Dynamic Contact/Impact Problems, Energy Conservation, and Planetary Gear Trains

A. Bajer and L. Demkowicz

Texas Institute for Computational and Applied Mathematics
The University of Texas at Austin
Austin, TX 78712

Abstract

The paper is a continuation of the work presented in [5], aimed at modeling of a general class of dynamic contact/impact problems for systems of rigid bodies. The contact is assumed to be frictionless, and the main challenge is to model the exchange between the kinetic and elastic energies in the system. This calls for a discretization scheme that conserves linear and angular momentum and, first of all, the total energy at the discrete level. Such a scheme has been presented in [5], and in this work we complement it with a very general treatment of the no-penetration condition for two elastic bodies. The approach is based on the concept of the sign (Rvachev) function, and it provides a natural generalization of the classical point-to-line condition [23]. The work has been motivated with the modeling of planetary gear trains and has resulted in the first (to our best knowledge) 2D Parallel Finite Element simulator for gear sets [2].

1 Introduction

The paper describes a research project which has led to the development of a numerical simulator which can be used to aid the design of planetary gear trains. The developed methodology can also be used for simulation of other mechanical systems which consist of rigid as well as elastic members interacting with each other through contact.

1.1 Motivation

Gears are most widely used means to transmit rotary motion from one shaft to another, and they can be found in almost all of today’s machines. Because they are so common, and they have been used for hundreds of years, we usually do not realize the significant effort necessary to design them. One of the most popular applications of gear sets can be found in vehicle transmission boxes where helical and spur gears are used. Due to the broad use of gear elements in many branches of
industry, constructing a reliable numerical model to simulate behavior of gear sets is desirable and could have a wide range of applications.

According to researchers from the automotive industry [8, 15], undesired vibration and noise generated by many different parts of a vehicle have their source in the transmission box. Namely, the rigid body vibrations caused by imperfections of gears can excite unwanted behavior of other vehicle components. The desire to design vehicles that are quieter and more comfortable, during the whole period of operation, has caused analysts and designers to seek new, more sophisticated models of dynamic behavior of gear systems.

The ultimate goal of our modeling effort is to predict the effect of wear and possible geometrical discrepancies on long time behavior of gear sets, in particular to predict the possibility of inducing unwanted rigid body vibrations of the whole set that would subsequently transmit to the rest of the vehicle. Studies of this type are usually done using simplified, mechanical, “spring-like” models represented by systems of ODE’s with a few degrees-of-freedom.

The investigation of governing equations to describe physical behavior observed in the rotating machinery has so far been mostly verified by direct experiments. It is the ultimate goal of our work to construct a finite element discrete model that would allow for a numerical verification of the various proposed simplified models, help in their derivation and, with the increasing power of contemporary computers, allow for direct simulations of the long time behavior of gear sets.

Simplified models. Existing models of gear sets assume that the gears are rigid except the teeth. The interaction between the gear teeth, so called mesh stiffness, is modeled as a spring. A detailed description of this kind of models can be found in the series of contributions by Kahraman [8, 10, 9, 3, 11], or a recent paper by Oh, Grosh and Barber [15].

In particular, in [9], Kahraman proposes a dynamic model for a whole planetary gear system as illustrated in Fig. 1. The set consists of the central gear called the sun gear \( S \), the carrier \( C \) with an arbitrary number of pinions \( p \) (also called planets) attached to it, and the external gear called the ring gear \( R \). This kind of gear sets are common in automatic transmissions. Multiple pinion sets allow for a load distribution between mating gears, what leads to a larger torque-to-weight ratio and the use of finer, thus quieter, gears. The equations of motion for this gear system are derived by considering sub-systems of meshing gear pairs, as if they were not interacting with each other, and then combining them together. The separation into the sub-systems allows for the use of simplified models considered in [12].

Energy and momentum conserving schemes. It was recognized early that, especially for long time simulations, it is absolutely necessary to construct discretization schemes that reflect the conservation principles of the continuous model, i.e. conserve the total energy, and linear and angular momentum when applicable.
Figure 1: Model of a planetary gear set.

For non-linear elastodynamics, such schemes were constructed in the pioneering works of Simo, Tarnov and Doblareé [19, 20], followed shortly by contribution of Crisfield and Shi [4].

The challenge of extending these schemes to the dynamic impact problems was undertaken by Laursen and Chawla [14] and Armero and Petocz [1]. The latter have designed a penalty approach, in which energy is "stored in a spring" during the contact and returned to the body after the separation.

The work presented in this paper is a follow up to conservative discretization of contact/impact problems for nearly rigid bodies presented in [5]. Contrary to [14], we do not attempt to develop a variational formulation for transient problem. Instead, we use the concept of the method of discretization in time [16], and convert the original transient problem into a series of the problems equivalent to a minimization of a quadratic energy functional with linear inequality constraints. Additionally, the discrete problem reflects the conservation properties of the continuous system: it conserves the total energy, and linear and angular momentum1, where applicable. Although derived in a different way, our scheme reduces effectively to that of Laursen and Chawla [14] and can be viewed as a reinterpretation of their discretization [5].

The main contribution of this paper is a generalization of the discretization scheme presented in [14] to the case of dynamic contact/impact of multiple elastic bodies of an arbitrary shape. The no-penetration condition is handled by means of the sign (Rvachev) function, allowing for considering bodies of arbitrary geometry, and providing a natural generalization of the “point to line” technique [23].

1The angular momentum is conserved “up to the linearization in the rotation angle”
The scope of the paper is as follows. We begin, in Section 2, with a short discussion of the kinematics, equations of motion, and energy conserving time discretization for elastic, nearly rigid bodies. The presentation of the no-penetration condition and the Rvachev function, proceeds in two steps. In Section 3 we discuss first the (simple) case of a single body impinging on a rigid obstacle of an arbitrary shape, and proceed with the discussion of the impact of two elastic bodies in Section 4. Elements of geometry of involute spur gear and the corresponding details of mesh generation are given in Section 5. In Section 6 we give details of a parallel implementation. Numerical simulations are presented in Section 7 and followed with conclusions. Finally, in Appendix, we discuss the equivalence of involved classical and variational formulations for a single time step problem.

2 Modeling of the motion of elastic, nearly rigid bodies.

In presenting the theory, we shall restrict ourselves to a simple case of two elastic bodies as illustrated in Fig. 2.

![Figure 2: Current configurations of two wheels.](image)

Two elastic wheels interact with each other through an elastic contact. The wheels, initially stress free, are subjected to an impulse represented by an initial velocity field corresponding to a rigid body motion. As a result of the impulse, the wheels bounce from each other. The goal is to determine the motion of the wheels.

**Kinematics.** We begin by representing position \( x^I \) of an arbitrary point on body \( I, I = 1, 2; \) identified with its material coordinates \( y^I \), at time \( t \), in the following form:

\[
x^I(y^I, t) = X^I(t) + A(\alpha^I(t))(y^I + u^I(y^I, t))
\]  

(1)
where:

- \( \mathbf{X}^I(t) \) denotes the position of the center of the wheel in the global system of coordinates \((x_1, x_2)\),
- \( \alpha^I(t) \) describes the current rotation of wheel \( I \), and \( A^I(\alpha^I) \) is the transformation (rotation) matrix from the material system of coordinates attached to the center of the wheel and the global system of coordinates \((x_1, x_2)\),

\[
A(\alpha^I) = \begin{pmatrix}
\cos \alpha^I & -\sin \alpha^I \\
\sin \alpha^I & \cos \alpha^I
\end{pmatrix},
\]

with derivative,

\[
B(\alpha^I) = \frac{d}{d\alpha^I} A(\alpha^I) = \begin{pmatrix}
\sin \alpha^I & -\cos \alpha^I \\
\cos \alpha^I & -\sin \alpha^I
\end{pmatrix},
\]

- \( \mathbf{u}^I(y^I, t) \) is the elastic displacement of wheel \( I \) measured in the material system of coordinates.

We will assume that displacement \( \mathbf{u}^I \) and its derivatives \( \nabla \mathbf{u}^I \) are small in the sense that we can neglect higher order terms in \( \mathbf{u} \) and its derivatives.

Differentiating (1) with respect to time, we get the velocity vector,

\[
\mathbf{v}^I(y^I, t) = \frac{d}{dt} \mathbf{x}^I(y^I, t) = \dot{\mathbf{X}}^I + \alpha^I \mathbf{B} \mathbf{y}^I + A(\alpha^I) \dot{\mathbf{u}}^I.
\]

**Equations of motion.** The following equations of motion are to be satisfied:

\[
\rho \frac{d \mathbf{u}^I}{dt} - A(\alpha^I) \text{div}_y \mathbf{\sigma}^{I}(\mathbf{u}^I) = 0.
\]

Here \( \rho \) denotes the mass density, and \( \mathbf{\sigma} \) is the stress tensor,

\[
\sigma_{ij}(\mathbf{u}^I) = 2\mu \varepsilon_{ij}(\mathbf{u}^I) + \lambda \varepsilon_{kk}(\mathbf{u}^I)
\]

\[
\varepsilon_{ij}(\mathbf{u}^I) = \frac{1}{2}(u^I_{i,j} + u^I_{j,i})
\]

where \( \mu \) and \( \lambda \) are Lame’s constants.

In order to obtain a variational form of the equations of motion we multiply equation (5) with a test function:

\[
\delta \mathbf{x}^I = \delta \mathbf{X}^I + B(\alpha^I) \mathbf{y}^I \delta \alpha^I + A(\alpha^I) \delta \mathbf{u}^I.
\]
Next, we integrate over the domain $\Omega^I$, and apply the divergence theorem to the elasticity operator:

$$ - \int_{\Omega^I} \mathbf{A}(\alpha^I) \, \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \delta \mathbf{X}^I \, dy^I = - \int_{\Omega^I} \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \mathbf{A}^T(\alpha^I) \, \delta \mathbf{X}^I \, dy^I \\
= - \int_{\partial \Omega^I} \mathbf{t}(\mathbf{u}^I) \cdot \mathbf{A}^T(\alpha^I) \, \delta \mathbf{X}^I = - \int_{\partial \Omega^I} \mathbf{A}(\alpha^I) \mathbf{t}(\mathbf{u}^I) \cdot \delta \mathbf{X}^I, \quad (8) $$

$$ - \int_{\Omega^I} \mathbf{A}(\alpha^I) \, \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \mathbf{B}(\alpha^I) \mathbf{y}^I \delta \alpha^I \, dy^I \\
= - \int_{\Omega^I} \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \mathbf{C} \mathbf{y}^I \delta \alpha^I \, dy^I = - \int_{\Omega^I} \mathbf{t}(\mathbf{u}^I) \cdot \mathbf{C} \mathbf{y}^I \delta \alpha^I, \quad (9) $$

where

$$ C = \mathbf{A}^T(\alpha^I) \mathbf{B}(\alpha^I) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (10) $$

is independent of $\alpha^I$,

$$ - \int_{\Omega^I} \mathbf{A}(\alpha^I) \, \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \mathbf{A}(\alpha^I) \, \delta \mathbf{u}^I \, dy^I \\
= - \int_{\Omega^I} \text{div}_y \mathbf{\sigma}(\mathbf{u}^I) \cdot \delta \mathbf{u}^I \, dy^I = \int_{\Omega^I} \mathbf{\sigma}(\mathbf{u}^I) \cdot \mathbf{\varepsilon}(\delta \mathbf{u}^I) - \int_{\partial \Omega^I} \mathbf{t}(\mathbf{u}^I) \cdot \delta \mathbf{u}^I, \quad (11) $$

where $\mathbf{t}(\mathbf{u}^I) = \mathbf{t}'$ is the stress vector on boundary $\partial \Omega^I$,

$$ t_i(\mathbf{u}^I) = \sigma_{ij}(\mathbf{u}^I) n_j, \quad (12) $$

with $\mathbf{n} = (n_j)$ denoting the outward normal unit vector to $\partial \Omega^I$ in the material system of coordinates. We arrive at the following weak form of the equations of motion:

$$ \int_{\Omega^I} \rho \frac{d\mathbf{u}^I}{dt} \cdot \delta \mathbf{X}^I - \int_{\partial \Omega^I} \mathbf{A}(\alpha) \mathbf{t}' \cdot \delta \mathbf{X}^I = 0 $$

$$ \int_{\Omega^I} \rho \frac{d\mathbf{u}^I}{dt} \cdot \mathbf{B}(\alpha^I) \mathbf{y}' \delta \alpha^I - \int_{\partial \Omega^I} \mathbf{t}' \cdot \mathbf{C} \mathbf{y}' \delta \alpha^I = 0 \quad (13) $$

$$ \int_{\Omega^I} \rho \frac{d\mathbf{u}^I}{dt} \cdot \mathbf{A}(\alpha^I) \delta \mathbf{u}^I + \int_{\Omega^I} \mathbf{\sigma}(\mathbf{u}^I) : \mathbf{\varepsilon}(\delta \mathbf{u}^I) - \int_{\partial \Omega^I} \mathbf{t}' \cdot \delta \mathbf{u}^I = 0. $$
**Rigid body limit.** In the limiting case of a rigid body, the elastic displacement $\mathbf{u}$ vanishes, and so does the variation $\delta \mathbf{u}$, eliminating thus the third equation in the (13).

\[
\begin{align*}
\left( \int_{\Omega^r} \rho \frac{d\mathbf{v}^f}{dt} \right) - \int_{\partial\Omega^r} A(\alpha^f) \mathbf{t}^f = 0
\end{align*}
\]

\[
\begin{align*}
\left( \int_{\Omega^r} \rho \frac{d\mathbf{v}^f}{dt} B(\alpha^f) \mathbf{y}^f \delta \alpha - \int_{\partial\Omega^r} \mathbf{t}^f \mathbf{C} \mathbf{y} \right) \delta \alpha^f = 0 \quad \text{(14)}
\end{align*}
\]

Note that the boundary terms in the first two equations, however, do not disappear, and correspond (after integration) to a resultant reactive force and moment, representing the interaction of the rigid body component with the rest of the structure. The formulation is also valid in the case when a part of the body is elastic and another part is rigid. Then, the elastic displacement vector $\mathbf{u}^f$ exists only over the elastic part, but velocity vector $\mathbf{v}^f$ is defined over the whole body.

Equations (14) represent the classical linear and angular momentum principles for a rigid body.

In the case of a body containing a rigid body component, displacement vector $\mathbf{u}$ in the neighboring elastic part is set to zero on the rigid/elastic interface, see Fig. 3. We shall assume that in the contact area (outer boundary of the wheels) the body is always elastic.

Equation (4), defining the velocity vector, will be also represented in the variational form,

\[
\int_{\Omega^r} \mathbf{v}^f \delta \mathbf{v}^f = \int_{\Omega^r} \left[ \dot{\mathbf{X}}^f + \dot{\alpha}^f B(\alpha^f) \mathbf{y}^f + A(\alpha^f) \dot{\mathbf{u}}^f \right] \delta \mathbf{v}^f, \forall \delta \mathbf{v}^f. \quad \text{(15)}
\]

![Figure 3: Homogeneous boundary condition for elastic displacement $\mathbf{u}$ on the rigid/elastic interface.](image-url)
Free motion: discretization in time and the conservation principles. In the case when no external forces are present, tractions \( t = 0 \), and the problem is driven by initial conditions:

\[
\begin{align*}
\mathbf{u}(x, 0) &= \mathbf{u}_0(x) \\
\mathbf{v}(x, 0) &= \mathbf{v}_0(x).
\end{align*}
\]

We recall that in this case, both linear and angular momenta and the total energy are conserved. The discretization in time fits a general energy-momentum framework proposed by Simo and Tarnow [20]. Discretized variational equality (15) takes now the following form:

\[
\int_{\Omega'} \frac{v_{t-1}^I + v_t^I}{2} \delta \mathbf{v}^I = - \int_{\Omega'} \left[ \frac{X_{n-1}^I - X_n^I}{\Delta t} + \frac{\alpha_{n-1}^I - \alpha_n^I}{\Delta t} \mathbf{B}(\alpha_{n-1}^I) \mathbf{y}^I + \mathbf{A}(\alpha_{n-1}^I) \left( \frac{\mathbf{u}_{n-1}^I - \mathbf{u}_n^I}{\Delta t} \right) \right] \delta \mathbf{v}^I \quad \forall \delta \mathbf{v}^I,
\]

and the discretized equations of motion are:

\[
\int_{\Omega'} \rho \frac{v_{t-1}^I - v_t^I}{\Delta t} \left[ \delta \mathbf{X}^I + \mathbf{B}(\alpha_{n-1}^I) \mathbf{y}^I \delta \alpha^I + \mathbf{A}(\alpha_{n-1}^I) \delta \mathbf{u}^I \right]
\]

\[
+ \int_{\Omega'} \sigma \left( \frac{\mathbf{u}_{n-1}^I + \mathbf{u}_n^I}{2} \right) : \varepsilon(\delta \mathbf{u}^I)
\]

\[
- \int_{\partial \Omega'} \mathbf{t} \left( \frac{\mathbf{u}_{n-1}^I + \mathbf{u}_n^I}{2} \right) \left[ \delta \mathbf{X}^I + \mathbf{B}(\alpha_{n-1}^I) \mathbf{y}^I \delta \alpha^I + \mathbf{A}(\alpha_{n-1}^I) \delta \mathbf{u}^I \right] = 0.
\]

Similarly, as on the continuous level, the discretization in time guarantees conservation of linear momentum, angular momentum (up to the linearization in rotation angle), and the total energy, all at the discrete level [5].

3 Modeling impact of an elastic body on an arbitrary rigid obstacle

The sign function and the no-penetration conditions. Let us assume first that body 2 (Fig. 2) is rigid \( (\mathbf{u}^2 = 0) \), and fixed in space \( (\mathbf{X}^2, \alpha^2 \text{ constant}) \). We want to construct a constraint for kinematic quantities \( \mathbf{X}^1, \alpha^1, \mathbf{u}^1 \) in terms of an algebraic inequality, expressing the fact that body 1
cannot penetrate body 2. Towards this goal, we begin by introducing a sign function \( F_1 \) such that:

\[
F_1(x^1) = \begin{cases} 
< 0, & x^1 \notin x^2(\Omega^2) \\
= 0, & x^1 \in x^2(\partial\Omega^2) \\
> 0, & x^1 \in x^2(\Omega^2).
\end{cases}
\]  

(19)

With the help of the sign function, the dynamic contact problem for a single elastic body and a rigid obstacle, consists of equations of motion (13) and the contact boundary conditions:

\[
F_1(x^1) \leq 0
\]

\[
F_1(x^1) < 0 \Rightarrow t(y^1) = 0
\]  

(20)

\[
F_1(x^1) = 0 \Rightarrow t(y^1) \cdot n \leq 0 \text{ and } t(y^1) \cdot \tau = 0.
\]

Here \( x^1 \) is again a function of rigid body displacement \( X^1(t) \), rotation \( \alpha^1(t) \) and elastic displacement \( u^1(y^1,t) \),

\[
x^1(y^1,t) = X^1(t) + A(\alpha^1(t))(y^1 + u^1(y^1,t)),
\]  

(21)

\( t(y^1) \) is the stress vector at contact point \( y^1 \), and \( n, \tau \) stand for the normal tangential unit vector to the current configuration or, equivalently speaking, to the obstacle.

**Linearization of the constraint.** In order to preserve the conservation properties of the time-discretization algorithm for the free body motion, the constraints must be linear. The linearization is based on the discretization in time and it involves several steps.

- Sign function \( F_1 \) is linearized around the previous time step solution \( x^1_{n-1} \):

\[
F_1(x^1) \approx F_1(x^1_{n-1}) + \frac{\partial F_1}{\partial x^1}(x^1_{n-1}) \cdot (x^1 - x^1_{n-1}).
\]  

(22)

- The linearized constraint is applied only at points \( y^1 \), where the body has come into contact, or where the penetration has occurred, i.e. for all \( y^1 \in \Gamma^c_n \), where

\[
\Gamma^c_n = \{ y^1 : F_1(x^1_{n-1}(y^1)) \geq 0 \}.
\]  

(23)

- The first term in expression (22) above is dropped. This is equivalent to allowing for some penetration that may have occurred at the previous time step. The amount of penetration can be reduced by selecting a smaller time step, or by an adaptive change of time step, when the penetration occurs.
• Formula (21) is linearized:

\[ x_n^1 = X_n^1 + A(\alpha_{n-1}^1)(y^1 + u_{n-1}^1) + (\alpha_n^1 - \alpha_{n-1}^1)B(\alpha_{n-1}^1)(y^1 + u_{n-1}^1). \]  

(24)

• The last term \( u_{n-1}^1 \) in the equation above is dropped:

\[ x_n^1 = X_n^1 + A(\alpha_{n-1}^1)(y^1 + u_{n-1}^1) + (\alpha_n^1 - \alpha_{n-1}^1)B(\alpha_{n-1}^1)y^1. \]  

(25)

The final form of the linearized constraint becomes:

\[
\delta F_1 (X_n^1, \alpha_n^1, u_n^1) = \frac{\partial F_1}{\partial x^1}(x_{n-1}^1) \cdot [X_n^1 - X_{n-1}^1 + (\alpha_n^1 - \alpha_{n-1}^1)B(\alpha_{n-1}^1)y^1 + A(\alpha_{n-1}^1)(u_n^1 - u_{n-1}^1)] \leq 0. \]  

(26)

The linearization of the constraint is illustrated in Fig. 4. For each point \( y^1 \in \Gamma_n^c \) where the contact or penetration at step \( n - 1 \) has occurred, i.e. \( F_1(x_{n-1}^1(X_{n-1}^1, \alpha_{n-1}^1, u_{n-1}^1(y^1))) \geq 0 \), the linearization process determines a straight line, passing through position \( x_{n-1}^1(y^1) \) at step \( n - 1 \), that represents a local approximation of the obstacle’s boundary. The direction of the line is uniquely determined and varies smoothly with the location of the point. For points that are located on the boundary of the obstacle, the line coincides with the tangent line to the obstacle’s boundary. Notice that the constraining line is not, in general, tangent to the configuration \( x_{n-1}^1(\Omega^1) \).

Figure 4: The linearized constraint at different points \( y^1 \).
Thus, effectively, the sign function specifies normal and tangential direction at all points $y^1 \in \Gamma_{n-1}^c$, even if they do not lie on the boundary of the obstacle. In practice, the constraint above is followed by a discretization in space, which is discussed later, and it is imposed at vertex nodes of a finite element mesh on the wheel boundary only. The construction of a specific function $F_1$ utilizes the concept of Rvachev function, and it is discussed later in this section.

**Formulation of the single time step problem.** Following the standard procedure [5] we reduce the solution of the single time step problem to the following variational inequality:

\[
\begin{aligned}
&\text{Find } u^1_n \text{ and } (X^1_n, \alpha^1_n, u^1_n) \text{ satisfying constraint (26) such that:} \\
&\int_{\Omega^1} \frac{u^1_{n-1} + u^1_n}{2} \delta v^1 = \\
&\int_{\Omega^1} \left[ \frac{X^1_n - X^1_{n-1}}{\Delta t} + \frac{\alpha^1_n - \alpha^1_{n-1}}{\Delta t} B(\alpha^1_{n-1}) y \\
&\quad + A(\alpha^1_{n-1}) \frac{u^1_n - u^1_{n-1}}{\Delta t} \right] \delta v^1 \quad \forall \delta v^1 \\
&\int_{\Omega^1} \frac{u^1_n - u^1_{n-1}}{\Delta t} \left[ (X^1_1 - X^1_n) + (\alpha^1_n - \alpha^1_1) B(\alpha^1_{n-1}) y \\
&\quad + A(\alpha^1_{n-1}) (u^1_n - u^1_1) \right] \\
&\quad + \int_{\Omega^1} \sigma \left( \frac{u^1_{n-1} + u^1_n}{2} \right) : \varepsilon(u^1_n - u^1_1) \geq 0 \\
&\forall (X^1, \alpha^1, u^1) \text{ satisfying constraint (26)}. 
\end{aligned}
\]

Variational inequality (27) is equivalent to the constrained minimization problem of a quadratic functional:

\[
J((X^1, \alpha^1, u^1, v^1)) = \frac{1}{2} b((X^1, \alpha^1, u^1, v^1), (X^1, \alpha^1, u^1, v^1)) - l(X^1, \alpha^1, u^1, v^1), 
\]

with the bilinear form:

\[
b((X^1, \alpha^1, u^1, v^1), (X^1, \alpha^1, u^1, v^1)) = \frac{1}{2} \int_{\Omega^1} \rho \varepsilon(v^1) : \varepsilon(v^1) + 2 \int_{\Omega^1} \rho \frac{X^1 + By^1 + Au^1}{\Delta t} v^1 \\
&\quad + \frac{1}{2} \int_{\Omega^1} \sigma(u^1) : \varepsilon(u^1),
\]

(29)
and the linear from:

\[ l(X^1, \alpha^1, u^1, v^1) = \frac{1}{2} \int_{\Omega^1} \rho v_{n-1}^1 v^1 + \int_{\Omega^1} \rho \frac{X_n^1 + B y^1 + A u_{n-1}^1}{\Delta t} v^1 \\
+ \int_{\Omega^1} \rho \frac{v_{n-1}^1}{\Delta t} [X^1 + B y^1 + A u^1] \\
+ \frac{1}{2} \int_{\Omega^1} \sigma(u_n^1) : \varepsilon(u^1). \]  

(30)

By making two substitutions

\[ (X^1, \alpha^1, u^1) = (X_{n-1}, \alpha_{n-1}, u_{n-1}), \]

and

\[ (X^1, \alpha^1, u^1) = (2X_n^1 - X_{n-1}^1, 2\alpha_n^1 - \alpha_{n-1}^1, 2u_n^1 - u_{n-1}^1), \]

in variational inequality (27), we obtain the identity stating that the total energy at time step \( n \), which a sum of the kinetic and elastic energies, is exactly equal to the total energy at time step \( n - 1 \).

**Rvachev’s function.** According to the standard CAD representations, complex domains are obtained via Boolean operations on a class of sets. Once, for each of the initial sets, a corresponding sign function has been defined, the idea of Rvachev [17, 18] allows for an automatic construction of the sign function for more complicated shapes resulting from Boolean operations on the initial sets.

![Figure 5: The concept of Rvachev function.](image)

For instance, if domains \( D_1 \) and \( D_2 \) (Fig. 5) are intersecting each other, the sign function for the common part \( D = D_1 \cap D_2 \) may be obtained by the formula:

\[ F_1 := (f_1 + f_2 - \sqrt{f_1^2 + f_2^2}) \cdot (f_1^2 + f_2^2). \]  

(31)
**Example: sign function for a tooth domain.** In practice, we shall apply the construction of Rvachev function to the simple case of a 'tooth' domain, resulting from the intersection of two half-planes (Fig. 6).

![Figure 6: Construction of Rvachev function.](Image)

Sign functions $f_1$ and $f_2$ for each of contributing half-planes are constructed in terms of coordinates of nodes $x_1$, $x_2$ and $x_3$.

$$f_1(z) = (z - x_1) \times (x_2 - x_1)$$

$$f_2(z) = (z - x_2) \times (x_3 - x_2).$$

(32)

In the case when nodes $x_1$, $x_2$, $x_3$ belong to a deforming (master) body, the case discussed next, their coordinates will be expressed in terms of the unknown rigid body and elastic displacements:

$$x_i = X_i + A(\alpha)(y_i + u_i)$$

(33)

Here $u_i$ is the elastic displacement of the $i$-th node.
Localization of the constraint. As noticed previously, the tangential and normal directions in linearized constraint (26), are determined by the gradient of sign function $F_1$, evaluated at the penetration (contact) point $x_1$,

$$\frac{\partial F_1}{\partial x_1} (x_{n-1}^1).$$

(34)

If point $x_{n-1}^1$ lies on the boundary of the obstacle, the tangent direction coincides with the tangent to the boundary of the domain. If, however, point $x_{n-1}^1$ has penetrated the obstacle, and it lies in the interior of it, gradient (34) depends on the shape of the entire boundary, and in view of this undesired dependence, it is practical to replace the whole boundary of the obstacle with a portion of it, and consider the sign function for that portion only. More precisely, we assume that the portion of the boundary is piecewise-analytic, and the curve can be uniquely extended to infinity (see Fig. 7). We shall refer to the described procedure as the localization of the sign (Rvachev) function.

In practice, we shall encounter only two different kinds of the localized portions of the obstacle boundary: a segment of a straight line, or a 'tooth' consisting of two segments of straight lines forming a cone (see Fig. 8). In the case of a straight line, our implementation effectively reduces to the standard point to line algorithm [23].

We emphasize that the proposed procedure of handling the no-penetration constraint through localization and linearization of the sign function is quite general, and it may be applied to more complicated geometries, and its higher order finite element approximations.

---

2If the point is close to the portion of the obstacle boundary that has been penetrated, the continuous dependence of the gradient allows to conclude that the gradient is somehow 'close' to the value on the boundary, but we have little control over the difference.
The localization procedure guarantees that the tangential direction in the linearized constraint depends only upon the selected portion of the boundary. In particular, for a single line segment, it coincides simply with the direction of the line, whereas for two segments, it interpolates smoothly between the two directions. Fig. 8 displays contour lines of the sign function for such a case.

4 Dynamic contact/impact problem for two elastic bodies

The quadratic functional to be minimized is obtained by summing up the corresponding contributions of two elastic bodies,

\[
J \left( (X^I, \alpha^I, u^I, v^I), I = 1, 2 \right) =
\sum_{I=1}^{2} \left\{ \frac{1}{2} \int_{\Omega^I} \rho \varepsilon^I + 2 \int_{\Omega^I} \rho \frac{X^I + By^I + Au^I}{\Delta t} v^I + \frac{1}{2} \int_{\Omega^I} \sigma(u^I) : \varepsilon(u^I) \right\} 
- \sum_{I=1}^{2} \left\{ \frac{1}{2} \int_{\Omega^I} \rho \varepsilon^I + \int_{\Omega^I} \rho \frac{X^I_{n-1} + By^I_{n-1} + Au^I_{n-1}}{\Delta t} v^I 
+ \int_{\Omega^I} \rho \frac{v^I_{n-1}}{\Delta t} \left[ X^I + By^I + Au^I \right] + \frac{1}{2} \int_{\Omega^I} \sigma(u^I_{n-1}) : \varepsilon(u^I) \right\}. \quad (35)
\]
The minimizer of the functional, subjected to constraint (40), discussed next, satisfies the following variational inequality:

\[
\left\{ \begin{array}{l}
\text{Find } \bm{v}_n^I \text{ and } (\bm{X}_n^I, \alpha_n^I, \bm{u}_n^I) \text{ satisfying constraint (40) such that:}

\int_{\Omega^I} \frac{\bm{v}_n^{I-1} + \bm{v}_n^I}{2} \delta \bm{v}^I = \\
\int_{\Omega^I} \left[ \frac{\bm{X}_n^I - \bm{X}_n^{I-1}}{\Delta t} \right. + \frac{\alpha_n^I - \alpha_n^{I-1}}{\Delta t} \, \bm{B}(\alpha_n^{I-1}) \, \bm{y} \\
+ A(\alpha_n^{I-1}) \left. \frac{\bm{u}_n - \bm{u}_n^{I-1}}{\Delta t} \right] \delta \bm{v}^I \quad \forall \delta \bm{v}^I, \ I = 1, 2 \\
+ \frac{1}{2} \sum_{I=1}^{2} \left\{ \int_{\Omega^I} \frac{\bm{v}_n^I - \bm{v}_n^{I-1}}{\Delta t} \left[ (\bm{X}_n^I - \bm{X}_n^{I-1}) + (\alpha_n^I - \alpha_n^{I-1}) \, \bm{B}(\alpha_n^{I-1}) \, \bm{y} \right. \\
+ A(\alpha_n^{I-1}) \left( \bm{u}_n^I - \bm{u}_n^{I-1} \right) \\
- \mu \frac{\bm{u}_n^I + \bm{u}_n^{I-1}}{2} \varepsilon \left( \bm{u}_n^I - \bm{u}_n^{I-1} \right) \right\} \geq 0 \\
\forall (\bm{X}_n^I, \alpha_n^I, \bm{u}_n^I), \ I = 1, 2 \text{ satisfying constraint (40)}. 
\end{array} \right.
\]  

(36)

A crucial part of the formulation lies in the implementation of the no-penetration condition. The strategy of applying the linearized constraint is similar to that for a rigid obstacle. We first identify points on the boundary of the slave body 1 that have come into contact or penetrated master body 2 at time step n − 1,

\[
\Gamma_{c,n}^1 = \left\{ \bm{y}^1 : F_1(\bm{x}_{n-1}^1(\bm{y}^1); \bm{X}_{n-1}^2, \alpha_{n-1}^2, \bm{u}_{n-1}^2) = 0 \right\}. 
\]  

(37)

For each point \( \bm{y}^1 \in \Gamma_{c,n}^1 \), we impose now a linearized form of the no-penetration constraint. We begin with the localization of the obstacle first. For each \( \bm{y}^1 \in \Gamma_{c,n}^1 \) we identify a corresponding portion of boundary \( \partial \Omega_{n-1}^2 \) that has been penetrated, and consider the corresponding, localized sign function \( F_1 \). Due to the localization, \( F_1 \) depends now on \( \bm{X}_i^2, \alpha_i^2 \) and elastic displacement \( \bm{u}_i^2 \) on the localized part of the boundary only; \( i = n - 1, n \). The linearized form of the constraint looks as follows:

\[
\delta F_1(\delta \bm{x}^1, \delta \bm{x}^2) = \frac{\partial F_1}{\partial \bm{x}^1} \cdot \delta \bm{x}^1 + \left( \frac{\partial F_1}{\partial \bm{x}^2}, \delta \bm{x}^2 \right) \leq 0, 
\]  

(38)

where

\[
\delta \bm{x}^I = \bm{X}_n^I - \bm{X}_{n-1}^I + \bm{B}(\alpha_n^{I-1}) \bm{y}^I(\alpha_n^I - \alpha_{n-1}^I) + A(\alpha_n^{I-1})(\bm{u}_n^I - \bm{u}_n^{I-1}), \ I = 1, 2 
\]  

(39)
with derivatives $\frac{\partial F_1}{\partial x_1}$ and $\frac{\partial F_1}{\partial x_2}$ evaluated at $(n-1)$-th time. Note that $\delta F_1$ is a function of $x_1$, but a functional of $x_2$. It is only the discretization of master body $\Omega_2$ that turns functional inequality (38) into a simple linear, algebraic inequality.

For the case of a two-segment portion of the boundary and linear finite elements, functional $F_1$ turns into a function of $X^2$, $\alpha^2$ and elastic displacements $u^2$ of nodes $A, B, C$ (Fig. 8). The final form of the linearized constraint applied at the discrete level takes the form:

$$
\delta F_1(\delta x^1; \delta x^2) = \frac{\partial F_1}{\partial x^1} \cdot \delta x^1 + \frac{\partial F_1}{\partial x_A^2} \cdot \delta x_A^2 + \frac{\partial F_1}{\partial x_B^2} \cdot \delta x_B^2 + \frac{\partial F_1}{\partial x_C^2} \cdot \delta x_C^2 \leq 0,
$$

(40)

with partial derivatives $\frac{\partial F_1}{\partial x_i}$ evaluated at level $n-1$.

Variational inequality (36) and constraint (38) are equivalent to a boundary-value problem involving a non-local contact condition. We delegate to the Appendix for a detailed discussion of this delicate issue.

**Discretization in space and the final algorithm.** We use linear continuous elements to approximate displacement field $u$, and linear discontinuous elements to approximate velocity field $v$. This is in accordance with the expected regularity of the solution: due to the elastic shock waves, both the velocity and the elastic stresses are discontinuous in time $t$ and material coordinate $y$. By building the rigid body motion directly into the kinematics, and using linear elements for the velocity field, we can reproduce the rigid body motion of the gears.

After the discretization in space, variational inequality (36) is solved by minimizing quadratic functional (35) with linear, inequality constraints (40). Additional linear constraints can also be imposed on the rigid body degrees of freedom ($X^I, \alpha^I$). This happens in the case of our final example of the planetary gear system, where the motion of planets is restricted by the rigid carrier. Because of the high computational cost of the minimization problem, solved at every time step, we perform static condensation of the unknowns, for which no constraints are imposed. First of all, the velocity degrees of freedom are eliminated on the element level. Next, all displacement degrees of freedom, on which no constrains are imposed, are eliminated. The final minimization problem involves only the rigid body motion degrees of freedom and elastic displacements of nodes in possible contact areas.

5  Elements of geometry of planetary gear sets and mesh generation

**A pair of meshing gears.** In the geometry of two meshing gears their pitch circles need to be tangent to each other, and the tangent point is called the pitch point, illustrated in Fig. 9. Thus their
velocities, if the centers of gears are fixed, must be equal,

\[
  r_1 \omega_1 = r_2 \omega_2. \tag{41}
\]

Here \(r_1\) and \(r_2\) are pitch radii, and \(\omega_1\) and \(\omega_2\) are angular velocities of the meshing gears. From equation (41) the following equalities can be derived,

\[
  \left| \frac{n_1}{n_2} \right| = \left| \frac{\omega_1}{\omega_2} \right| = \frac{N_2}{N_1}, \tag{42}
\]

where \(n_i\) is a rotation speed (rpm), \(N_i\) is the number of teeth.

![Nomenclature of meshing gears.](image)

The line which crosses point P under the angle \(\phi\) to the line tangent to the pitch circle, is called the pressure line or the line of action. The pressure angle \(\phi\) is standardized and, in the temporary gears, has values of 20 or 25 degrees. If the gears are meshing with each other, they must have the same pressure angle. The circle tangent to the pressure line is called the base circle. Since the radius of the base circle \(r_b\) is perpendicular to the line of action at the tangent point, then

\[
r_b = r \cos(\phi)
\]

where \(r\) is a radius of the pitch circle.

To minimize vibrations, we want the force which acts on the gear teeth to be constant; varying
neither in magnitude nor in direction. This requirement is met by the involute tooth form. The involute is constructed on the base circle (see Fig. 10). Consider a base circle with the radius \( r_b \), on which a string is wrapped up. The center of the circle is at point \( O \) and passes through points \( A \) and \( B \). When we start unwrapping the string, a free end which was originally at point \( A \) marks a curve. This curve defines the involute. The length of the segment of straight line \( BC \) is equal to the length of arch \( AB \), because it is the part of the string which was unwrapped from the part of the circle between points \( A \) and \( B \).

\[ |AB| = r_b(\theta + \psi) \]

Also the segment \( BC \) is perpendicular to the radius of the base circle \( r_b \) and point \( B \), so

\[ \rho^2 = r_b^2 + r_b^2(\theta + \psi)^2, \]

thus,

\[ \theta + \psi = \sqrt{\rho^2/r_b^2 - 1}. \]

Taking into account that \( CBO \) is a right triangle, we have:

\[ \tan \psi = \theta + \psi. \]

Combining two equations above,

\[ \psi = \tan^{-1}(\sqrt{\rho^2/r_b^2 - 1}). \quad (43) \]

Next we can express the angle \( \theta \) in terms of \( \psi \), which depends on radius \( \rho \):

\[ \theta = \tan \psi - \psi. \quad (44) \]

For a more detailed description of generating the involute curve we refer to [22, 21]. The involute tooth form, which is the most commonly used in spur gears, allows for the driven gear to rotate at a constant angular velocity, provided the angular velocity of the driving gear is constant. The constant direction of the contact force is determined by the pressure line.

**Geometric modeling.** The finite element mesh generated for the planetary gear train is shown in Fig. 11. The geometry of the set was defined within a Geometrical Modeling Package (GMP) [6]. The package allows for arbitrary geometrical entities (curves, figures) to be defined by a user. Using this feature, the involute curve was added to the package. The parameterization of the involute curve takes the following steps:

- Given \( \xi \in [0, 1] \) determine \( \rho = (1 - \xi)r_b + \xi r_a \), where \( r_a \) is a radius of the addendum circle,
\[ \psi = \tan^{-1}\left(\sqrt{\frac{r_b^2}{r_b^2 - 1}}\right), \]
\[ \theta = \tan \psi - \psi, \]
\[ \rho \text{ and } \theta \text{ determine the position of the point on the involute curve, expressed in the polar coordinate system.} \]

Figure 10: Construction of an involute curve.

Please note, that the tooth profiles in the ring are concave instead of convex, as for the other gears in the system. After some modifications, the algorithm described above can be use to parameterize the concave tooth profiles too. Thus the geometry of the gear train is defined exactly. An approximation of the geometry takes place during the mesh generation. The developed simulator was based on 2Dhp90 - a general \( hp \)-adaptive code, which uses an algebraic mesh generator. This generator maintains an interface with GMP, which allows for adding nodes (during the adaptation process) on the exact geometry.

In our example the module \( m = 2.5 \text{mm}, \) which is the ratio of the pitch circle diameter to the number of teeth. The module is common for all the gears in the system. We have assumed the following numbers of teeth:

\[ N_S = 30, \ N_P = 20, \ N_R = 70 \]
which has enforced the radii:

\[ r_S = 75\text{mm}, \quad r_P = 50\text{mm}, \quad r_R = 175\text{mm}, \quad r_C = 125\text{mm}. \]

6 Parallel implementation of the algorithm

A contact search algorithm. One of the most challenging difficulties, which we had to overcome in order to develop the parallel simulator, was the design of an algorithm which determines finite element nodes which have come into contact. The problem is caused by the fact that the contact occurs between two gears which have been assigned to different processors, and the data structures for these gears are on the different processors, too. Also the large rigid body motion of the system makes this task more complicated. Our goal was to design a reliable algorithm which requires a minimum amount of communication between the processors. The general flow of the algorithm which determines possible contact nodes on the teeth of any two meshing gears can be described as follows:

- Using MPI library exchange between the processors the global positions of the gears, i.e. center coordinates \( X^I \), and rotation angles \( \alpha^I \).
- Knowing the center coordinates of the two meshing gears \( X^I \), determine the coordinates of the pitch point, which is the theoretical tangent point of the pitch circles of the two mating gears.
- Based on the position of the pitch point and the global rotation angle \( \alpha^I \), find the teeth which are closest to the pitch point. These teeth are in contact between each other.
- Get the equation of the line of action which passes through the pitch point. The angle between the line of action and the line tangent to the pitch circle is equal to the pressure angle (see Fig. 9).
- For each of the gears, knowing the direction of the gear rotation, determine the side of the tooth which is in contact, and find a node on that side which is closest to the line of action (see Fig. 12). This node is a possible contact node, because the theoretical contact node slides along the line of action.
- Also determine neighbors of the possible contact node. If the tooth is treated as a master body (obstacle), the neighboring nodes are used to define the no-penetration region.
Figure 11: Finite element mesh of a planetary gear system.

The algorithm described above allows for selecting possible contact nodes with a minimum amount of communication between processors - only the rigid body parameters for the gears - $X^I$, $\alpha^I$ need to be exchanged. This makes the process fast, and minimizes the risk of a possible bottle
neck, when the code is generalized to run on a larger number of processors.

**The final, one-step algorithm.** As it was mentioned at the beginning of this section, every gear is assigned to one processor, and the minimization problem is solved on a separate node. A general flow of the parallel algorithm is presented in Fig. 13. At every time step, we start with determining the possible contact nodes using the algorithm described above. Next, using a frontal solver, we eliminate all degrees of freedom except the ones which are associated with the possible contact nodes and the rigid body motion ($X^l$, $\alpha^l$). Because of the discontinuous approximation for the velocity field, the elimination of those degrees of freedom can be done on the element level. At the end of this process we end up with a condensed system containing a small fraction of the total number of degrees of freedom corresponding to the original model. For example, if the original problem has about 15 thousand unknowns (on one processor), it is reduced to a system with less than 30 unknowns. The corresponding small matrices are sent then from every gear to a designated processor where the minimization takes place. But first, the global matrix must be assembled, and the possible contribution from the rigid members (carrier) are added. Now, when the data from all
Figure 13: Flow chart of the parallel implementation of the one time step algorithm.

gears is gathered at the common processor, we are able to determine for which nodes, from the
list of possible contact nodes, the no-penetration conditions are to be imposed and construct these constraints. Next, the minimization problem is (sequentially) solved using Goldfarb-Idnani method [7] implemented by Spellucci (spellucci@mathematik.th-darmstadt.de). This method finds a minimum of a positive-definite quadratic functional with linear equality and inequality constraints. Then, the solution of the minimization problem is sent back to the appropriate processors, where the backward substitution is performed. During the minimization process, which is done on the designated processor, all other nodes are idling. Because the minimization problem is small, its solution takes about 1000 times less time than the condensation of degrees of freedom on the gear processors. Thus the fact that the minimization problem is solved sequentially should not affect a possible speed up, which can be achieved by assigning each gear to more than one processor. If the minimization problem becomes an issue anyway, the most expensive part of the minimization algorithm - Cholesky decomposition, can be done in parallel to reduce the idling time of the gear processors.

The algorithm was implemented on Cray T3E using MPI, but machine dependent features were not used, so it can be run on other parallel machines which support MPI.

7 Numerical simulations

7.1 Two gear system: initial velocity problem

The first gear train we have simulated consists of two gears (Fig. 14). The gears have 30 and 20 teeth respectively. Their common module is 2.5mm, thus their radii are 375 and 250mm. Both gears are made of steel; Young modulus $E = 2$ GPa, and Poisson ratio $\nu = 0.3$, and mass density $\rho = 7830$ kg/m$^3$. The gears are supported at their centers; they can rotate freely, but can move neither in the horizontal nor in the vertical direction. This support was modeled by imposing equality constrains on $X^1$, $X^2$ - the positions of the gear centers. In this problem we impose an initial angular velocity $\omega = 5750$ rpm on the left gear, while the right gear is at rest. Both gears are initially stress free. Such high velocity impacts are not encountered in practice, usually the gears are accelerated gradually, but we use this example to test the conservative properties of the integration scheme. For this problem (on the continuous level) the total energy is conserved, but the angular momentum is not.

Fig. 15 shows the described gears with a generated finite element mesh used in this numerical experiment. All elements are linear and, as in the previous examples, we use the linear discontinuous elements for the velocity field. Figures 16 and 17 present selected consecutive still frames of the motion of the gears. When a tooth of one of the gears impacts on a tooth of the second gear, an elastic wave starts propagating throughout the bodies. In the figures, the propagating wave is represented by von Mises stress distribution. The full animation for this problem can be found on
the web page: http://www.ticam.utexas.edu/~jedrek.

In Fig. 18 the total, potential and kinetic energy are presented as a function of time. Initially all energy is stored in the form of kinetic energy, and as a result of impacts, a part of kinetic energy is transformed into the potential energy. Although both potential (elastic) and kinetic energy oscillate throughout the process, the sum of them is absolutely constant - almost up to the number of significant digits. This proves that the energy conserving time integration scheme has been successfully implemented. Figure 19 presents the angular velocities for both gears. It can be seen that velocities oscillate around certain average values.

7.2 Two gear system: moment induced problem

In this numerical experiment we consider the same set of gears as before, but this time we impose stationary initial conditions at $t = 0$, and the gears are driven by accelerating moment $M_1$ imposed on the left gear (Fig. 20). The moment grows linearly from zero until it achieves a certain value, and then it remains constant (45). On the right gear a braking moment, proportional to the angular velocity, is imposed (46). As before, the bodies are stress free at the beginning.
The same finite element mesh (Fig. 15) and approximation, as in the previous problem, was used. Fig. 21 presents consecutive still frames of the motion of the gears. Because the gears are accelerated at a much slower rate, the induced stresses are much lower than in the previous case. As it can be seen in Fig. 22, a marginal amount of energy is transformed into the potential energy. In contrast to the previous example, the separation between teeth after a contact almost does not
Figure 16: Two gear system: propagation of elastic waves induced by the impact and represented by von Mises stress distribution.
Figure 17: Two gear system: the propagation of elastic waves induced by the impact and represented by von Mises stress distribution.
Figure 18: Two gear system, initial velocity problem: energy.

Figure 19: Two gear system, initial velocity problem: frequency.

occur. Fig. 23 shows angular velocities of both gears. Again, the full animation for this problem can be found on the web page: http://www.ticam.utexas.edu/~jedrek.

To analyze the convergence of the proposed method, we solved the moment induced problem with the same data, but on a twice finer mesh. Figures 24 and 25 compare kinetic and potential energies, as well as angular velocities, for both meshes. It can be noticed that, by refining the mesh,
we were able to reduce oscillations significantly.

7.3 The planetary gear train

Finally, we performed numerical simulations of the full planetary gear system shown in Fig. 1. The train consists of a sun, four planets, a ring and a carrier. The dimensions of this gear train were described in Chapter 5. The gears are made with the same material as in the previous two cases. The system is driven by an accelerating moment imposed on the sun. A braking moment is applied to the carrier.

As it was mentioned before the carrier is a rigid member and its equations of motion can be derived from equations (27):

$$MC^n = \Delta t M \left( V_{C}^{n-1} + \frac{1}{\Delta t} \dot{C}^{n-1} \right), \quad (47)$$

where

- $C^n = (X^n_C, \alpha_C^n)$ - is the position of the carrier center and the rotation angle at time step $n$,
- $M$ - is mass matrix

$$M = \begin{pmatrix} M_C & 0 & 0 \\ 0 & M_C & 0 \\ 0 & 0 & I_C \end{pmatrix}, \quad (48)$$
with $M_C, I_C$ denoting mass of the carrier and the moment of inertia, respectively.
The planets are attached to the carrier in such a way that they are allowed to rotate freely, but they can move neither in the horizontal nor vertical direction with respect to the carrier. This kind of joint is modeled by enforcing the following equality constraint:

$$ X_i = X_C + R_C \left( \cos(\alpha_i + \alpha_C), \sin(\alpha_i + \alpha_C) \right), $$

(49)
Figure 24: Two gear system, moment induced problem: comparison of the kinetic and potential energies obtained on the coarse (black) and finer (blue) mesh.

![Energy Plot](image)

Figure 25: Two gear system, moment induced problem: comparison of angular velocities obtained on both meshes.

![Velocity Plot](image)

where

- $X_i$ - is the position of the center of the $i$-th planet,
• $R_C$ - is the carrier radius,

• $\alpha_{Bi}$ - is the initial position of the $i$-th planet on the carrier.

Fig. 26 presents selected consecutive still frames of the motion of the gear train. Again, the full animation for this problem can be found on the web page: http://www.ticam.utexas.edu/~jedrek.

Fig. 27 presents the kinetic and potential energy of the planetary gear train as a function of time. In 28 we can see angular velocities of the sun and the carrier. For the perfect, rigid gears we can determine the ratio of these angular velocities. For this system the ratio is equal $\frac{10}{3}$. In Fig. 29 we can see the ratio of computed angular velocities of the sun and the carrier. As one could expect, it oscillates around the value for the rigid system.

8 Conclusions

We have presented a methodology for simulating gear trains. The dynamic behavior of gears is modeled as a general dynamic/contact impact problem for multibody systems consisting of elastic/nearly rigid bodies and rigid components, interacting with each other through contact, and possibly joint connections. The numerical results illustrate the feasibility of the approach. We have considered only 2D models but the entire approach can be generalized to 3D kinematics and geometries.

At minimum, the gear simulator can be used to calibrate the simplified models, replacing the costly experimental verification. We hope that, with increasing capabilities of computing, the simulations can become a practical tool for designing gear sets, and will set a foundation for studying the effects of wear and possible geometrical discrepancies on unwanted vibrations, and overall performance of transmission boxes.

References


Figure 26: Planetary gear system, moment induced problem: gear configuration at consecutive time steps.
Figure 27: Planetary gear system: kinetic and potential energy.

Figure 28: Planetary gear system: angular velocities of the sun and the carrier.


Figure 29: Planetary gear system: the ratio of angular velocities of the sun and the carrier.


Appendix

Classical formulation for the one step impact problem for two elastic bodies

We examine now in detail boundary conditions that correspond to constraint (40) and variational formulation (36). We shall follow the standard strategy [13] to recover first equations of motion, next traction boundary conditions, and finally, the contact condition.

**Step 1.** We rewrite the elastic energy term in the form:

\[
\int_{\Omega'} \sigma \left( \frac{u_{n-1}^I + u_n^I}{2} \right) : \varepsilon \left( u' - u_n^I \right) = \\
\int_{\Omega'} \left[ A^T (\alpha_{n-1}^I) \sigma \left( \frac{u_{n-1}^I + u_n^I}{2} \right) A(\alpha_{n-1}^I) \right] : \varepsilon \left( \delta x^I \right) = \\
- \int_{\Omega'} \left[ A(\alpha_{n-1}^I) \text{div} y^I \sigma \left( \frac{u_{n-1}^I + u_n^I}{2} \right) \right] \cdot \delta x^I \\
+ \int_{\partial \Omega'} \left[ A(\alpha_{n-1}^I) \delta \left( \frac{u_{n-1}^I + u_n^I}{2} \right) \right] \cdot \delta x^I, \quad (50)
\]

where

\[
\delta x^I = X^I - X_{n-1}^I + B (\alpha_{n-1}^I) y^I (\alpha^I - \alpha_{n-1}^I) + u' - u_{n-1}^I, \quad I = 1, 2. \quad (51)
\]

**Step 2.** Select \( u^I \) on boundary \( \partial \Omega^I \) in such a way that the virtual displacement vector vanishes on boundary \( \partial \Omega^I \), but takes arbitrary values inside of \( \Omega^I \). Trading the virtual displacement vector for its negative, we use the Fourier lemma to ‘recover’ equations of motion:

\[
\frac{v_n^I - v_{n-1}^I}{\Delta t} - A(\alpha_{n-1}^I) \text{div} y^I \sigma \left( \frac{u_{n-1}^I + u_n^I}{2} \right) = 0 \quad \text{in} \quad \Omega^I, \quad I = 1, 2. \quad (52)
\]
Step 3. Once we know that the equations of motion are satisfied, we integrate the elastic energy term by parts to conclude that the following variational inequality must be satisfied by the stress vectors on boundaries $\partial \Omega^I$,

$$\sum_{I=1}^{2} \int_{\partial \Omega^I} \left[ A(\alpha_{n-1}) t \left( \frac{u_{n-1}^I + u_n^I}{2} \right) \right] \cdot \delta x^I \geq 0,$$

$$\forall (X^I, \alpha^I, u^I), \ I = 1, 2 \text{ satisfying constraint (40).} \ (53)$$

Step 4. Recall that constraint (40) is imposed on $\Gamma_{c,n}^1$ and values of the virtual displacement of the slave body are arbitrary on remaining part of the boundary $\Gamma_{t,n}^1$. Index $n$ emphasizes that parts $\Gamma_{c,n}^1$ and $\Gamma_{t,n}^1$ of the boundary change at every time step. Consequently, we have recovered the traction free boundary condition on $\Gamma_{t,n}^1$,

$$t \left( \frac{u_{n-1}^1 + u_n^1}{2} \right) = 0 , \ (54)$$

and the first integral in (53) reduces to part $\Gamma_{c,n}^1$ of the boundary only.

Step 5. Partial derivative $\frac{\partial F_1}{\partial x^1}$ in the linearized constraint (40) is determining tangential and normal directions at point $y^1 \in \Gamma_{c,n}^1$. Partial derivative $\frac{\partial F_1}{\partial x^2}$ is a linear functional acting on the virtual displacement of the master body, and its particular form depends upon the construction of a concrete sign function. Using an integral representation theorem for the derivative $\frac{\partial F_1}{\partial x^1}$, we can rewrite constraint (40) in the following form:

$$\delta x^1 \cdot n^1(y^1) + \int_{\Gamma_2(y^1)} K(y^1, y^2) \cdot \delta x^2(y^2) \ dy^2 \leq 0. \ (55)$$

Here $K(y^1, y^2)$ represents the interaction between the virtual normal displacement of the first body at $y^1$, and the virtual displacement vector of the second body at $y^2$. The integration extends only over a part $\Gamma_2(y^1)$ of boundary $\partial \Omega^2$ that has participated in the definition of the localized sign function.

Step 6. For all points on part $\Gamma_{c,n}^1$ where the constraint has not been active, the corresponding virtual displacement can take arbitrary values on that part of the boundary and, consequently, the stress vector must vanish,

$$\delta F_1(y^1) < 0 \Rightarrow t \left( \frac{u_{n-1}^1 + u_n^1}{2} \right) = 0 , \ (56)$$

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**Step 7.** For points \( y^1 \) where the constraint has been active, we can relate the normal component of the virtual displacement at \( y^1 \) with the virtual displacement on \( \Gamma_c^2(y^1) \),

\[
\delta x^1 \cdot n^1(y^1) = - \int_{\Gamma_c^2(y^1)} K(y^1, y^2) \cdot \delta x^2(y^2) \, dy^2.
\]  

Consequently, the tangential stress at \( y^1 \) vanishes (no restriction on the tangential component of the stress vector), and we end up with the following form of variational inequality (53):

\[
- \int_{\delta \Gamma_1(y^1) = 0} \left[ A(\alpha_{n-1}^1) t\left(\frac{u_{n-1}^1 + u_n^1}{2}\right) \right] \cdot n^1(y^1) \int_{\Gamma_c^2(y^1)} K(y^1, y^2) \cdot \delta x^2(y^2) \, dy^2
\]
\[+ \int_{\partial \Omega^2} \left[ A(\alpha_{n-1}^1) t\left(\frac{u_{n-1}^2 + u_n^2}{2}\right) \right] \cdot \delta x^2(y^2) \, dy^2 \geq 0. \quad (58)
\]

Rearranging the order of integration in the first part, we get:

\[
\int_{\Gamma_1^2} \left[ A(\alpha_{n-1}^1) t\left(\frac{u_{n-1}^2 + u_n^2}{2}\right) \right] \cdot \delta x^2(y^2) dy^2 + \int_{\Gamma_1^1} A(\alpha_{n-1}^1) t\left(\frac{u_{n-1}^1 + u_n^1}{2}\right) \cdot \delta x^2(y^2) dy^2 \geq 0 \quad (59)
\]

Here

\[
\Gamma_1^2 = \bigcup_{y^1 : \delta \Gamma_1(y^1) = 0} \Gamma_c^2(y^1) \quad \text{and} \quad \Gamma_1^2 = \partial \Omega^2 - \Gamma_1^2. \quad (60)
\]

Consequently, due to the arbitrariness of virtual displacement \( \delta x^2 \), the stress vector vanishes on \( \Gamma_1^2 \). On \( \Gamma_c^2 \) we get a **nonlocal compatibility condition** for the normal stress:

\[
A(\alpha_{n-1}^2) t\left(\frac{u_{n-1}^2 + u_n^2}{2}\right) = \int_{\Gamma_c^1(y^1)} \left( A(\alpha_{n-1}^1) t\left(\frac{u_{n-1}^1 + u_n^1}{2}\right) \right) \cdot n^1(y^1) K(y^1, y^2) \, dy^1 \quad (61)
\]

where

\[
\Gamma_c^1(y^2) = \{ y^1 : y^2 \in \Gamma_c^2(y^1) \} \quad (62)
\]

is the collection of all points \( y^1 \) that interact through the constraint with point \( y^2 \).
**Step 8.** We set $\delta x^2 = 0$ in inequality (58) and use the fact that $\delta x^1 \cdot n \leq 0$ to conclude that:

$$
\left[ A\left( a_{n-1}^1 \right) t \left( \frac{u_{n-1}^1 + u_n^1}{2} \right) \right] \cdot n^1 \leq 0.
$$

**Step 9.** Boundary condition (61) specifies the stress vector at $y^2$ as a function of normal stress on $\Gamma^1_c$. This is a vector identity. If we select $n^2$ to be a unit vector of the vectors on both sides, we can argue that (61) relates the normal stress at $\ell^2$ (corresponding to this specific direction) to the normal stresses on $\Gamma^1_c$. The corresponding tangential stress vanishes. The sign of this normal stress will depend upon the behavior of coupling term $K (y^1, y^2)$.

By exactly reversing the argument, we can show that all the recovered boundary conditions and the equilibrium equations imply the satisfaction of variational inequality (36). The bottom line of these lengthy, but classical considerations is that the linearized constraint involves displacement of the slave body at $y^1$ and displacements of the master body at all points $y^2 \in \Gamma^1_c (y^1)$ that are involved in the construction of the localized sign function for the obstacle. As a consequence, the corresponding compatibility condition for the normal stress is non-local, and it may violate the condition that the normal stresses on the master body must be negative.

**Discretization.** It is illuminating to see how the non-local compatibility condition for stresses (61) is realized at the discrete level. As mentioned before, after discretizing both bodies with linear elements, in practice we encounter only two situations: point to line, and point to wedge (two lines) contact. We shall illustrate now the corresponding relation for the discrete nodal contact forces for the point to line case.

In the situation illustrated in Fig. 30, the sign function for the half-plane determined by points $A$ and $B$ is given by the formula:

$$
F = (x_B - x_A) \times (x - x_A).
$$

Point $x$ represents a node on the slave body that has penetrated the master body through line $AB$. The no penetration, point to line, condition imposed on node $x$ is expressed in terms of position of the penetrating point $x$, and positions of points defining the discretize obstacle - $A$ and $B$, (comp. equation 40). By enforcing that constraint, discrete nodal forces occur at $x, A, B$. These forces must satisfy the following equilibrium equation:

$$
F_x + F_A + F_B = 0
$$

It follows from the construction of Rvachev function (64) that force $F_x$ is perpendicular to line $AB$. In other words, even though point $x$ has penetrated line $AB$ by some finite depth, it is line
Figure 30: Discretization of the constraint: point to line case.

\(AB\) that defines tangential and normal direction at point \(x\). Contrary to the nodal contact force \(F_x\), forces \(F_A\) and \(F_B\) representing the response (reaction) of the obstacle (master body) on the penetrating node \(x\) are no longer perpendicular to \(AB\), unless point \(x\) is located exactly on line \(AB\).

Additionally, if a neighbor node of \(x\) on the slave body (not shown in the Fig. 30) penetrates the master body as well, these may create an additional contribution to the nodal forces \(F_A, F_B\).

Whereas it is difficult to specify the direction of forces \(F_A, F_B\) a-priori, we should notice that with decreasing time step, the penetration will decrease as well, and point \(x\) will converge to line \(AB\). Forces \(F_A, F_B\) become then perpendicular to line \(AB\).

In the case of a point to wedge (two lines) contact, the discretize obstacle is defined by three points and all of them (plus the penetrating point \(x\)) are involved in the definition of a constraint. Consequently, the self-equilibrated nodal forces occur at all these nodes.