Finite Element Gradient Superconvergence for Nonlinear Elliptic Problems

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FINITE ELEMENT GRADIENT SUPERCONVERGENCE
FOR NONLINEAR ELLIPTIC PROBLEMS

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Abstract

We consider second order nonlinear elliptic boundary value problems in two- and three-dimensional domains with Dirichlet boundary conditions approximated using finite element schemes based on linear and quadratic triangular (tetrahedral) finite elements. Superconvergence of the tangential derivatives at Gauss points on edges of elements is proved for these nonlinear problems and quasiuniform partitions of the domain.

1. Introduction

In finite element applications the derivative of the approximate solution is frequently a quantity of primary interest. This is the case, for instance, in determining the heat and mass fluxes, velocities and strains or stresses. Simply differentiating the approximate solution on an element will generate an approximation to the gradient but the accuracy and asymptotic rate will be inferior to that of the primary variable. However, it is now well known that certain superconvergence properties of this approximate gradient may hold at special points in the element or that simple post-processing schemes such as locally averaging can yield superconvergent extraction
formulas. In turn, this implies that good gradient approximations can be computed without resorting to the expense of computations on finer grids. Moreover, these superconvergent properties may be useful in constructing a posteriori error indicators for adaptive refinement.

Superconvergence has been the subject of theoretical investigations by several authors primarily for linear elliptic problems - see e.g. Douglas and Dupont [13, 14], Douglas, Dupont and Wheeler [15], Nitsche and Schatz [30], Bramble and Schatz [3]. Superconvergence of the gradient for bilinear and biquadratic quadrilateral elements is proved by Zlamal [35, 36]. Generalization of these results for polynomials of higher degree can be found in Lesaint and Zlamal [27]. Linear triangular elements are considered by Andreev [1], Levine [28] and linear tetrahedrons by Kantchev and Lazarov [25]. The results of Andreev, Lazarov [2] and Pehlivanov [31, 32] extend the previous results for quadratic triangular and tetrahedral isoparametric elements in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, see also Schatz, Sloan, and Wahlbin [33, 34].

Numerical experiments strongly suggest that most of the superconvergence results for the linear problems may be carried out over to reasonably well behaved nonlinear problems. Recently, a number of studies related to superconvergence in nonlinear elliptic problems have appeared: for a class of minimal surface problems - Chen [4]; for semilinear problems - Chow and Lazarov [7], Chow, Carey, and Lazarov [6]; for second and fourth order equations in rectangular 2-D domain and rectangular finite elements - Dautov, Lapin, and Lyashko [12], Dautov and Lapin [11], Dautov [10]. Since simplex elements do not have the cartesian or tensor-product "symmetry" of the rectangular elements, analysis is more difficult and the theoretical results concerning simplicial elements are rare. However, superconvergence of linear triangular elements for strongly nonlinear problems and uniform partitions has been studied by Kantchev [24].

Here we consider second order nonlinear elliptic boundary value problems in two- and three-dimensional domains with Dirichlet boundary conditions. The finite element solutions of problems of this type are studied in Ciarlet, Schultz, and Varga [9], Frehse and Rannacher [19, 20], Glowinski and Marrocco [22, 23], Melkes [29], Korneev [26], Feistauer and Ženíšek [18], Chow [5].

In the present paper finite element schemes, based on linear and quadratic triangular and tetrahedral finite elements, are investigated and we extend the gradient superconvergence results in the following directions:
(i) for quadratic simplicial elements in $\mathbb{R}^n$, $n = 2, 3$;

(ii) for isoparametric elements;

(iii) for quasiuniform partitions.

We prove superconvergence of the gradient under assumptions of regularity of the nonlinear coefficients, ellipticity and monotonicity. Note that these assumptions are quite natural and close to the assumptions in Feistauer and Ženíšek [18]. Excellent examples of practical problems which satisfy our assumptions can also be found in Feistauer [16, 17], Glowinski [21], Glowinski and Marrocco [22], and Chow [5].

An outline of this paper is as follows: In Section 2 we introduce some notations and state our main assumptions. Some results from Ciarlet [8] and Feistauer and Ženíšek [18] are listed as preliminary results in Section 3. Several estimates for the interpolant that are crucial to the superconvergence analysis are proved in Section 4. First, we consider the corresponding linear problem and prove the estimate

$$|a^{\text{lin}}(u - u_I, v_h)| \leq Ch^{k+1} \|u\|_{k+2,\Omega} |v_h|_{1,\Omega}$$

for any function $v_h$ from the finite dimensional space (see Theorem 4.1 and Theorem 4.2). Here $u_I$ is the standard finite element interpolant. Next, we prove the following estimate for nonlinear problems (Theorem 4.4)

$$|a(u, v) - a(u_I, v)| \leq Ch^{k+1} \left(1 + \|u\|_{k+2,\Omega}\right) \|u\|_{k+2,\Omega} |v_h|_{1,\Omega},$$

where $k$ equals 1 for linear elements and 2 for quadratic elements.

All results are derived for regular and $k$-quasiuniform partitions. The error due to the numerical integration is investigated in Theorem 4.5. Finally in Section 5, we obtain the main result of this paper:

$$|u - u_h|_{1,h}^{\ast} \leq Ch^{k+1} \left(\|u\|_{k+2,\Omega} \left(1 + \sum_{s=1}^{k} \|u\|_{k+2,\Omega}^{s} \right) + \|f\|_{k+1,\Omega}\right),$$

where $\cdot |_{1,h}^{\ast}$ is a discrete seminorm constructed by using the tangential derivatives at Gauss points of order $k$ on the element edges.
2. Notations and Problem Formulation

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\Gamma$. Denote by $W^m_p(\Omega)$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$, the Sobolev space on $\Omega$, provided with the usual norm $\| \cdot \|_{m, p, \Omega}$ and seminorms $| \cdot |_{i, p, \Omega}$, $0 \leq i \leq m$, and by $H^m(\Omega)$ the Sobolev space $W^m_2(\Omega)$ with norm $\| \cdot \|_{m, \Omega}$ and seminorms $| \cdot |_{i, \Omega}$, $0 \leq i \leq m$. The symbol $P_k(G)$ stands for the space of polynomials of degree at most $k$ over $G$. Let

$$V = H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \}. \quad (2.1)$$

We consider the following nonlinear problem with Dirichlet boundary conditions: Find $u \in V$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \in V, \quad (2.2)$$

where

$$a(u, v) = \int_\Omega \sum_{i=1}^n a_i(x, \nabla u) \partial_i v \, dx, \quad (f, v) = \int_\Omega f v \, dx. \quad (2.3)$$

Here the nonlinear coefficients $a_i$ depend on $x \in \Omega$ and $\nabla u$. All results below hold in the general case when $a_i$ depend on $x$, $u$ and $\nabla u$ (see Remark 5.1).

Notations $\alpha$ and $\beta$ are used for any multi-indices such that

$\Omega \subset \mathbb{R}^2 : \alpha \in \{ (\alpha_1, \alpha_2, 0, 0) : \alpha_i \geq 0 \}, \beta \in \{ (0, 0, \beta_1, \beta_2) : \beta_i \geq 0 \},$

$\Omega \subset \mathbb{R}^3 : \alpha \in \{ (\alpha_1, \alpha_2, \alpha_3, 0, 0, 0) : \alpha_i \geq 0 \}, \beta \in \{ (0, 0, 0, \beta_1, \beta_2, \beta_3) : \beta_i \geq 0 \}.$

Throughout the paper $k$ is the degree of polynomials in the finite element spaces (see (2.8)).

The following assumptions will be made concerning $a(\cdot, \cdot)$:

(H1) The functions $a_i(x, \xi), x \in \Omega, \xi \in \mathbb{R}^n$, are continuous in $\Omega \times \mathbb{R}^n$; there exists a constant $C > 0$ such that for $1 \leq i \leq n$

$$|a_i(x, \xi)| \leq C \left( \sum_{j=1}^n |\xi_j| \right) \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \quad (2.4)$$

(H2) The derivatives $\partial^\alpha (\partial^\beta a_i)$, $1 \leq i \leq n$, for $|\alpha| + |\beta| \leq k + 1$, $|\alpha| \leq k$, $|\beta| \geq 1$, are continuous and bounded in $\Omega \times \mathbb{R}^n$; there exists a constant $C > 0$ such that

$$\left| \partial^\alpha (\partial^\beta a_i) \right|_{0, \infty, \Omega \times \mathbb{R}^n} = \sup_{x \in \Omega, \xi \in \mathbb{R}^n} \left| \partial^\alpha (\partial^\beta a_i(x, \xi)) \right| \leq C. \quad (2.5)$$
(H3) The following inequality holds

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \xi_j}(x, \xi) \eta_i \eta_j \geq \mu \sum_{j=1}^{n} \eta_j^2$$

for all \( x \in \Omega \) and \( \xi, \eta \in \mathbb{R}^n \), \( \mu > 0 \) is a constant independent of \( x, \xi \) and \( \eta \).

(H4) The functions \( \partial^\alpha a_i, 1 \leq |\alpha| \leq k + 1, 1 \leq i \leq n \), are continuous in \( \Omega \times \mathbb{R}^n \); there exists a constant \( C > 0 \) such that

$$|\partial^\alpha a_i(x, \xi)| \leq C \left( \sum_{j=1}^{n} |\xi_j| \right)$$

for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \).

Remark 2.1 It can be proved that under assumptions H1, H2 with \( |\alpha| = 0 \) and \( |\beta| = 1 \), H3, and \( f \in L^2(\Omega) \) there exists a unique solution of problem (2.1) (see Feistauer and Ženíšek [18]).

Remark 2.2 In the general case of nonconstant coefficients it is necessary to introduce some quadrature formulas to evaluate \( a(\cdot, \cdot) \) and \( (f, \cdot) \) approximately. The assumption H4 will appear only in the analysis of the error due to the numerical integration.

Consider the natural partition of \( \mathbb{R}^2 \) into unit squares. Then let each of these squares be divided into two triangles using one of the diagonals. These diagonals must be oriented identically. Hence we obtain a partition of \( \mathbb{R}^2 \) into isosceles triangles (see Fig. 2.1). In the three-dimensional case, we start from the natural partition of \( \mathbb{R}^3 \) into unit cubes. Then, for each cube, we project one of the cube’s celestial diagonals onto each face. Thus each cube is divided into six tetrahedra. Again, as in \( \mathbb{R}^2 \), these celestial diagonals must be oriented identically (see Fig. 2.2). We denote these partitions by \( \hat{T} \).

Next, let the whole domain \( \Omega \) be covered by isoparametric linear or quadratic triangular (tetrahedral) elements \( K \in T_h \) (\( T_h \) is the triangulation of \( \Omega \)). We assume that for each \( K \in T_h \) there exists \( \hat{K} \in \hat{T} \) and invertible mapping \( F_K:\hat{x}\in\hat{K}\rightarrow F_K(\hat{x})\in K \) such that \( F_K(\hat{K}) = K \) and the components of \( F_K \) are polynomials of degree \( k \); i.e. \( (F_K)_j \in P_k(\hat{K}), 1 \leq j \leq n \), where

\[
k = \begin{cases} 
    1 & \text{for linear elements} \\
    2 & \text{for quadratic elements}
\end{cases}
\]

(2.8)
As commonly used, we have the correspondence \( \hat{v}(\hat{x}) = v(F_K(\hat{x})) \). Define the finite element space

\[
V_h = \left\{ v_h \in C^0(\Omega) : v_h|_K = \hat{v}_h|_K \in P_k(\hat{K}) \quad \forall K \in T_h, \quad v_h = 0 \text{ on } \Gamma \right\} . \tag{2.9}
\]

Since we allow the use of isoparametric elements, some of the elements may be curvilinear. Denote by \( d_i, 1 \leq i \leq n + 1 \), the vertices of any element \( K \) and by \( \hat{F}_K \) the unique affine mapping, which satisfies \( \hat{F}_K(d_i) = d_i, 1 \leq i \leq n + 1 \). Here \( \hat{d}_i, 1 \leq i \leq n + 1 \), are the vertices of the corresponding element \( \hat{K} \). Let \( d_{ij} = (d_i + d_j)/2 \), \( \hat{d}_{ij} = (\hat{d}_i + \hat{d}_j)/2, 1 \leq i < j \leq n + 1 \). The family of partitions \( \{T_h\} \) is termed regular if the following holds for all \( K \in T_h \) (see Ciarlet [8]):

(i) there exists a constant \( \sigma \) such that

\[
h_K/\rho_K \leq \sigma
\]

where \( h_K \) is the diameter of \( K \) and \( \rho_K \) is the diameter of the inscribed circle (sphere) in \( K \);

(ii) the quantities \( h_K \) approach zero;

(iii) for quadratic elements

\[
\text{dist}(d_{ij}, \hat{d}_{ij}) \leq Ch_K^2 ,
\]

where \( \hat{d}_{ij} = \hat{F}_K(d_{ij}), 1 \leq i < j \leq n + 1 \).

Define

\[
|F_K|_{m,\infty,\hat{K}} = \max_{1 \leq i \leq n} \max_{|\gamma|=m} |\partial^\gamma (F_K)|_{0,\infty,\hat{K}} ,
\]
where $\gamma = (\gamma_1, \gamma_2)$ for $\Omega \subset \mathbb{R}^2$ ( $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ for $\Omega \subset \mathbb{R}^3$ ). We say that the partition $T_h$ of $\Omega$ is 1-quasiuniform if for every two elements $K_1$ and $K_2$ which have a common side (face) the following inequality is fulfilled:

$$
|F_{K_1} - F_{K_2}|_{1,\infty, \partial K_1 \cap \partial K_2} \equiv \max_{1 \leq i \leq n} \max_{|\gamma| = 1} |\partial^\gamma (F_{K_1})_i - \partial^\gamma (F_{K_2})_i|_{0,\infty, \partial K_1 \cap \partial K_2} \leq C h^2 ,
$$

(2.10)

where the constant $C$ does not depend on $h$. The partition $T_h$ is 2-quasiuniform when, in addition to (2.10), we have

$$
|F_{K_1} - F_{K_2}|_{2,\infty, \partial K_1 \cap \partial K_2} \leq C h^3 .
$$

(2.11)

Obviously, any uniform mesh satisfies these assumptions. Also, any mesh with nodes which are obtained by a smooth mapping of uniform mesh nodes satisfies (2.10) and (2.11) (see Zlamal [36], Levine [28], Pehlivanov [32]). Of course, such 1- or 2-quasiuniform meshes impose a restriction on the domain $\Omega$. Later we relax (2.10)
and (2.11) to obtain results for more general domains but at the expense of a lower order of convergence.

Now we can formulate the discrete problem which corresponds to the problem (2.2): Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h)_h \quad \text{for all} \quad v_h \in V_h ,$$

(2.12)

where the index $h$ in $a_h(\cdot, \cdot)$, $(f, \cdot)_h$ indicates that we have applied some quadrature formula with positive coefficients in order to calculate the integrals in $a(u_h, v_h)$ and $(f, v_h)$.

3. Preliminary Results

We shall use the following notations:

$$J_{F_K}(\hat{x}) = \det (\partial_j(F_K)\hat{x})_{i,j=1}^n , \quad J_{F_K}^{-1}(x) = (J_{F_K}(\hat{x}))^{-1} ,$$

for any element $K \in T_h$.

Theorem 3.1 (Ciarlet [8, Theorems 3.1.3 and 4.3.3]) \textit{Let the partition of $\Omega$ be regular. Then the following estimates hold:}

$$|F_K|_{1, \infty, K} \leq C h_K , \quad |F_K|_{2, \infty, K} \leq C h_K^2 , \quad |F_K|_{3, \infty, K} = 0 ,$$

$$|F_K^{-1}|_{1, \infty, K} \leq C h_K^{-1} , \quad |F_K^{-1}|_{2, \infty, K} \leq C h_K^{-1} , \quad |F_K^{-1}|_{3, \infty, K} \leq C h_K^{-1} ,$$

$$|J_{F_K}|_{0, \infty, K} \leq C \text{meas}(K) , \quad |J_{F_K}^{-1}|_{0, \infty, K} \leq C (\text{meas}(K))^{-1} .$$

Define the following bilinear form

$$a^{\text{lin}}(\eta, v) = \int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) \partial_i \eta \partial_j v \, dx .$$

(3.1)

Suppose that the matrix of coefficients $A = (a_{ij}(x))_{i,j=1}^n$, $x \in \Omega$, is positive definite and the coefficients $a_{ij}$ are bounded; i.e. there exist constants $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 \zeta^T \zeta \leq A \zeta \leq \alpha_2 \zeta^T \zeta .$$
for all vectors $\zeta \in \mathbb{R}^n$ and all $x \in \Omega$.

For any element $K$ we have

$$
\int_K \sum_{i,j=1}^n a_{ij}(x) \partial_i w \partial_j v \, dx = \int_K \sum_{i,j=1}^n b_{ij}(\hat{x}) \partial_i \hat{w} \partial_j \hat{v} \, d\hat{x},
$$

(3.2)

where

$$
B = (b_{ij})_{i,j=1}^n, \quad B(\hat{x}) = DF^{-1}(x) \hat{A}(\hat{x}) (DF^{-1}(x))^T J_F(\hat{x}).
$$

(3.3)

Here we have used the notations

$$
DF^{-1}(x) = (\partial_j(F^{-1})_i)_{i,j=1}^n,
$$

$$
\hat{A} = (\hat{a}_{ij})_{i,j=1}^n, \quad A = (a_{ij})_{i,j=1}^n, \quad \hat{a}_{ij}(\hat{x}) = a_{ij}(F_K(\hat{x})).
$$

Theorem 3.2 Let $a_{ij} \in W_{k+1}^1(\Omega)$. Then:

$$
|B|_{s,\infty, K} \leq Ch^{s-2} \text{meas}(K) \quad \text{for} \quad 0 \leq s \leq k + 1,
$$

(3.4)

$$
|b_{ij}(\hat{x}) - b_{ij}(\hat{x}_0)| \leq Ch^{-1} \text{meas}(K),
$$

(3.5)

for all $\hat{x} \in \hat{K}, \hat{x}_0 \in \hat{K}$.

Theorem 3.3 (Feistauer and Ženíšek [18]) The following implications hold:

(i) $H2 \Rightarrow$ the form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}^1$ is Lipschitz continuous; i.e. there exists a constant $C$ such that:

$$
|a(v, z) - a(w, z)| \leq C\|v - w\|_{1,\Omega}\|z\|_{1,\Omega} \quad \text{for all} \quad v, w \in H^1(\Omega).
$$

(3.6)

(ii) $H3 \Rightarrow$ the form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}^1$ is $H^1(\Omega)$ strongly monotone with respect to the seminorm $|\cdot|_{1,\Omega}$; i.e. there exists a constant $\mu > 0$ such that

$$
|a(v, v - w) - a(w, v - w)| \geq \mu |v - w|_{1,\Omega}^2 \quad \text{for all} \quad v, w \in H^1(\Omega).
$$

(3.7)

(iii) $H2$ and the condition that the quadrature formula with positive coefficients is exact for $p \in P_0 \Rightarrow$ the forms $a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}^1$ are uniformly $V_h$-Lipschitz continuous; i.e. there exists $C > 0$ such that:

$$
|a_h(v_h, z_h) - a_h(w_h, z_h)| \leq C\|v_h - w_h\|_{1,\Omega}\|z_h\|_{1,\Omega} \quad \text{for all} \quad v_h, w_h \in V_h.
$$

(3.8)
(iv) \( H^3 \) and the condition that the quadrature formula with positive coefficients is exact for \( p \in P_0 \Rightarrow \) the forms \( a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}^1 \) are uniformly \( V_h \)-strongly monotone with respect to the seminorm \( | \cdot |_{1, \Omega} \); i.e. there exists a constant \( \mu > 0 \) such that:

\[
|a_h(v_h, v_h - w_h) - a_h(w_h, v_h - w_h)| \geq \mu |v_h - w_h|_{1, \Omega}^2 \quad \text{for all} \quad v_h, w_h \in V_h .
\]

\[(3.9)\]

4. Basic Estimates

Several estimates are now developed that will be used subsequently in Section 5 to obtain the desired superconvergence results. First, we construct interpolation estimates for the bilinear functional \( a^\text{lin}(\cdot, \cdot) \) where \( k \)-quasiuniform partitions of the domain \( \Omega \) are required (Theorems 4.1 and 4.2). This condition is relaxed in Theorem 4.3 to piecewise \( k \)-quasiuniform partitions at the expense of a lower order of convergence. Finally, quadrature formula error estimates are established in Theorem 4.5.

Denote

\[
\| \eta \|_{m, \infty, T_h} = \max_{K \in T_h} \| \eta \|_{m, \infty, K} .
\]

**Theorem 4.1** Let \( T_h \) be a regular and \( 1 \)-quasiuniform partition of linear finite elements, the coefficients \( a_{ij} \in C^0(\Omega) \) and \( \| a_{ij} \|_{1, \infty, T_h} \leq C \). Let \( u \in H^3(\Omega) \) and \( u_I \) be the standard finite element interpolant of \( u \). Then the following estimate holds:

\[
|a^\text{lin}(u - u_I, v_h)| \leq Ch^2 \| u \|_{3, \Omega} |v_h|_{1, \Omega} \quad \text{for all} \quad v_h \in V_h .
\]

**Proof:** The case \( \Omega \subset \mathbb{R}^2 \) is studied in Andreev [1], Levine [28], and the three-dimensional case for uniform partitions in Kantchev and Lazarov [25]. Here we consider the three-dimensional case for \( 1 \)-quasiuniform partitions.

Since \( C^\infty(\bar{\Omega}) \) is dense in \( H^3(\Omega) \), it is sufficient to prove (4.1) for \( u \in C^\infty(\bar{\Omega}) \). Denote \( w = u - u_I \). We have

\[
a^\text{lin}(w, v_h) = \sum_{i,j=1}^n \sum_{K \in T_h} \int_K a_{ij}(x) \partial_i w \partial_j v_h \, dx
\]

\[
= \sum_{i,j=1}^n \sum_{K \in T_h} \int_{\hat{K}} b_{ij}(\hat{x}) \partial_i \hat{w} \partial_j \hat{v}_h \, d\hat{x} ,
\]
where \( b_{ij} \) are defined in (3.2). We shall prove that

\[
\sum_{K \in T_h} \int_{K_m} b_{p1}(\hat{x}) \partial_{\hat{p}} \hat{w} \partial_1 \hat{v}_h \, d\hat{x} \leq C h^2 \| u \|_{3, \Omega} |v_h|_{1, \Omega}
\]  

(4.2)

for \( 1 \leq p \leq 3 \). The remaining cases are treated in the same way.

Consider any tetrahedron in \( \hat{T} \). Then it has an edge parallel to the \( \hat{x}_1 \)-axis. Denote by \( \hat{K}_1, \hat{K}_2, \ldots, \hat{K}_6 \) the tetrahedra which share this edge, and by \( U \) their union. Let \( \hat{d}_i, 1 \leq i \leq 8 \), be the vertices of \( U \) (see Fig. 4.1). Then \( \hat{d}_m, \hat{d}_{m+1}, \hat{d}_7, \hat{d}_8 \) are the vertices of \( \hat{K}_m, 1 \leq m \leq 6 \). Further all indices with \( m \) are interpreted as modulo 6.

Let \( G \) be an arbitrary point from edge \( \hat{d}_7 \hat{d}_8 \) and for notational convenience let us denote \( b_{p1, \hat{K}_m} = g_m \). We have

\[
\int_{\hat{K}_m} b_{p1, \hat{K}_m}(\hat{x}) \partial_{\hat{p}} \hat{w} \partial_1 \hat{v}_h \, d\hat{x} = \int_{\hat{K}_m} (g_m(\hat{x}) - g_m(G)) \partial_{\hat{p}} \hat{w} \partial_1 \hat{v}_h \, d\hat{x}
\]

\[
+ g_m(G) \partial_1 \hat{v}_h(G) \int_{\hat{K}_m} \partial_{\hat{p}} \hat{w} \, d\hat{x},
\]

(4.3)

since \( \hat{v}_h \in P_1(\hat{K}_m) \). From Theorem 3.2 we get

\[
\int_{\hat{K}_m} (g_m(\hat{x}) - g_m(G)) \partial_{\hat{p}} \hat{w} \partial_1 \hat{v}_h \, d\hat{x} \leq C h^2 \| u \|_{2, \hat{K}_m} |v_h|_{1, \hat{K}_m}.
\]

Hence

\[
\sum_{K_m \in T_h} \int_{\hat{K}_m} b_{p1, \hat{K}_m}(\hat{x}) \partial_{\hat{p}} \hat{w} \partial_1 \hat{v}_h \, d\hat{x} = I_1 + \sum_{K_m \in T_h} g_m(G) \partial_1 \hat{v}_h(G) \int_{\hat{K}_m} \partial_{\hat{p}} \hat{w} \, d\hat{x},
\]

(4.4)

where

\[
I_1 \leq C h^2 \| u \|_{2, \Omega} |v_h|_{1, \Omega}.
\]

(4.5)

It remains to estimate the second term on the right in (4.4). For each tetrahedron \( \hat{K}_m, 1 \leq m \leq 6 \), we construct a functional \( H(m; \hat{u}) \) such that the functional

\[
L(m; \hat{u}) = \partial_1 \hat{v}_h(G) \int_{\hat{K}_m} \partial_{\hat{p}} \hat{w} \, d\hat{x} + H(m; \hat{u})
\]

(4.6)

vanishes for polynomials of second degree, i.e.

\[
L(m; \hat{u}) = 0 \quad \text{for all} \quad \hat{u} \in P_2(\hat{K}_m).
\]

(4.7)
Figure 4.1: The tetrahedra $\hat{K}_1, \ldots, \hat{K}_6$
First, define the following functionals

\[
\phi_m(\gamma) = \int_{K_m} \partial_p (\hat{x}^\gamma - (\hat{x})^\gamma) \, d\hat{x}
\]  

(4.8)

for each multi-index \( \gamma = (\gamma_1, \gamma_2, \gamma_3), |\gamma| = \gamma_1 + \gamma_2 + \gamma_3 = 2, \) where \( \hat{x}^\gamma = \hat{x}_1^{\gamma_1} \hat{x}_2^{\gamma_2} \hat{x}_3^{\gamma_3}. \)

Let

\[
\mu_1(\gamma; 1) = 0, \quad \mu_1(\gamma; 2) = \phi_1(\gamma).
\]

For \( 2 \leq m \leq 6 \) we define recursively:

\[
\mu_m(\gamma; m) = -\mu_{m-1}(\gamma; m),
\]

\[
\mu_m(\gamma; m + 1) = \phi_m(\gamma) + \mu_{m-1}(\gamma; m).
\]

Further we set

\[
S_m(\gamma; m) = \frac{1}{\gamma! \text{meas}(\tau(m, 7, 8))} \int_{\tau(m, 7, 8)} \partial^\gamma \hat{u}|_{K_m} \, d\tau,
\]

\[
S_m(\gamma; m + 1) = \frac{1}{\gamma! \text{meas}(\tau(m + 1, 7, 8))} \int_{\tau(m + 1, 7, 8)} \partial^\gamma \hat{u}|_{K_m} \, d\tau.
\]

where \( \tau(m, 7, 8) \) is the triangle with vertices \( \hat{d}_m, \hat{d}_7, \hat{d}_8, 1 \leq m \leq 6 \). Let us note that if \( \hat{u} = (\hat{x})^\delta, |\delta| = 2, \) then

\[
S_m(\gamma; m) = S_m(\gamma; m + 1) = \begin{cases} 
1 & \text{for } \gamma = \delta \\
0 & \text{for } \gamma \neq \delta
\end{cases}
\]

Finally, setting

\[
\theta = -\partial_t \hat{w}_h(G),
\]

\[
R_m(\gamma; m) = \theta S_m(\gamma; m) \mu_m(\gamma; m),
\]

\[
R_m(\gamma; m + 1) = \theta S_m(\gamma; m + 1) \mu_m(\gamma; m + 1),
\]

we can construct the desired functional

\[
H(m; \hat{u}) = \sum_{|\gamma| \leq 2} (R_m(\gamma; m) + R_m(\gamma; m + 1)).
\]  

(4.9)
Then property (4.7) holds. This property can now be utilized in (4.4) as follows

\[
\sum_{K_m \in T_h} \int_{K_m} b_{p_1,k_m}(\hat{x}) \partial_p \hat{w} \partial_t \hat{v}_h \, d\hat{x} = I_1 + \sum_{K_m \in T_h} g_m(G) \left( \partial_t \hat{v}_h(G) \int_{K_m} \partial_p \hat{w} d\hat{x} + H(m; \hat{u}) \right) - \sum_{K_m \in T_h} g_m(G) H(m; \hat{u}) = I_1 + I_2 - \sum_{K_m \in T_h} g_m(G) H(m; \hat{u}),
\]

(4.10)

where

\[
I_2 = Ch^2 \|u\|_{3,\Omega} \|v_h\|_{1,\Omega}.
\]

(4.11)

Using the quasiuniformity of the partition, the last term can be estimated by

\[
\sum_{K_m \in T_h} g_m(G) H(m; \hat{u}) \leq Ch^2 \|u\|_{3,\Omega} \|v_h\|_{1,\Omega}.
\]

(4.12)

Substituting (4.5), (4.11) and (4.12) into (4.10) we get the desired estimate (4.2).

If one of the six tetrahedra of \(U\) is not an image of a finite element in \(T_h\) then \(\hat{\partial}_7 \hat{\partial}_8\) is an image of a part of the boundary. But then \(\theta = 0\), since \(v \in V_h \subset H^1(\Omega)\).

This completes the proof. □

**Theorem 4.2** Let \(T_h\) be a regular and 2-quasiuniform partition of quadratic finite elements, the coefficients \(a_{ij} \in C^0(\Omega)\) and \(\|a_{ij}\|_{1,\infty,T_h} \leq C\). Let \(u \in H^4(\Omega)\) and \(u_I\) be the standard finite element interpolant of \(u\). Then the following estimate holds:

\[
|a^{\text{lin}}(u - u_I, v_h)| \leq Ch^3 \|u\|_{4,\Omega} \|v_h\|_{1,\Omega} \text{ for all } v_h \in V_h. \quad (4.13)
\]

**Remark 4.1** The case \(\Omega \in \mathbb{R}^2\) is considered by Andreev and Lazarov [2] and the case \(\Omega \subset \mathbb{R}^3\) by Pehlivanov [31, 32]. □

**Remark 4.2** The fact that \(v_h = 0\) on \(\Gamma\) is crucial for the validity of (4.1) and (4.13). Suppose that \(v_h\) does not vanish on \(\Gamma\), i.e.

\[
v_h \in \hat{V}_h = \left\{ v_h \in C^0(\Omega) : v_h|_K = \hat{v}_h|_K \in P_h(K) \quad \forall K \in T_h \right\}.
\]

(4.14)
Then we have the estimate

\[ \sum_{K_m \in T_h} g_m(G) H(m; \hat{u}) \leq C h^{k+1/2} \|u\|_{k+3/2, \Omega} \|v_h\|_{1, \Omega} + \nu_T , \]

where \( \nu_T \) denotes terms defined on \( \Gamma \). Since the dimension of \( \Gamma \) is \( n - 1 \), the terms on the boundary elements can be estimated by \( C h^{k+1/2} \) (see Pehlivanov [32]). Hence

\[ \sum_{K_m \in T_h} g_m(G) H(m; \hat{u}) \leq C h^{k+1/2} \|u\|_{k+3/2, \Omega} \|v_h\|_{1, \Omega} \quad (4.15) \]

and

\[ |a_{\text{lin}}(u - u_I, v_h)| \leq C h^{k+1/2} \|u\|_{k+3/2, \Omega} \|v_h\|_{1, \Omega} \quad \text{for all} \quad v_h \in \tilde{V}_h . \tag{4.16} \]

We say that the partition \( T_h \) of \( \Omega \) is piecewise \( k \)-quasiuniform if the domain \( \Omega \) can be divided into subdomains and each of these subdomains has a \( k \)-quasiuniform partition (see (2.8), (2.10), and (2.11)). Obviously, the class of domains with piecewise \( k \)-quasiuniform partitions is much larger than the class of domains with global \( k \)-quasiuniform partition. We have the following result:

**Theorem 4.3** Let \( T_h \) be a regular and piecewise \( k \)-quasiuniform partition of \( \Omega \). Let \( u \in H^{k+3/2}(\Omega) \). Then estimate (4.16) holds.

**Theorem 4.4** Let \( T_h \) be a regular and \( k \)-quasiuniform partition of \( \Omega \). Let \( u \in H^{k+2}(\Omega) \) and \( H1, H2 \) be satisfied. Then

\[ |a(u, v_h) - a(u_I, v_h)| \leq C h^{k+1} \|u\|_{k+2, \Omega} \|v_h\|_{1, \Omega} \cdot Q_1 \quad (4.17) \]

for all \( v_h \in V_h \), where \( Q_1 = 1 + \|u\|_{k+2, \Omega} \).

**Proof.** Let \( w = u - u_I \). We have

\[
\begin{align*}
a(u, v_h) - a(u_I, v_h) &= \sum_{K \in T_h} \sum_{i=1}^n \int_K (a_i(x, \nabla u) - a_i(x, \nabla u_I)) \partial_i v_h \, dx \\
&= \sum_{K \in T_h} \sum_{i,j=1}^n \int_K (\hat{a}_i(x, \nabla \hat{u} DF^{-1}(x)) - \hat{a}_i(x, \nabla \hat{u}_I DF^{-1}(x))) c_{ij} \partial_j \hat{v}_h \, d\hat{x} . \tag{4.18}
\end{align*}
\]
where $\tilde{C} = (c_{ij})_{i,j=1}^{n} = (DF^{-1}(x))^T F(\tilde{x})$. Let $x \in \Omega$ and $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n)$ belong to $\mathbb{R}^n$. Let $g_i(t) \equiv \dot{a}_i(\tilde{x}, p + t(q - p)), t \in [0, 1]$. Then

$$g_i(0) = \dot{a}_i(\tilde{x}, p), \quad g_i(1) = \dot{a}_i(\tilde{x}, q) , \quad (4.19)$$

$$g_i'(t) = \sum_{l=1}^{n} \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, p + t(q - p)) (q_l - p_l) , \quad (4.20)$$

$$\dot{a}_i(\tilde{x}, q) - \dot{a}_i(\tilde{x}, p) = \int_{0}^{1} g_i'(t) \, dt . \quad (4.21)$$

Let $\sigma_{ij}$ be independent of $t$. Multiplying (4.21) by $\sigma_{ij}$ we get

$$(\dot{a}_i(\tilde{x}, q) - \dot{a}_i(\tilde{x}, p))\sigma_{ij} = \int_{0}^{1} g_i'(t)\sigma_{ij} \, dt . \quad (4.22)$$

Substituting $q = \nabla \dot{u}DF^{-1}(x), p = \nabla \dot{u}_lDF^{-1}(x)$ and $\sigma_{ij} = c_{ij}\partial_j\dot{\nu}_h$ in (4.22) we obtain:

$$\dot{a}_i(\tilde{x}, \nabla \dot{u}DF^{-1}(x)) - \dot{a}_i(\tilde{x}, \nabla \dot{u}_lDF^{-1}(x)) \quad c_{ij}\partial_j\dot{\nu}_h$$

$$= \sum_{l=1}^{n} \int_{0}^{1} \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}_lDF^{-1}(x) + t\nabla \dot{w}DF^{-1}(x))(q_l - p_l)c_{ij}\partial_j\dot{\nu}_h \, dt$$

$$= S_1 + S_2 , \quad (4.23)$$

where

$$S_1 = \sum_{l=1}^{n} \left( \int_{0}^{1} \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}_lDF^{-1}(x) + t\nabla \dot{w}DF^{-1}(x)) \, dt \right.$$

$$- \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}DF^{-1}(x))(q_l - p_l)c_{ij}\partial_j\dot{\nu}_h ,$$

$$S_2 = \sum_{l=1}^{n} \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}DF^{-1}(x))(q_l - p_l)c_{ij}\partial_j\dot{\nu}_h .$$

Let us set (for fixed $i$ and $l$)

$$r(t, s) = \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}DF^{-1}(x) + s (\nabla \dot{u}_lDF^{-1}(x) + t\nabla \dot{w}DF^{-1}(x) - \nabla \dot{u}DF^{-1}(x))) .$$

Then

$$r(t, 0) = \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}DF^{-1}(x)) ,$$

$$r(t, 1) = \frac{\partial \dot{a}_i}{\partial \xi_l}(\tilde{x}, \nabla \dot{u}_lDF^{-1}(x) + t\nabla \dot{w}DF^{-1}(x)) ,$$

16
and we can express $S_1$ as

$$S_1 = \sum_{i=1}^{n} \int_0^1 (r(t, 1) - r(t, 0)) \, dt \, (q_i - p_i) c_{ij} \partial_j \hat{v}_h.$$  \hspace{1cm} (4.24)

Furthermore,

$$r(t, 1) - r(t, 0) = \int_0^1 \frac{\partial r}{\partial s} \, ds$$

$$= \int_0^1 \left( \frac{\partial^2 \hat{a}_i}{\partial \xi_1 \partial \xi_1}, \ldots, \frac{\partial^2 \hat{a}_i}{\partial \xi_i \partial \xi_n} \right)^T \, ds$$

$$\cdot (\nabla \hat{u}_1 DF^{-1}(x) + t \nabla \hat{w} DF^{-1}(x) - \nabla \hat{u} DF^{-1}(x)) \hspace{1cm} (4.25)$$

Using (4.24), (4.25), and assumption H2, we get

$$\int_K S_1 \, d\hat{x} \leq C \left( \sum_{|\beta|=2} |\partial^\beta \hat{a}_i|_{0, \infty, K \times \mathbb{R}^n} \right)$$

$$\sum_{i=1}^{n} \left| \int_0^1 \int_K (t - 1) \nabla (\hat{u} - \hat{u}_1) DF^{-1}(x) (q_i - p_i) c_{ij} \partial_j \hat{v}_h \, d\hat{x} \, dt \right|$$

$$\leq Ch^{-3} \text{meas}(K) |\hat{u} - \hat{u}_1|_{1,4, K} ||\hat{v}_h||_{1,K}$$

$$\leq Ch^{-3} \text{meas}(K) |\hat{u}_1|_{k+1,4,K} ||\hat{v}_h||_{1,K}$$

$$\leq Ch^{2k} \|u\|_{k+1,4,K} ||v_h||_{1,K}. \hspace{1cm} (4.26)$$

Then

$$\sum_{K \in T_h} \sum_{i,j=1}^{n} \int_K S_1 \, d\hat{x} \leq Ch^{2k} \|u\|_{k+1,4,\Omega} ||v_h||_{1,\Omega}$$

$$\leq Ch^{2k} \|u\|_{k+2,\Omega} ||v_h||_{1,\Omega}, \hspace{1cm} (4.27)$$

where the continuous embedding $W^{k+2,2}(\Omega) \subset W^{k+1,4}(\Omega)$ was used. From (4.18), (4.23), and (4.27),

$$a(u, v_h) - a(u_I, v_h)$$

$$\leq Ch^{2k} \|u\|_{k+2,\Omega} ||v_h||_{1,\Omega}$$

$$+ \left| \sum_{K \in T_h} \sum_{i,j=1}^{n} \sum_{l=1}^{n} \int_K \frac{\partial a_i}{\partial \xi_i} (\hat{x}, \nabla \hat{u} DF^{-1}(x)) (q_i - p_i) c_{ij} \partial_j \hat{v}_h \, d\hat{x} \right|. \hspace{1cm} (4.28)$$
Now we consider the second term on the right in (4.28). Define (for fixed $i$ and $\ell$)
\[
g(x) = \frac{\partial a_{i}}{\partial \xi_{\ell}}(x, \nabla u) = \frac{\partial \hat{a}_{i}}{\partial \xi_{\ell}}(\hat{x}, \nabla \hat{u} D^{-1}(x)) = \hat{g}(\hat{x}) .
\]

Note that $g(x) \in C^0(\Omega)$ for $u \in H^2(\Omega)$. Following the same steps as in Theorem 4.1, we have to estimate terms of the form
\[
S_3 = \sum_{K \in \mathcal{T}_h} \left( \int_{\hat{K}} \hat{g}(\hat{x}) - \hat{g}(\hat{x}_0))(q_i - p_l)c_{i\ell} \partial_{\ell} \hat{v}_h \, d\hat{x} \right) (4.29)
\]
and
\[
S_4 = \sum_{K \in \mathcal{T}_h} \left( \int_{\hat{K}} \hat{g}(\hat{x}_0)(q_i - p_l)c_{i\ell} \partial_{\ell} \hat{v}_h \, d\hat{x} \right) , (4.30)
\]
where $\hat{x}_0$ corresponds to a fixed point in $\hat{K}$ (see (4.3)). Using the quasiuniformity of the partition and assumption $H2$,
\[
S_4 \leq Ch^{k+1} \left( \sum_{i=1}^{n} \sum_{|\beta|=1} \left| \partial^\beta a_i \right|_{0,0,\Omega \times \mathbb{R}^n} \right) \|u\|_{k+2,\Omega} \|v_h\|_{1,\Omega} . (4.31)
\]

In order to estimate $S_3$ we will use the same technique as in the estimation of $S_1$. More specifically,
\[
S_3 \leq \sum_{K \in \mathcal{T}_h} Ch^{-1} \text{meas}(K) \|\hat{g}(\hat{x}) - \hat{g}(\hat{x}_0))(q_i - p_l)\|_{0,\hat{K}} \|\hat{v}_h\|_{1,\hat{K}} \\
\leq \sum_{K \in \mathcal{T}_h} Ch^{-1} \text{meas}(K) \|\hat{g}(\hat{x}) - \hat{g}(\hat{x}_0)\|_{0,4,\hat{K}} \|q_i - p_l\|_{0,4,\hat{K}} \|\hat{v}_h\|_{1,\hat{K}} \\
\leq \sum_{K \in \mathcal{T}_h} Ch^{-2} \text{meas}(K) \|\hat{g}\|_{1,4,\hat{K}} \|\hat{u} - \hat{u}_{I_{1,4,\hat{K}}}\|_{1,4,\hat{K}} \|\hat{v}_h\|_{1,\hat{K}} \\
\leq \sum_{K \in \mathcal{T}_h} Ch^{-2} \text{meas}(K) h(\text{meas}(K))^{-1/4} \|g\|_{1,4,\hat{K}} \|\hat{u}\|_{k+1,4,\hat{K}} \|\hat{v}_h\|_{1,\hat{K}} \\
\leq \sum_{K \in \mathcal{T}_h} Ch^{-2} \text{meas}(K) \left( h + h(\text{meas}(K))^{-1/4} \|u\|_{2,4,\hat{K}} \right) \|\hat{u}\|_{k+1,4,\hat{K}} \|\hat{v}_h\|_{1,\hat{K}} \\
\leq Ch^{k+1} \sum_{K \in \mathcal{T}_h} (\text{meas}(K))^{1/4} \|u\|_{k+1,4,\hat{K}} \|v_h\|_{1,\hat{K}}
\]
We obtain the desired estimate (4.17) from (4.28), (4.31), and (4.32).

**Remark 4.3** A similar result to that in Theorem 4.4 also holds for piecewise $k$-quasiuniform partitions but with $1/2$ order of convergence less.

**Theorem 4.5** Let $T_h$ be a regular partition of $\Omega$ and the quadrature formula on $\hat{K}$ be exact for $P_{2k-1}(\hat{K})$. Let $u_I \in V_h$ be the nodal interpolant of $u$. If $u \in H^{k+1}(\Omega)$, $f \in H^{k+1}(\Omega)$, and assumptions H1, H2, H4 are satisfied then

$$
|a(u_I, v_h) - a_h(u_I, v_h)| \leq C h^{k+1} \|u\|_{k+1, \Omega} |v_h|_{1, \Omega} \cdot Q_2
$$

and

$$
|(f, v_h) - (f, v_h)_h| \leq C h^{k+1} \|f\|_{k+1, \Omega} |v_h|_{1, \Omega}
$$

for all $v_h \in V_h$, where

$$Q_2 = 1 \quad \text{for linear elements } (k = 1)$$

and

$$Q_2 = 1 + \|u\|_{3, \Omega}^2 + \|u\|_{3, \Omega}^2 \quad \text{for quadratic elements } (k = 2).$$

**Proof.** Let us estimate the quadrature formula error over an element $K \in T_h$:

$$E_K = \sum_{i=1}^{n} \left( \int_K a_i(x, \nabla u_I) \partial_i v_h \, dx - \sum_{l=1}^{L} \omega_{l,K} \left( a_i(\cdot, \nabla u_I) \partial_i v_h \right) (b_{l,K}) \right)$$

$$= \sum_{i,j=1}^{n} \left( \int_K \hat{a}_i(\hat{x}, \nabla \hat{u}_I DF^{-1}) c_{ij} \partial_j \hat{v}_h \, d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l \left( \hat{a}_i(\cdot, \nabla \hat{u}_I DF^{-1}) c_{ij} \partial_j \hat{v}_h \right) \left( \hat{b}_l \right) \right),$$

where $(c_{ij})_{i,j=1}^{n} = (DF^{-1}(x))^T J_F(\hat{x})$. Here $\hat{\omega}_l$ and $\hat{b}_l$ are the weights and the nodes of the quadrature formula on $\hat{K}$,

$$\omega_{l,K} = \hat{\omega}_l J_{F_K}(\hat{b}_l), \quad b_{l,k} = F_K(\hat{b}_l).$$
We have
\[ |E_K| \leq C |\tilde{v}_h|_{1,\infty,K} \sum_{i,j=1}^n |\tilde{a}_{ij}|_{1,\infty,K} \quad (4.35) \]

From the embedding \( H^2(\tilde{K}) \subset C^0(\tilde{K}) \),
\[ |\tilde{a}_{ij}|_{0,\infty,K} \leq C |\tilde{a}_{ij}|_{2,\tilde{K}} \]
Then using the exactness of the quadrature formula and the Bramble–Hilbert lemma we get
\[ |E_K| \leq C |\tilde{v}_h|_{1,\tilde{K}} \sum_{i,j=1}^n |\tilde{a}_{ij}|_{k+1,\tilde{K}} \]
\[ \leq C |\tilde{v}_h|_{1,\tilde{K}} \sum_{i,j=1}^n \sum_{s=0}^{k+1} |\tilde{a}_i|_{s,K} |c_{ij}|_{k+1-s,\infty,K} \]
\[ \leq Ch^{k+1} |\tilde{v}_h|_{1,\tilde{K}} \sum_{i=1}^n \sum_{s=0}^{k+1} |a_i|_{s,K} \quad (4.36) \]
where the estimate
\[ |c_{ij}|_{k+1-s,\infty,K} \leq Ch^{k-s} \text{meas}(K) \quad \text{for} \quad 0 \leq s \leq 3, \quad 1 \leq i, j \leq n, \]
was used. Consider linear elements first. From H1, H2, and H4 we get
\[ |a_i(x, \nabla u_I)|_{s,K} \leq C |u_I|_{1,K} \]
\[ \leq C \left( |u_I - u|_{1,K} + |u|_{1,K} \right) \]
\[ \leq C \|u\|_{2,K} \quad (4.37) \]
for \( 0 \leq s \leq 2 \). Hence
\[ |E_K| \leq Ch^2 \|u\|_{2,K} |v_h|_{1,K} \quad (4.38) \]
and estimate (4.33) follows immediately with \( Q_2 = 1 \).

Now we consider quadratic elements. Taking into account H1, H2, and H4, the seminorms \( |a_i(x, \nabla u_I)|_{s,K} \) are bounded as follows:
\[ |a_i(x, \nabla u_I)|_{0,K} \leq C |u_I|_{1,K} \quad (4.39) \]
\[ |a_i(x, \nabla u_I)|_{1,K} \leq C \left( |u_I|_{1,K} + |u_I|_{2,K} \right) \quad (4.40) \]
\[ |a_i(x, \nabla u_I)|_{2,K} \leq C \left( |u_I|_{1,K} + |u_I|_{2,K} + |u_I|_{2,4,K}^2 \right) \quad (4.41) \]
\[ |a_i(x, \nabla u_I)|_{3,K} \leq C \left( |u_I|_{1,K} + |u_I|_{2,K} + |u_I|_{2,4,K}^2 + |u_I|_{2,6,K}^3 \right) \quad (4.42) \]
In the same way as in (4.37),
\[
|u_i|_{1,K} \leq C\|u\|_{2,K}, \\
|u_i|_{2,K} \leq C\|u\|_{2,K}, \\
|u_i|_{2,4,K} \leq C\|u\|_{2,4,K}, \\
|u_i|_{2,6,K} \leq C\|u\|_{2,6,K}.
\]
(4.43) (4.44) (4.45) (4.46)

Then, from (4.36) and (4.39)-(4.46),
\[
|E_K| \leq Ch^3 \left(\|u\|_{2,K} + \|u\|_{2,4,K}^2 + \|u\|_{2,6,K}^3\right) |v_h|_{1,K}.
\]

Summation over all elements leads to
\[
\sum_{K \in T_h} |E_K| \leq C h^3 \sum_{K \in T_h} \left(\|u\|_{2,K} + \|u\|_{2,4,K}^2 + \|u\|_{2,6,K}^3\right) |v_h|_{1,K}
\]
\[
\leq C h^3 \left(\|u\|_{2,\Omega} + \|u\|_{2,4,\Omega}^2 + \|u\|_{2,6,\Omega}^3\right) |v_h|_{1,\Omega}
\]
\[
\leq C h^3 \left(\|u\|_{2,\Omega} + \|u\|_{3,\Omega}^2 + \|u\|_{3,\Omega}^3\right) |v_h|_{1,\Omega}
\]
\[
\leq C h^3 \left(1 + \|u\|_{3,\Omega} + \|u\|_{3,\Omega}^2\right) \|u\|_{3,\Omega} |v_h|_{1,\Omega},
\]
(4.47)

where the embedding $W^{3,2}(\Omega) \subset W^{2,6}(\Omega)$ was used. This completes the proof of (4.33). Estimate (4.34) follows the proof of Theorems 4.1.5 and 4.4.5 in Ciarlet [8].

\[\square\]

5. Superconvergence Estimates

Let $g_0 = 0$ denote the Gauss point of first order in the interval $[-1, 1]$; i.e. the root of the Legendre polynomial of first degree $L_1(\hat{x}) = \hat{x}$, $\hat{x} \in [-1, 1]$. Also, denote by $g_1 = -1/\sqrt{3}$, $g_2 = 1/\sqrt{3}$ the Gauss points of second order; i.e. the roots of the Legendre polynomial $L_2(\hat{x}) = (3\hat{x}^2 - 1)/2$, $\hat{x} \in [-1, 1]$. Let $e_{ij}$, $1 \leq i < j \leq n + 1$, be the edges of $K \in T_h$ (note that some of the edges may be curvilinear). Then on each edge $e_{ij}$ of $K$ we have one Gauss point of first order $g_{0,ij}$ and two Gauss points of second order $g_{1,ij}$ and $g_{2,ij}$ (first, we map the interval $[-1, 1]$ onto the edge $\hat{e}_{ij}$ of $\hat{K}$, and then we use the mapping $F_K : \hat{K} \to K$). Let $\tau_{0,ij}$ be the unit vector along the tangent of $e_{ij}$ at the point $g_{0,ij}$. Correspondingly, $\tau_{1,ij}$ and $\tau_{2,ij}$ are the unit tangential
vectors at the points $g_{1,ij}$ and $g_{2,ij}$. Next, let us define the following seminorms: when $k = 1$ (linear elements)

$$|v|_{1,h}^* = \left( \sum_{K \in T_h} \text{meas}(K) \sum_{1 \leq i < j \leq n+1} \left( \frac{\partial v}{\partial \tau_{0,ij}}(g_{0,ij}) \right)^2 \right)^{1/2};$$

when $k = 2$ (quadratic elements)

$$|v|_{1,h}^* = \left( \sum_{K \in T_h} \text{meas}(K) \sum_{1 \leq i < j \leq n+1} \left( \left( \frac{\partial v}{\partial \tau_{1,ij}}(g_{1,ij}) \right)^2 + \left( \frac{\partial v}{\partial \tau_{2,ij}}(g_{2,ij}) \right)^2 \right)^{1/2}. $$

**Theorem 5.1** Let $v_h \in V_h$. Then

$$|v_h|_{1,h}^* \leq C |v_h|_{1,\Omega}. $$

*Proof.* It is sufficient to show that

$$|v_h|_{1,h,K}^* \leq C |v_h|_{1,K}. $$

Since the norms in finite-dimensional space are equivalent we obtain:

$$\left( |v_h|_{1,h,K}^* \right)^2 \leq C \text{meas}(\bar{K}) |v_h|_{1,\infty,K}^2 \leq C \text{meas}(\bar{K}) h^{-2} |v_h|_{1,\infty,K}^2 \leq C \text{meas}(\bar{K}) h^{-2} |v_h|_{1,K}^2 \leq C \text{meas}(\bar{K}) h^{-2} (\text{meas}(\bar{K}))^{-1} h^2 |v_h|_{1,k}^2 \leq C |v_h|_{1,K}^2. \quad \Box$$

Now we are in a position to formulate the main result in the paper.

**Theorem 5.2** Let $u$ and $u_h$ be the solutions of the problems (2.2) and (2.12) respectively. Let the following assumptions be satisfied:

(i) $u \in H^{k+2}(\Omega) \cap H_0^1(\Omega)$, $f \in H^{k+1}(\Omega)$;

(ii) $H1-H4$;

(iii) $T_h$ is regular and $k$-quasiuniform;

(iv) the quadrature formula with positive coefficients is exact for $P_{2k-1}(\bar{K})$. 

22
Then the following estimate holds

\[ |u - u_h|_{1,h}^* \leq C h^{k+1} \left( \|u\|_{k+2,\Omega} \cdot Q + \|f\|_{k+1,\Omega} \right), \]  

(5.1)

where

\[ Q = 1 + \sum_{s=1}^{k} \|u\|^s_{k+2,\Omega} \cdot .\]

Proof. Let \( u_I \) be the nodal interpolant of \( u \). From Theorem 5.1 we get:

\[ |u - u_h|_{1,h}^* \leq |u - u_I|_{1,h}^* + |u_I - u_h|_{1,h}^* \]

\[ \leq |u - u_I|_{1,h}^* + C |u_I - u_h|_{1,\Omega} \]  

(5.2)

Let \( v_h \in V_h \). Then

\[ a_h(u_I, v_h) - a_h(u_h, v_h) \]
\[ = a_h(u_I, v_h) - a(u, v_h) + a(u_I, v_h) - a(u, v_h) + a(u, v_h) - a_h(u_h, v_h) \]
\[ = a_h(u_I, v_h) - a(u, v_h) + a(u_I, v_h) - a(u, v_h) + (f, v_h) - (f, v_h)_h \]  

(5.3)

From Theorems 4.4 and 4.5

\[ |a(u_I, v_h) - a(u, v_h)| \leq C h^{k+1} \|u\|_{k+2,\Omega} |v_h|_{1,\Omega} \cdot Q_1 \]  

(5.4)

\[ |a(u_I, v_h) - a_h(u_I, v_h)| \leq C h^{k+1} \|u\|_{k+1,\Omega} |v_h|_{1,\Omega} \cdot Q_2 \]  

(5.5)

\[ |(f, v_h) - (f, v_h)_h| \leq C h^{k+1} \|f\|_{k+1,\Omega} |v_h|_{1,\Omega} \]  

(5.6)

Estimates (5.3)–(5.6) yield

\[ |a_h(u_I, v_h) - a_h(u_h, v_h)| \leq C h^{k+1}(\|u\|_{k+2,\Omega} \cdot Q + \|f\|_{k+1,\Omega}) |v_h|_{1,\Omega} \]  

(5.7)

Since the forms \( a_h(\cdot, \cdot) \) are uniformly \( V_h \)-strongly monotone, we have for \( v_h = u_I - u_h \)

\[ \mu |u_I - u_h|_{1,\Omega}^2 \leq a_h(u_I, u_I - u_h) - a_h(u_h, u_I - u_h) \]
\[ \leq C h^{k+1}(\|u\|_{k+2,\Omega} \cdot Q + \|f\|_{k+1,\Omega}) |u_I - u_h|_{1,\Omega} \]

Hence

\[ |u_I - u_h|_{1,\Omega} \leq C h^{k+1}(\|u\|_{k+2,\Omega} \cdot Q + \|f\|_{k+1,\Omega}) \]  

(5.8)
Now we deal with the term $|u-u_I|_{1,h}^*$. The discrete seminorm $|u-u_I|_{1,h}^*$ is constructed in such a way that the following estimate holds:

$$|u-u_I|_{1,h}^* = \left( \sum_{K \in T_h} \left( |u-u_I|_{1,h,K}^* \right)^2 \right)^{1/2} \leq C h^{k+1} \|u\|_{k+2,\Omega}$$

The desired result follows from (5.2), (5.8) and (5.9).

**Remark 5.1** The same superconvergence estimate holds when the nonlinear coefficients in (2.3) have the form $a_i(x,u,\nabla u)$, $1 \leq i \leq n$, with the following changed assumptions:

(H1*) The functions $a_i(x,\xi)$, $x \in \Omega$, $\xi = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, are continuous in $\Omega \times \mathbb{R}^{n+1}$ and for $1 \leq i \leq n$

$$|a_i(x,\xi)| \leq C \left( \sum_{j=0}^{n} |\xi_j| \right) \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{n+1}.$$

(H2*) The derivatives $\partial^\alpha (\partial^\beta a_i)$ for $|\alpha| + |\beta| \leq k + 1$, $|\alpha| \leq k$, $|\beta| \geq 1$, $1 \leq i \leq n$, are continuous and bounded in $\Omega \times \mathbb{R}^{n+1}:

$$|\partial^\alpha (\partial^\beta a_i(x,\xi))| \leq C \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{n+1}.$$

(H3*) The following inequality holds

$$\sum_{i=1}^{n} \sum_{j=0}^{n} \frac{\partial a_i}{\partial \xi_j}(x,\xi) \eta_i \eta_j \geq \mu \sum_{j=1}^{n} \eta_j^2 \text{ for all } x \in \Omega \text{ and } \xi, \eta \in \mathbb{R}^{n+1},$$

where $\mu > 0$ is a constant independent of $x, \xi$ and $\eta$.

(H4*) The functions $\partial^\alpha a_i$, $|\alpha| = 1, 2, 3, 1 \leq i \leq n$, are continuous in $\Omega \times \mathbb{R}^{n+1}$; there exists a constant $C > 0$ such that

$$|\partial^\alpha a_i(x,\xi)| \leq C \left( \sum_{j=0}^{n} |\xi_j| \right) \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{n+1} \quad \square$$
Conclusions

In this work we develop superconvergent estimates for the tangential derivatives along edges for 2D and 3D simplex finite elements applied to nonlinear elliptic problems. The results extend previous work – in this case to include quadratic elements with curved edges and quasiuniform partitions. These superconvergent results are proved under assumptions of monotonicity of the nonlinear elliptic operator and regularity of the nonlinear coefficients and problem solution. As with the corresponding estimates for the linear case, there are some restrictions on the amount of local distortion in the mesh both with respect to deviation of curved edges away from linearity and also in the shape of adjacent elements that share a common edge or face. Further, these results can be augmented by post-processing procedures to compute continuous superconvergent gradient approximations.

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27


