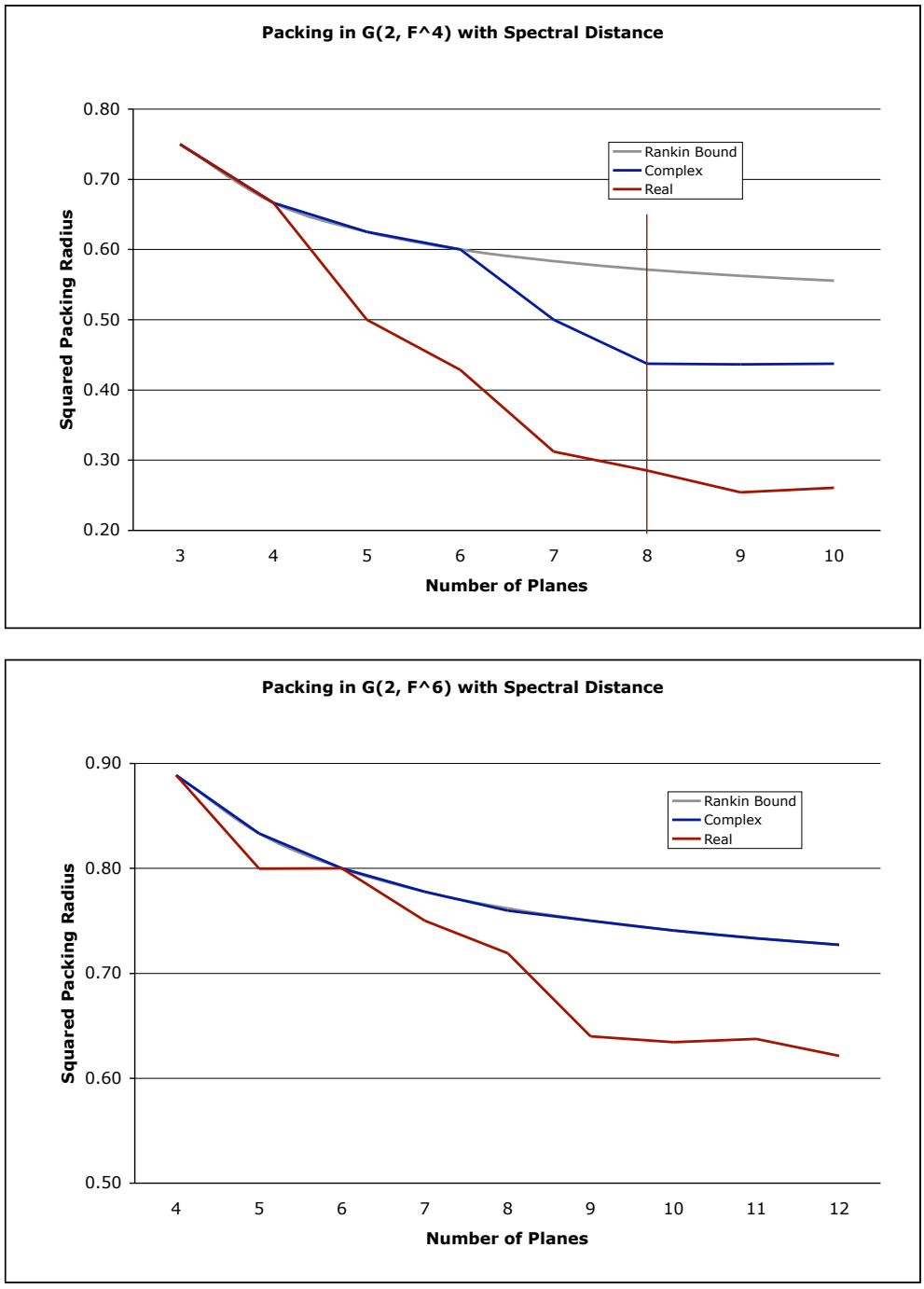

FIGURE 3. PACKING IN GRASSMANNIANS WITH SPECTRAL DISTANCE: The red line indicates the best real packings obtained by the author, while the blue line indicates the best complex packings obtained. The Rankin Bound (A.2) is depicted in gray. The vertical red line marks the largest number of real subspaces for which the Rankin Bound is attainable.

		Packing Radii	
	N	$\mathbb{F} = \mathbb{R}$	$\mathbb{F} = \mathbb{C}$
$\mathbb{G}(2, \mathbb{F}^4)$ Fubini–Study Distance	3	1.0000	1.0000
	4	1.0000	1.0000
	5	1.0000	1.0000
	6	1.0000	1.0000
	7	0.8933	0.8933
	8	0.8447	0.8559
	9	0.8196	0.8325
	10	0.8176	0.8216
	11	0.7818	0.8105
	12	0.7770	0.8033
$\mathbb{G}(2, \mathbb{F}^5)$ Fubini–Study Distance	3	1.0000	1.0000
	4	1.0000	1.0000
	5	1.0000	1.0000
	6	0.9999	1.0000
	7	1.0000	0.9999
	8	1.0000	0.9999
	9	1.0000	1.0000
	10	0.9998	1.0000
	11	0.9359	0.9349
	12	0.9027	0.9022

TABLE 8. PACKING IN GRASSMANNIANS WITH FUBINI–STUDY DISTANCE: Real packings ($\mathbb{F} = \mathbb{R}$) compared with complex packings ($\mathbb{F} = \mathbb{C}$). The number of points is denoted by N. The packing radius of a configuration is the Fubini–Study distance (4.3) between the closest pair of subspaces. We have scaled the distance by $2/\pi$ so that it ranges between zero and one.

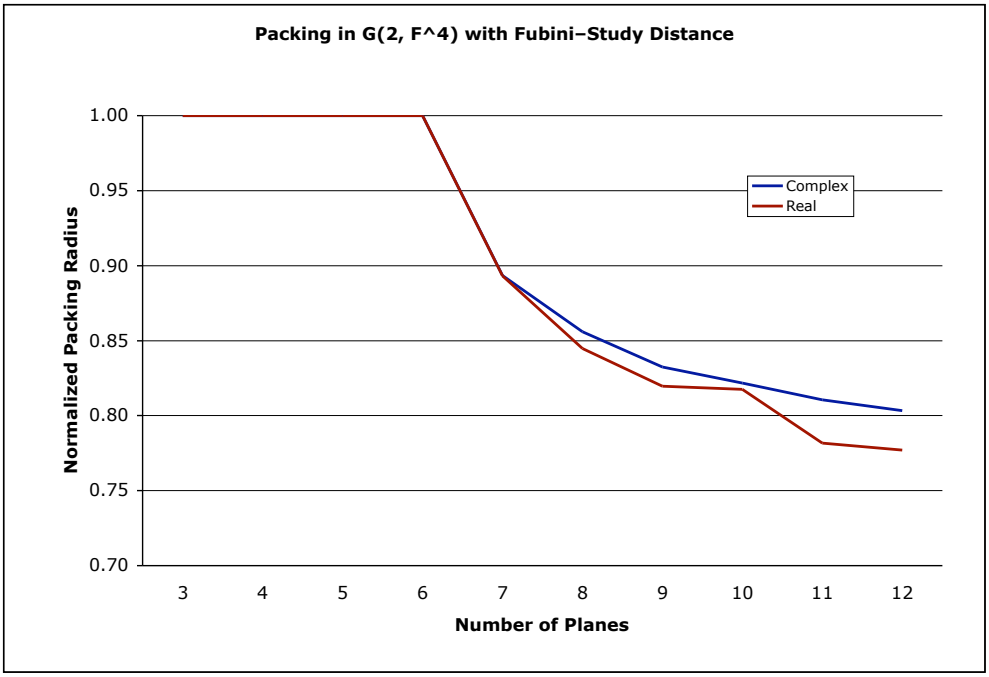

FIGURE 4. PACKING IN GRASSMANNIANS WITH FUBINI–STUDY DISTANCE: The red line indicates the best real packings obtained by the author, while the blue line indicates the best complex packings obtained.

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