NEW HDG METHODS FOR THE STOKES AND OSEEN EQUATIONS*

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Abstract. In this work, we derive new hybridized discontinuous Galerkin methods for the Stokes and Oseen equations. The schemes are based on the first order schemes defined using the velocity gradient as an auxiliary variable. For the Stokes equations, through an upwind HDG methodology, we define four HDG schemes, differing only in the definition of the numerical flux. One of the schemes uses the velocity as the trace unknown, which is related to existing methods for the velocity-pressure-gradient form of the Stokes equations. It is known that for these schemes, modifications are required to so that the local solver uniquely defines the pressure. One modification requires that the global trace system be solved iteratively, while the other modification introduces additional elementwise constant global unknowns and renders the trace system a saddle point system. Of our three new schemes, one scheme uses the *tangential* velocity and an additional scalar as trace unknowns. This scheme has the unique advantage that the HDG local solver is well-posed without modification. For the Oseen equations, we also define four upwind HDG schemes. Again, one is related to existing schemes, while the other three are new, one with the advantage of having a well-posed local solver without modification. For the advantageous schemes, we prove well-posedness, demonstrate numerical convergence, and compare the results to those of the existing schemes.

Key words. zzzFILL, zzzTHIS, zzzIN

AMS subject classifications. zzzFILL, zzzTHIS, zzzIN

1. Introduction. In this paper we propose three new hybridized discontinuous Galerkin (HDG) formulations for the Stokes equations and three new HDG formulations for the Oseen equations. The hybridization technique and post-processing have been proposed to reduce computational costs of saddle-point problems and to improve the accuracy of numerical solutions [1]. HDG methods were developed by Cockburn, coauthors, and others to mitigate the computational costs of classical discontinuous Galerkin (DG) methods. They have been proposed for various types of PDEs including, but not limited to, Poisson-type equations [7, 9, 15, 10], the Stokes equation [6, 14], the Oseen equations [5], and the incompressible Navier-Stokes equations [16].

In HDG discretizations, the coupled unknowns are single-valued traces introduced on the mesh skeleton, i.e., the faces, and for high order implicit systems the resulting matrix is substantially smaller and sparser compared to standard DG approaches. Once they are solved for, the volume DG unknowns can be recovered in an element-by-element fashion, completely independent of one another. Therefore HDG methods have an intrinsic structure for parallel computing which is essential for large scale applications. Nevertheless, devising an HDG method for coupled PDE systems is challenging because construction of a consistent and robust HDG flux is nontrivial. We adopt the upwind HDG framework proposed in [2, 4, 3] since it provides a systematic construction of HDG methods for a large class of PDEs.

In this section, we outline the basic concepts of HDG in the context of a general class of PDEs and review the upwind HDG framework [2]. The reader can refer to Appendix A for the common notation used throughout this work. Consider the

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abstract first order system of PDEs

$$\nabla \cdot \mathbf{F}(\boldsymbol{u}) + \mathbf{C}\boldsymbol{u} := \frac{\partial \boldsymbol{u}}{\partial t} + \sum_{l=1}^{d} \frac{\partial \mathbf{F}_l(\boldsymbol{u})}{\partial x_l} + \mathbf{C}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

where the vector $\mathbf{F}_l = \mathbf{A}^l \boldsymbol{u}$ is the lth component of the flux, $\boldsymbol{u} \in \mathbb{R}^m$ is the unknown solution, and $\boldsymbol{f}$ is a forcing term. For simplicity, the matrices $\mathbf{A}^l$ are assumed to be continuous across Ω.

Formally, multiplying (1.1) by an elementwise continuous test function, integrating over every element K of a finite element mesh $\mathcal{T}_h$, and integrating by parts, we have

$$-\left(\mathbf{F}(\boldsymbol{u}), \nabla \boldsymbol{v}\right)_K + \left(\mathbf{C}\boldsymbol{u}, \boldsymbol{v}\right)_K + \left\langle \mathbf{F}(\boldsymbol{u}) \cdot \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial K} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_K . \tag{1.2}$$

The boundary term $\boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{n}$ can be written as $\boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{n} = \mathbf{A}\boldsymbol{u}$, where

$$\mathbf{A} := \sum_{l=1}^{d} \mathbf{A}^l n_l. \tag{1.3}$$

The treatment of this boundary term in the numerical scheme is what differentiates HDG and traditional DG. Working now with discrete (polynomial) function spaces, replacing the boundary term by a single-valued flux that depends on the solution $\boldsymbol{u}_h$ on each side of the interface, $\mathbf{F}_h^* = \mathbf{F}_h^*(\boldsymbol{u}_h^-, \boldsymbol{u}_h^+)$ gives a steady-state DG scheme

$$-\left(\mathbf{F}(\boldsymbol{u}_h), \nabla \boldsymbol{v}\right)_K + \left(\mathbf{C}\boldsymbol{u}_h, \boldsymbol{v}\right)_K + \left\langle \mathbf{F}_h^*(\boldsymbol{u}_h^-, \boldsymbol{u}_h^+) \cdot \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial K} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_K . \tag{1.4}$$

For steady-state problems and time-dependent problems with implicit time discretization, the DG scheme (1.4) leads to a system where all the unknowns are globally coupled. Instead, to construct an HDG scheme, we introduce the trace quantity $\widehat{\boldsymbol{u}}_h$ and replace the flux on the boundary in (1.2) by a one sided HDG flux $\widehat{\mathbf{F}}_h = \widehat{\mathbf{F}}_h(\boldsymbol{u}_h^-, \widehat{\boldsymbol{u}}_h)$, which gives

$$-\left(\mathbf{F}(\boldsymbol{u}_h), \nabla \boldsymbol{v}\right)_K + \left(\mathbf{C}\boldsymbol{u}_h, \boldsymbol{v}\right)_K + \left\langle \widehat{\mathbf{F}}_h\left(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h\right) \cdot \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial K} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_K . \tag{1.5}$$

To close the system, we enforce that the normal flux is (weakly) continuous across element interfaces,

$$\left\langle \widehat{\mathbf{F}}_h\left(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h\right) \cdot \boldsymbol{n}, \widehat{\boldsymbol{v}} \right\rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0 \tag{1.6}$$

for test functions $\widehat{\boldsymbol{v}}$ that are continuous on each skeleton face (but are discontinuous at skeleton face interfaces). The HDG scheme comprises the local solver (1.5), the transmission or conservation conditions (1.6), and boundary conditions, which are prescribed through the trace unknowns on the domain boundary. The main point of the upwind HDG framework [2] is the definition of the HDG flux. The Godunov flux is traditionally written as

$$\mathbf{F}^* \cdot \boldsymbol{n}^- = \frac{1}{2}\left[\boldsymbol{F}(\boldsymbol{u}^-) + \boldsymbol{F}(\boldsymbol{u}^+)\right] \cdot \boldsymbol{n}^- + \frac{1}{2}\left|\mathbf{A}\right|\left(\boldsymbol{u}^- - \boldsymbol{u}^+\right), \tag{1.7}$$

but can also be written in terms of the upwind state $\boldsymbol{u}^*$ as

$$\mathbf{F}^* \cdot \boldsymbol{n} = \boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{n} + \left|\mathbf{A}\right|\left(\boldsymbol{u} - \boldsymbol{u}^*\right). \tag{1.8}$$

This one-sided expression of the Godunov flux leads naturally to the definition of the HDG flux by treating the upwind state $\boldsymbol{u}^*$ as an unknown $\widehat{\boldsymbol{u}}$,

$$\widehat{\mathbf{F}}_h \cdot \boldsymbol{n} = \boldsymbol{F}(\boldsymbol{u}_h) \cdot \boldsymbol{n} + |\mathbf{A}| \left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right), \tag{1.9}$$

where we have assumed that $\mathbf{A}$ admits an eigendecomposition $\mathbf{RDR}^{-1}$. Here $\mathbf{D}$ is a diagonal matrix of eigenvalues and $|\mathbf{A}| := \mathbf{R}\,|\mathbf{D}|\,\mathbf{R}^{-1}$ where $|\mathbf{D}|$ is $\mathbf{D}$ with each entry replaced with its absolute value. Thus, the upwind HDG framework provides a unified methodology by which to derive parameter-free HDG schemes by hybridizing the Godunov flux. We refer the reader to [2] for more details. It may appear that we have m trace variables that must be solved for, but we can reduce the number of trace unknowns when we consider each PDE specifically, as will be demonstrated in sections 2 and 3.

For linear systems, the HDG scheme (1.5) and (1.6) gives rise to the following matrix equations, where $\mathbb{U}$ represents the vector degrees of freedom of $\boldsymbol{u}_h$, and $\widehat{\mathbb{U}}$ represents the vector degrees of freedom of $\widehat{\boldsymbol{u}}_h$,

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix}^* \begin{Bmatrix} \mathbb{U} \\ \widehat{\mathbb{U}} \end{Bmatrix} = \begin{Bmatrix} \mathbb{F}_l \\ \mathbb{F}_g \end{Bmatrix}. \tag{1.10}$$

Here, the subscripts l and g stand for local and global, respectively. Nonzero terms in $\mathbb{F}_g$ may result, for example, depending on the boundary conditions and how they are enforced.

The power of HDG comes from the following.

- The HDG flux is one-sided, i.e., for a given element, the flux depends only on the solution in that element and the neighboring skeleton faces. Together with the fact that the discontinuous basis functions are local to one element, this implies that $\mathbb{A}$ is *block diagonal*.
- If the local solver $(\widehat{\boldsymbol{u}}_h, \boldsymbol{f}) \mapsto \boldsymbol{u}_h$ given by (1.5) is well-posed, then $\mathbb{A}$ is *invertible*.

A consequence of these two points is that we can easily eliminate $\mathbb{U}$ from (1.10) by a static condensation procedure, and write

$$\mathbb{U} = \mathbb{A}^{-1}\left[\mathbb{F}_l - \mathbb{B}\widehat{\mathbb{U}}\right]. \tag{1.11}$$

The global system (1.10) then reduces to

$$\underbrace{\left(\mathbb{D} - \mathbb{C}\left[\mathbb{A}\right]^{-1}\mathbb{B}\right)}_{\mathbb{K}}\widehat{\mathbb{U}} = \underbrace{\mathbb{F}_g - \mathbb{C}\left[\mathbb{A}\right]^{-1}\mathbb{F}_l}_{\mathbb{F}}. \tag{1.12}$$

In practice, $\mathbb{K}$ and $\mathbb{F}$ are formed by a local assembly procedure, $\widehat{\mathbb{U}}$ is solved for from the reduced global system (1.12), and then $\mathbb{U}$ is recovered in an element by element fashion from (1.11).

2. Stokes Equations.

In this section, we construct HDG methods for the Stokes equations based on the upwind HDG framework proposed in [2]. The HDG methods are based on the first order Stokes system defined through an auxiliary variable based on the velocity gradient. Through the use of this framework, we derive four different HDG schemes. One of the schemes is related to or is precisely the one defined in [14, 2]. The other schemes are new in this work. We prove well-posedness of two schemes that seem to be particularly useful, and present numerical results for these two schemes, showing that they give practically identical results.

2.1. Construction of Upwind HDG Schemes. For notation used in this section and throughout this work, see Appendix A. The Stokes equations in dimensionless form read

$$-\frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1b}$$

where $\mathrm{Re} := \frac{\rho u_0 l_0}{\mu}$ is the Reynolds number, ρ is the fluid density, u_0 is a characteristic speed, l_0 is a characteristic length scale, and μ is the dynamic viscosity of the fluid. All parameters are assumed to be constant. We consider the boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \partial\Omega_D, \tag{2.2a}$$

$$-\frac{1}{\mathrm{Re}}\nabla u \cdot \boldsymbol{n} + p\boldsymbol{n} = \boldsymbol{f}_N \quad \text{on } \partial\Omega_N, \tag{2.2b}$$

where $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$. In the case that $\partial\Omega_N = \emptyset$, the compatibility condition on the Dirichlet boundary data $\int\limits_{\partial\Omega} \boldsymbol{u}_D \cdot \boldsymbol{n} = 0$ should be satisfied, and we have to impose an additional constraint on the pressure. We choose this constraint to be the zero mean pressure $\int\limits_{\Omega} p = 0$. For simplicity, we consider the case where $\partial\Omega_D \neq \emptyset$.

Toward applying the upwind HDG framework outlined in [2], we first put (2.1) into first order form through the definition of an auxiliary variable. We have multiple choices as to how to define the auxiliary variable, leading to different HDG formulations. In this work, we define the auxiliary variable $\mathbf{L}$ through the velocity gradient, leading to a velocity-gradient-pressure formulation:

$$\mathrm{Re}\mathbf{L} - \nabla \boldsymbol{u} = 0, \tag{2.3a}$$

$$-\nabla \cdot \mathbf{L} + \nabla p = \boldsymbol{f}, \tag{2.3b}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.3c}$$

To define a general HDG scheme for the Stokes equations, we multiply (2.3) by a test function, integrate over the computational domain, integrate by parts, replace the boundary terms with a not-necessarily-single-valued HDG flux, then weakly enforce the single valuedness of the HDG flux. HDG schemes defined in this manner for (2.3) will take a general form consisting of the local equations

$$\mathrm{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \boldsymbol{u}_h^* \otimes \boldsymbol{n}, \mathbf{G}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4a}$$

$$\left(\mathbf{L}_h, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(p_h, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle -\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h}, \tag{2.4b}$$

$$-\left(\boldsymbol{u}_h, \nabla q\right)_{\mathcal{T}_h} + \left\langle \boldsymbol{u}_h^* \cdot \boldsymbol{n}, q\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4c}$$

the conservation equations

$$\left\langle \boldsymbol{u}_h^* \otimes \boldsymbol{n}, \widehat{\mathbf{G}}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.4d}$$

$$-\left\langle -\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.4e}$$

$$-\left\langle \boldsymbol{u}_h^* \cdot \boldsymbol{n}, \widehat{q}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.4f}$$

and the boundary conditions

$$\left\langle \boldsymbol{u}_h^*, \widehat{\boldsymbol{w}}\right\rangle_{\partial\Omega_D} = \left\langle \boldsymbol{u}_D, \widehat{\boldsymbol{w}}\right\rangle_{\partial\Omega_D}, \tag{2.4g}$$

$$\left\langle -\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n}, \widehat{\boldsymbol{w}}\right\rangle_{\partial\Omega_N} = \left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{w}}\right\rangle_{\partial\Omega_N}. \tag{2.4h}$$

In all of the HDG schemes we will derive, the discontinuous polynomial spaces in which we seek the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ and to which their corresponding test functions $(\mathbf{G}, \boldsymbol{v}, q)$ belong are as follows:

$$\mathbf{G}_h := \left\{\mathbf{G} \in \left[L^2(\Omega)\right]^{d\times d} \,:\, \mathbf{G}|_K \in \mathbf{G}_h(K)\right\}, \tag{2.5a}$$

$$\boldsymbol{V}_h := \left\{\boldsymbol{v} \in \left[L^2(\Omega)\right]^{d} \,:\, \boldsymbol{v}|_K \in \boldsymbol{V}_h(K)\right\}, \tag{2.5b}$$

$$Q_h := \left\{q \in L^2(\Omega) \,:\, q|_K \in Q_h(K)\right\}, \tag{2.5c}$$

where $\mathbf{G}_h(K)$, $\boldsymbol{V}_h(K)$, $Q_h(K)$ are total-degree or tensor-product finite element spaces defined on K that we assume to be of equal polynomial order $k \geq 1$.

The quantities $\boldsymbol{u}_h^*$ and $-\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n}$ are yet-to-be-defined, not-necessarily-single-valued numerical fluxes, which are function of the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ and trace variables $\left(\widehat{\mathbf{L}}_h, \widehat{\boldsymbol{u}}_h, \widehat{p}_h\right)$. The trace variables reside in discontinuous polynomial spaces defined on the mesh skeleton, as do the interior test functions $\left(\widehat{\mathbf{G}}, \widehat{\boldsymbol{v}}, \widehat{q}\right)$ and boundary test function $\widehat{\boldsymbol{w}}$. In what follows, we derive different choices for $\boldsymbol{u}_h^*$ and $-\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n}$ and analyze schemes that result from some specific choices. The fluxes we derive will have a minimal number of trace unknowns (d scalar unknowns) so that not all of the trace unknowns $\left(\widehat{\mathbf{L}}_h, \widehat{\boldsymbol{u}}_h, \widehat{p}_h\right)$ (and their corresponding test functions) will exist as unknowns (and test functions). Related to this is the fact that not all of the conservation equations (2.4d)–(2.4f) must be explicitly enforced, as some will be automatically satisfied depending on the choice of the numerical flux. Additionally, the boundary test function $\widehat{\boldsymbol{w}}$ will have a natural association with the interior skeleton test functions among $\left(\widehat{\mathbf{G}}, \widehat{\boldsymbol{v}}, \widehat{q}\right)$ that do exist in the scheme. These points will be made clearer after we derive the HDG numerical fluxes.

The first order system (2.3) fits into the general framework (1.1), and is symmetric hyperbolic. Indeed, choosing the ordering of unknowns as the column vector $\boldsymbol{U} := (\mathrm{vec}\,(\mathbf{L})\,; \boldsymbol{u}; p)$, we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{n}\otimes_K\mathbf{I} & \mathbf{0} \\ -\boldsymbol{n}^\top\otimes_K\mathbf{I} & \mathbf{0} & \boldsymbol{n} \\ \mathbf{0} & \boldsymbol{n}^\top & 0 \end{bmatrix}. \tag{2.6}$$

We can perform the eigendecomposition $\mathbf{A} = \mathbf{R}\mathbf{D}\mathbf{R}^{-1}$, where $\mathbf{D}$ is a diagonal matrix comprising the eigenvalues of $\mathbf{A}$, and $\mathbf{R}$ is a matrix whose columns are the eigenvectors corresponding those eigenvalues. Defining $|\mathbf{D}|$ by taking the absolute value of each eigenvalue in $\mathbf{D}$, we can define $|\mathbf{A}| := \mathbf{R}\,|\mathbf{D}|\,\mathbf{R}^{-1}$. It can be shown that for the Stokes system we have

$$|\mathbf{A}| = \begin{bmatrix} \mathbf{N}\otimes_K\left(\frac{1}{\tau_t^S}\mathbf{T} + \frac{1}{\tau_n^S}\mathbf{N}\right) & \mathbf{0} & -\frac{1}{\tau_n^S}\boldsymbol{n}\otimes_K\boldsymbol{n} \\ \mathbf{0} & \tau_t^S\mathbf{T} + \tau_n^S\mathbf{N} & \mathbf{0} \\ -\frac{1}{\tau_n^S}\boldsymbol{n}^\top\otimes_K\boldsymbol{n}^\top & \mathbf{0} & \frac{1}{\tau_n^S} \end{bmatrix}, \tag{2.7}$$

where $\tau_t^S := 1$ and $\tau_n^S := \sqrt{2}$. Later, we will consider more general parameters τ_t and τ_n than τ_t^S and τ_n^S which give the upwind flux. This allows us to generalize the upwind scheme, to define simpler schemes, and to make connections to existing HDG methods. We define the normal upwind flux $\boldsymbol{F}_n^*$ as a column vector

$\boldsymbol{F}_n^* := (\mathrm{vec}\,(-\boldsymbol{u}^* \otimes \boldsymbol{n})\,; -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}; \boldsymbol{u}^* \cdot \boldsymbol{n})$. Since there is a one-to-one correspondence between $\mathrm{vec}\,(-\boldsymbol{u}^* \otimes \boldsymbol{n})$ and $-\boldsymbol{u}^* \otimes \boldsymbol{n}$, we also identify $\boldsymbol{F}_n^*$ with the triple

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\boldsymbol{u}^* \otimes \boldsymbol{n} \\ -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} \\ \boldsymbol{u}^* \cdot \boldsymbol{n} \end{bmatrix}. \tag{2.8}$$

In this way, we can write the exact upwind flux in its one-sided form, $\boldsymbol{F}_n^* = \mathbf{A}\boldsymbol{U} + |\mathbf{A}|\,(\boldsymbol{U} - \boldsymbol{U}^*)$, as

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\boldsymbol{u} \otimes \boldsymbol{n} + \left(\frac{1}{\tau_t^S}\mathbf{T} + \frac{1}{\tau_n^S}\mathbf{N}\right)(\mathbf{L} - \mathbf{L}^*)\,\mathbf{N} - \frac{1}{\tau_n^S}(p - p^*)\,\mathbf{N} \\ -\mathbf{L}\boldsymbol{n} + p\boldsymbol{n} + \left(\tau_t^S\mathbf{T} + \tau_n^S\mathbf{N}\right)(\boldsymbol{u} - \boldsymbol{u}^*) \\ \boldsymbol{u} \cdot \boldsymbol{n} - \frac{1}{\tau_n^S}\boldsymbol{n} \cdot [(\mathbf{L} - \mathbf{L}^*)\,\boldsymbol{n}] + \frac{1}{\tau_n^S}(p - p^*) \end{bmatrix}. \tag{2.9}$$

At this point, we can eliminate "starred quantities" from the right side of (2.9) with the aim of defining an HDG flux with minimal trace unknowns. It turns out that we can do so in a way that naturally leads to four different forms of the upwind flux, each with d scalar starred quantities. The key to reducing the number of trace unknowns is the following relations between the upwind states.

LEMMA 2.1. *The following relationships between the upwind states hold:*

$$\tau_t^S\mathbf{T}\,(\boldsymbol{u} - \boldsymbol{u}^*) = \mathbf{T}\,(\mathbf{L} - \mathbf{L}^*)\,\boldsymbol{n}, \tag{2.10a}$$

$$\tau_n^S\mathbf{N}\,(\boldsymbol{u} - \boldsymbol{u}^*) = -\mathbf{N}\,[-(\mathbf{L} - \mathbf{L}^*)\,\boldsymbol{n} + (p - p^*)\,\boldsymbol{n}]\,. \tag{2.10b}$$

Proof. The claims follow directly from equating the tangential components of the left and right sides of the second term of (2.9), and doing the same for the normal components. ∎

Note that we arrive at the same expressions by equating the left and right sides of the first term of (2.9). Equating the third term gives the expression (2.10b). That is to say that (2.10a) and (2.10b) are the only two relations we can discover from (2.9).

Using (2.10a) to eliminate either $\mathbf{T}\boldsymbol{u}^*$ or $\mathbf{T}\mathbf{L}^*\boldsymbol{n}$, and using (2.10b) to eliminate either $\mathbf{N}\boldsymbol{u}^*$ or $\mathbf{N}\,(-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n})$, we arrive at the following four forms of the upwind flux.

The $\boldsymbol{u}^*$ flux: The quantity $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$ can be eliminated from (2.9) so that (2.9) can be written as

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\boldsymbol{u}^* \otimes \boldsymbol{n} \\ -\mathbf{L}\boldsymbol{n} + p\boldsymbol{n} + \left(\tau_t^S\mathbf{T} + \tau_n^S\mathbf{N}\right)(\boldsymbol{u} - \boldsymbol{u}^*) \\ \boldsymbol{u}^* \cdot \boldsymbol{n} \end{bmatrix}. \tag{2.11}$$

The $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$ flux: The quantity $\boldsymbol{u}^*$ can be eliminated from (2.9) so that (2.9) can be written as

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\boldsymbol{u} \otimes \boldsymbol{n} + \left(\frac{1}{\tau_t^S}\mathbf{T} + \frac{1}{\tau_n^S}\mathbf{N}\right)(\mathbf{L} - \mathbf{L}^*)\,\mathbf{N} - \frac{1}{\tau_n^S}(p - p^*)\,\mathbf{N} \\ -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} \\ \boldsymbol{u} \cdot \boldsymbol{n} - \frac{1}{\tau_n^S}\boldsymbol{n} \cdot [(\mathbf{L} - \mathbf{L}^*)\,\boldsymbol{n}] + \frac{1}{\tau_n^S}(p - p^*) \end{bmatrix}. \tag{2.12}$$

The $(\mathbf{T}\boldsymbol{u}^*, f^*)$ flux: The quantities $\mathbf{T}\mathbf{L}^*\boldsymbol{n}$ and $\mathbf{N}\boldsymbol{u}^*$ can be eliminated from (2.9) so that (2.9) can be written as

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\mathbf{N}\boldsymbol{u} \otimes \boldsymbol{n} - \mathbf{T}\boldsymbol{u}^* \otimes \boldsymbol{n} - \frac{1}{\tau_n^S}(-\boldsymbol{n} \cdot [\mathbf{L}\boldsymbol{n}] + p - f^*)\,\mathbf{N} \\ -\mathbf{T}\,(\mathbf{L}\boldsymbol{n}) + f^*\boldsymbol{n} + \tau_t^S\mathbf{T}\,(\boldsymbol{u} - \boldsymbol{u}^*) \\ \boldsymbol{u} \cdot \boldsymbol{n} + \frac{1}{\tau_n^S}(-\boldsymbol{n} \cdot [\mathbf{L}\boldsymbol{n}] + p - f^*) \end{bmatrix}, \tag{2.13}$$

where $f^* := -\boldsymbol{n} \cdot [\mathbf{L}^* \boldsymbol{n}] + p^*$.

The $(\mathbf{N}\boldsymbol{u}^*, \mathbf{T}\mathbf{L}^*\boldsymbol{n})$ flux: The quantities $\mathbf{N}(-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n})$ and $\mathbf{T}\boldsymbol{u}^*$ can be eliminated from (2.9) so that (2.9) can be written as

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\mathbf{N}\boldsymbol{u}^* \otimes \boldsymbol{n} - \mathbf{T}\boldsymbol{u} \otimes \boldsymbol{n} - \frac{1}{\tau_t^S}\mathbf{T}(-\mathbf{L} + \mathbf{L}^*)\mathbf{N} \\ (-\boldsymbol{n} \cdot [\mathbf{L}\boldsymbol{n}] + p)\boldsymbol{n} + \mathbf{T}(-\mathbf{L}^*\boldsymbol{n}) + \tau_n^S \mathbf{N}(\boldsymbol{u} - \boldsymbol{u}^*) \\ \boldsymbol{u}^* \cdot \boldsymbol{n} \end{bmatrix}. \tag{2.14}$$

Finally, in order to define numerical fluxes

$$\boldsymbol{F}_{n,h}^* := \begin{bmatrix} -\boldsymbol{u}_h^* \otimes \boldsymbol{n} \\ -\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n} \\ \boldsymbol{u}_h^* \cdot \boldsymbol{n} \end{bmatrix} \tag{2.15}$$

to be used in the HDG scheme (2.4), we append a subscript h to the terms in (2.11)–(2.14) and replace the starred quantities on the right side of (2.11)–(2.14) with hatted unknown quantities residing on the mesh skeleton. Additionally we replace τ_t^S and τ_n^S with τ_t and τ_n, which, from the well-posedness analysis, can be freely chosen positive values. This gives the following numerical fluxes.

The $\widehat{\boldsymbol{u}}_h$ flux:

$$\boldsymbol{F}_{n,h}^* := \begin{bmatrix} -\widehat{\boldsymbol{u}}_h \otimes \boldsymbol{n} \\ -\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + (\tau_t\mathbf{T} + \tau_n\mathbf{N})(\boldsymbol{u} - \widehat{\boldsymbol{u}}_h) \\ \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n} \end{bmatrix}. \tag{2.16}$$

The $\widehat{\boldsymbol{f}}_h$ flux (where $\widehat{\boldsymbol{f}}_h$ approximates $-\mathbf{L}^*\tilde{\boldsymbol{n}} + p^*\tilde{\boldsymbol{n}}$):

$$\boldsymbol{F}_{n,h}^* := \begin{bmatrix} -\left(\boldsymbol{u}_h + \left(\frac{1}{\tau_t}\mathbf{T} + \frac{1}{\tau_n}\mathbf{N}\right)\left(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} - \mathrm{sgn}\widehat{\boldsymbol{f}}_h\right)\right) \otimes \boldsymbol{n} \\ \mathrm{sgn}\widehat{\boldsymbol{f}}_h \\ \boldsymbol{u}_h \cdot \boldsymbol{n} + \frac{1}{\tau_n}\left(-\boldsymbol{n} \cdot [\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{\boldsymbol{f}}_h \cdot \tilde{\boldsymbol{n}}\right) \end{bmatrix}. \tag{2.17}$$

The $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (where $\widehat{f}_h$ approximates $-\boldsymbol{n} \cdot [\mathbf{L}^*\boldsymbol{n}] + p^*$):

$$\boldsymbol{F}_{n,h}^* := \begin{bmatrix} -\left(\left(\widehat{\boldsymbol{u}}_h^t + \mathbf{N}\boldsymbol{u}_h\right) + \frac{1}{\tau_n}\left(-\boldsymbol{n} \cdot [\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{f}_h\right)\boldsymbol{n}\right) \otimes \boldsymbol{n} \\ \widehat{f}_h\boldsymbol{n} - \mathbf{T}\mathbf{L}_h\boldsymbol{n} + \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right) \\ \boldsymbol{u}_h \cdot \boldsymbol{n} + \frac{1}{\tau_n}\left(-\boldsymbol{n} \cdot [\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{f}_h\right) \end{bmatrix}. \tag{2.18}$$

The $(\widehat{u}_h^{\tilde{n}}, \widehat{\boldsymbol{f}}_h^t)$ flux (where $\widehat{\boldsymbol{f}}_h^t$ approximates $-\mathbf{T}\mathbf{L}^*\tilde{\boldsymbol{n}}$):

$$\boldsymbol{F}_{n,h}^* := \begin{bmatrix} -\left(\widehat{u}_h^{\tilde{n}}\tilde{\boldsymbol{n}} + \boldsymbol{u}_h^t + \frac{1}{\tau_t}\left(-\mathbf{T}\mathbf{L}_h\boldsymbol{n} - \mathrm{sgn}\widehat{\boldsymbol{f}}_h^t\right)\right) \otimes \boldsymbol{n} \\ \mathrm{sgn}\widehat{\boldsymbol{f}}_h^t + \mathbf{N}(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n}) + \tau_n\left(\mathbf{N}\boldsymbol{u}_h - \widehat{u}_h^{\tilde{n}}\tilde{\boldsymbol{n}}\right) \\ \mathrm{sgn}\widehat{u}_h^{\tilde{n}} \end{bmatrix}. \tag{2.19}$$

It can be shown that any of the fluxes (2.16)–(2.19) are suitable for use in the HDG scheme (2.4), some being more practical than others. It should also be noted that it is not necessary to use the same flux on all skeleton faces. It may be convenient to use one flux on the skeleton faces that are on the interior of the computational domain and a different flux for each part of the boundary corresponding to a different boundary condition. For example, the $\widehat{\boldsymbol{u}}_h$ flux (2.16) can be used to directly prescribe Dirichlet boundary conditions of type (2.2a), the $\widehat{\boldsymbol{f}}_h$ flux (2.17) can be used to directly prescribe boundary conditions of type (2.2b), and the $(\widehat{u}_h^{\tilde{n}}, \widehat{\boldsymbol{f}}_h^t)$ flux (2.19) can be used to directly prescribe the conditions for "mirror" symmetry boundary conditions. If it is possible to treat the boundary conditions in this manner, all boundary skeleton unknowns decouple from the interior skeleton unknowns, thereby keeping the number of coupled unknowns in the system to a minimum.

Recall that in order to realize one of the advantages of HDG, the volume unknowns must be uniquely defined by the trace unknowns; that is, the local solver must be well posed. It can be shown that, without modifications, schemes using (2.16) and (2.19) only define the pressure p_h up to a constant. Similarly, (2.17) only defines the velocity $\widehat{\boldsymbol{u}}_h$ up to constant. On the other hand, (2.18) defines the all of the volume unknowns uniquely. In the following sections, we explicitly define schemes based on $\widehat{\boldsymbol{u}}_h$ flux (2.16) and modifications that ensure uniqueness of the local solver. This is the "standard" flux for the velocity gradient based HDG scheme for the Stokes equations. We also define a new scheme based on the flux (2.18) that requires no modifications for well-posedness of the local solver. We do not pursue HDG schemes based on (2.17) and (2.19), as they do not appear to offer benefits compared to the other schemes.

2.2. HDG Schemes Using the $\widehat{\boldsymbol{u}}_h$ Flux. In this section, we define an upwind HDG scheme based on (2.16), which recovers schemes developed in [6, 2]. For the sake of this discussion, we use (2.16) on all skeleton faces. The discontinuous polynomial space in which we seek the trace unknowns $\widehat{\boldsymbol{u}}_h$ is

$$\widehat{\boldsymbol{V}}_h := \left\{ \widehat{\boldsymbol{v}} \in \left[L^2(\mathcal{E}_h)\right]^d \ : \ \widehat{\boldsymbol{v}}|_e \in \widehat{\boldsymbol{V}}_h(e) \right\}, \tag{2.20}$$

where $\widehat{\boldsymbol{V}}_h(e)$ is a polynomial space defined on e that is assumed to be of the same polynomial order k as the volume unknowns.

With the numerical flux (2.16), the enforcement of the Dirichlet boundary condition (2.4g) simplifies to an L^2 projection of the Dirichlet boundary data to the trace unknown on $\partial\Omega_D$, thereby decoupling the trace unknowns on $\partial\Omega_D$ from the rest of the unknowns. Then we can decompose the trace unknown

$$\widehat{\boldsymbol{u}}_h = \widehat{\boldsymbol{u}}_h^i + \widehat{\boldsymbol{u}}_h^D \tag{2.21}$$

where $\widehat{\boldsymbol{u}}_h^D$ is defined on $\partial\Omega_D$ as the L^2 projection of the boundary data,

$$\left\langle \widehat{\boldsymbol{u}}_h^D, \widehat{\boldsymbol{v}} \right\rangle_{\partial\Omega_D} = \langle \boldsymbol{u}_D, \widehat{\boldsymbol{v}} \rangle_{\partial\Omega_D} \quad \text{for all } \widehat{\boldsymbol{v}} \in \widehat{\boldsymbol{V}}_h(e) \text{ for all } e \in \partial\Omega_D, \tag{2.22}$$

and $\widehat{\boldsymbol{u}}_h^i$ is the trace unknown $\widehat{\boldsymbol{u}}_h$ restricted to $\mathcal{E}_h \backslash \partial\Omega_D$. Note that in writing (2.21) we identify $\widehat{\boldsymbol{u}}_h^i$ and $\widehat{\boldsymbol{u}}_h^D$ with their extensions by zero to $\mathcal{E}_h$. Then $\widehat{\boldsymbol{u}}_h^i$ resides in the polynomial space

$$\widehat{\boldsymbol{V}}_h^i := \left\{ \widehat{\boldsymbol{v}} \in \left[L^2(\mathcal{E}_h \backslash \partial\Omega_D)\right]^d \ : \ \widehat{\boldsymbol{v}}|_e \in \widehat{\boldsymbol{V}}_h(e) \right\}. \tag{2.23}$$

With this in place, we write the HDG scheme as follows.

Formulation 2.2. Find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i$ such that the local equations

$$\mathrm{Re}\,(\mathbf{L}_h, \mathbf{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla\cdot\mathbf{G})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \mathbf{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.24a}$$

$$-(\nabla\cdot\mathbf{L}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} + \langle \mathbf{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.24b}$$

$$-(\boldsymbol{u}_h, \nabla q)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.24c}$$

and the conservation equation and Neumann boundary condition

$$-\langle -\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \mathbf{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = -\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\rangle_{\partial\Omega_N} \tag{2.24d}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}^i_h$, where

$$\mathbf{S} := \tau_t\mathbf{T} + \tau_n\mathbf{N}, \tag{2.25}$$

and $\widehat{\boldsymbol{u}}^D_h$ is defined by (2.22). If $\partial\Omega_N = \emptyset$, we additionally require the zero mean pressure conditions for the uniqueness of the pressure

$$(p_h, 1)_{\mathcal{T}_h} = 0. \tag{2.26}$$

Some comments are in order. First, using the flux (2.16), the conservation conditions (2.4d) and (2.4f) are automatically satisfied, and so we do not need to explicitly include these equations in the formulation. Second, the conservation condition (2.4e) and the Neumann boundary condition (2.4h) (where we associate $\widehat{\boldsymbol{w}}$ with $\widehat{\boldsymbol{v}}$) are combined in (2.24d). Third, we have integrated by parts the terms in (2.4e) in order to write the scheme in a concise manner that reveals the symmetric and skew symmetric terms. Finally, it is not necessary to decompose $\widehat{\boldsymbol{u}}_h$ into the coupled "interior" unknowns and the decoupled Dirichlet boundary unknowns in (2.24a)–(2.24c). We can recouple (2.22) to the rest of the system, but that would change the matrix structure of the trace system that we comment on in the following discussions.

In the following, we discuss the well-posedness of Formulation 2.2.

THEOREM 2.3. *(well-posedness of Formulation 2.2)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$ (which is true in particular for $\tau_t = \tau^S_t$ and $\tau_n = \tau^S_n$). Then Formulation 2.2 is well-posed in the sense that given $\boldsymbol{f}$, $\boldsymbol{u}_D$, and $\boldsymbol{f}_N$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h$.*

Proof. It is sufficient to prove that if $\boldsymbol{f}$, $\boldsymbol{u}_D$, and $\boldsymbol{f}_N$ are zero, then the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h)$ is zero. We can rewrite Formulation 2.2 as: find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}^i_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}^i_h$ such that

$$\begin{aligned}
&a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}^i_h\right), (\mathbf{G}, \boldsymbol{v}, \widehat{\boldsymbol{v}})\right)\\
&\quad + a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}^i_h\right), (\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})\right) = l\,(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}^i_h$, where

$$\begin{aligned}
a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}^i_h\right), (\mathbf{G}, \boldsymbol{v}, \widehat{\boldsymbol{v}})\right) &= \mathrm{Re}\,(\mathbf{L}_h, \mathbf{G})_{\mathcal{T}_h} + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{v}\rangle_{\partial\Omega_D}\\
&\quad + \left\langle \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}^i_h\right), \boldsymbol{v} - \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D},
\end{aligned}$$

$$a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i\right), (\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})\right) = (\boldsymbol{u}_h, \nabla\cdot\mathbf{G})_{\mathcal{T}_h} - (\nabla\cdot\mathbf{L}_h, \boldsymbol{v})_{\mathcal{T}_h}$$
$$+ (\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla q)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h^i, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} + \langle \mathbf{L}_h\boldsymbol{n}, \widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D}$$
$$+ \left\langle \widehat{\boldsymbol{u}}_h^i\cdot\boldsymbol{n}, q\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \langle p_h, \widehat{\boldsymbol{v}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D},$$

and

$$l(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}) = \left\langle \widehat{\boldsymbol{u}}_h^D, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\Omega_D} + (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}$$
$$+ \left\langle \mathbf{S}\widehat{\boldsymbol{u}}_h^D, \boldsymbol{v}\right\rangle_{\partial\Omega_D} - \left\langle \widehat{\boldsymbol{u}}_h^D\cdot\boldsymbol{n}, q\right\rangle_{\partial\Omega_D} - \langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\rangle_{\partial\Omega_N}.$$

Setting $\boldsymbol{f} = \boldsymbol{0}$, $\boldsymbol{u}_D = \boldsymbol{0}$ (and therefore $\widehat{\boldsymbol{u}}_h^D = \boldsymbol{0}$), and $\boldsymbol{f}_N = \boldsymbol{0}$ gives $l = 0$. Setting $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}) = \left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i\right)$ gives $a_{skew} = 0$ leaving only the symmetric terms,

$$\operatorname{Re}(\mathbf{L}_h, \mathbf{L}_h)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h^i\right), \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h^i\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{u}_h\rangle_{\partial\Omega_D} = 0. \tag{2.27}$$

All of the terms in the previous expression are nonnegative and as a consequence must be zero. Thus $\mathbf{L}_h = \boldsymbol{0}$ in $\mathcal{T}_h$, $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h$ on $\mathcal{E}_h\setminus\partial\Omega_D$, and $\boldsymbol{u}_h = \boldsymbol{0}$ on $\partial\Omega_D$.

Integration by parts reveals that equation (2.24a) reduces to $(\nabla u_h, \mathbf{G})_{\mathcal{T}_h} = 0$ and since $\nabla\boldsymbol{V}_h \subset \mathbf{G}_h$, we set $\mathbf{G} = \nabla u_h$ to conclude that u_h is elementwise constant. But since $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h$ on $\mathcal{E}_h^o$ and $\widehat{\boldsymbol{u}}_h$ is single valued on $\mathcal{E}_h^o$, $\boldsymbol{u}_h$ is continuous across each internal interface, and therefore $\boldsymbol{u}_h$ is globally constant. Since $\widehat{\boldsymbol{u}}_h$ is zero on $\partial\Omega_D$ we conclude $\boldsymbol{u}_h = \boldsymbol{0}$ and $\widehat{\boldsymbol{u}}_h = \boldsymbol{0}$.

Then (2.24b) reduces to $(\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} = 0$, and since $\nabla Q_h \subset \boldsymbol{V}_h$, we can conclude p_h is elementwise constant. Since (2.24d) reduces to $\langle p_h\boldsymbol{n}, \widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega}$ for $\widehat{\boldsymbol{v}}$ with support on $\mathcal{E}_h^o$, then p_h is globally continuous and globally constant. In the case that $\partial\Omega_N \neq \emptyset$, we have $\langle p_h\boldsymbol{n}, \widehat{\boldsymbol{v}}\rangle_{\partial\Omega_N} = 0$ implies that $p_h = 0$ on $\partial\Omega_N$ and therefore that $p_h = 0$ everywhere. Otherwise the zero mean discrete pressure condition (2.26) implies p_h is zero. ∎

We next prove that the local solver, (2.24a)–(2.24c), in Formulation 2.2 determines the local pressure p_h only up to an elementwise constant.

THEOREM 2.4. *(well-posedness of the local solver of Formulation 2.2) Suppose that $\tau_t > 0$ and $\tau_n > 0$. Given $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h/\mathcal{P}_0(\mathcal{T}_h)$ to the local equations (2.24a)–(2.24c).*

Proof. It is sufficient to restrict our attention to a single element, and prove that if $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$ are zero, then the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ is zero. We can rewrite the local solver defined by (2.24a)–(2.24c) restricted to one element as find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$ such that

$$\begin{aligned}\operatorname{Re}(\mathbf{L}_h, \mathbf{G})_K + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{v}\rangle_{\partial K} + (\boldsymbol{u}_h, \nabla\cdot\mathbf{G})_K - (\nabla\cdot\mathbf{L}_h, \boldsymbol{v})_K + (\nabla p_h, \boldsymbol{v})_K \\ - (\boldsymbol{u}_h, \nabla q)_K = (\boldsymbol{f}, \boldsymbol{v})_K + \langle \mathbf{S}\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\rangle_{\partial K} + \langle \widehat{\boldsymbol{u}}_h, \mathbf{G}\boldsymbol{n}\rangle_{\partial K} - \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q\rangle_{\partial K}\end{aligned} \tag{2.28}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$. Setting $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$ to zero, and setting $(\mathbf{G}, \boldsymbol{v}, q) = (\mathbf{L}_h, \boldsymbol{u}_h, p_h)$, we have

$$\operatorname{Re}(\mathbf{L}_h, \mathbf{L}_h)_K + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{u}_h\rangle_{\partial K} = 0. \tag{2.29}$$

Thus $\mathbf{L}_h = \mathbf{0}$ in K and $\boldsymbol{u}_h = \mathbf{0}$ on ∂K.

Integrating by parts what remains of (2.24a) gives that $\boldsymbol{u}_h$ is constant in K, and since $\boldsymbol{u}_h = \mathbf{0}$ on ∂K, that $\boldsymbol{u}_h = \mathbf{0}$ in K. Integrating (2.24b) by parts gives that p_h is constant in K. ◻

2.3. Modifications for Local Solver Invertibility. As we saw in the previous section, given $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$, the local solver (2.24a)–(2.24c) of the HDG Formulation 2.2 does not uniquely define the pressure p_h in Q_h. The reason for this can be seen as follows. It is known that the Stokes equations with only Dirichlet boundary conditions must be equipped with an additional condition on the pressure, usually the zero mean pressure condition, in order to be well-posed. The local solver of Formulation 2.2 can be interpreted as solving the Dirichlet problem on each element with $\widehat{\boldsymbol{u}}_h$ as the boundary data. From what we know about the Dirichlet problem for the Stokes equations, we could not have expected that this local problem would be well-posed. An HDG scheme whose local (element) problem is not well-posed is not particularly useful, as it loses one of the main advantages of HDG methods as compared to DG methods – the ability to condense the volume (DG) unknowns out of the global linear system to have a resulting global system with a reduced number of unknowns. Therefore, Formulation 2.2 must be modified in order to be useful.

There are two methods in the literature for addressing this issue [14]. One method is a direct method that involves the introduction of additional global unknowns. The other method is an iterative method, involving pseudotime, that does not change the number of unknowns. We review those methods here before introducing a new method in the next section that uses a different form of the HDG flux to avoid this issue all together.

2.3.1. The Augmented Lagrangian Approach. The Augmented Lagrangian approach for Stokes HDG schemes introduced in [14]. It is described by adding a pseudotime derivative to (2.3c) as

$$\frac{\partial p}{\partial \tau} + \nabla \cdot \boldsymbol{u} = 0, \tag{2.30}$$

providing an initial condition $p(\tau = 0) = p_0$, then solving for the steady state solution with an HDG spatial discretization of (2.3a), (2.3b), and (2.30), with an implicit Euler temporal discretization, and with the choice of $p_0 = 0$. Altering Formulation 2.2 in such a manner, we have the following formulation describing a single pseudotime step.

Formulation 2.5. Find $(\mathbf{L}_h^k, \boldsymbol{u}_h^k, p_h^k, \widehat{\boldsymbol{u}}_h^{i,k})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i$ such that the local equations

$$\operatorname{Re}\left(\mathbf{L}_h^k, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^k, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h^k, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} = 0, \tag{2.31a}$$

$$-\left(\nabla \cdot \mathbf{L}_h^k, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^k, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^k - \widehat{\boldsymbol{u}}_h^k\right), \boldsymbol{v} \right\rangle_{\partial \mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.31b}$$

$$\frac{1}{\Delta \tau}\left(p_h^k, q\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h^k, \nabla q\right)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{u}}_h^k \cdot \boldsymbol{n}, q \right\rangle_{\partial \mathcal{T}_h} = \frac{1}{\Delta \tau}\left(p_h^{k-1}, q\right)_{\mathcal{T}_h}, \tag{2.31c}$$

and the conservation equation and Neumann boundary condition

$$-\left\langle -\mathbf{L}_h^k \boldsymbol{n} + p_h^k \boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h^k - \widehat{\boldsymbol{u}}_h^k\right), \widehat{\boldsymbol{v}} \right\rangle_{\partial \mathcal{T}_h \setminus \partial \Omega_D} = -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}} \right\rangle_{\partial \Omega_N} \tag{2.31d}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i$, where $\widehat{\boldsymbol{u}}_h^D$ is defined by (2.22) and $\mathbf{S}$ is defined by (2.25).

In the above, k represents the pseudotime step number. Finally, [14] describes a stopping criterion for the pseudotime iterations,

$$\frac{\left\| p_h^k - p_h^{k-1} \right\|}{\left\| p_h^k \right\|} < \epsilon. \tag{2.32}$$

Algorithm 2.1 describes the solution procedure. We emphasize here that $\Delta\tau$ and ϵ

Algorithm 2.1 Augmented Lagrangian solution procedure.

```
choose Δτ and ε
set p_h^0 = 0, k = 1
while true do
  solve for (L_h^k, u_h^k, p_h^k, û_h^k) using Formulation 2.5
  if (2.32) is true then
    break
  end if
  k ← k + 1
end while
```

must be chosen. We also remark that the stopping criterion (2.32) will not be useful as it is written if the exact pressure is zero. To handle such cases, it may be useful to add a small positive parameter (whose magnitude must be chosen) to the denominator of (2.32).

Some remarks are in order. First, it can be seen that the local solver associated with Formulation 2.5 is well-posed. Indeed, repeating the arguments in the proof for Theorem 2.4, now with p_h^{k-1} as an additional forcing function, instead of (2.29) we will have

$$\mathrm{Re}\left(\mathbf{L}_h^k, \mathbf{L}_h^k\right)_K + \left\langle \mathbf{S}\boldsymbol{u}_h^k, \boldsymbol{u}_h^k \right\rangle_{\partial K} + \frac{1}{\Delta\tau}\left(p_h^k, p_h^k\right)_K = 0, \tag{2.33}$$

which allows us to conclude $p_h^k = 0$. Second, forming the condensed global system (in terms of $\widehat{\boldsymbol{u}}_h^i$ only) gives a global system

$$A\widehat{U}^k = F^{k-1}, \tag{2.34}$$

where the matrix A is symmetric and positive definite. See Appendix B for details.

2.3.2. The Average Edge Pressure Approach. A direct (as opposed to iterative) approach to modifying Formulation 2.2 to obtain a well-posed local solver is given in [14]. The method involves introducing a global unknown representing an elementwise average edge-pressure. We give a slightly different presentation here with implementation using a Lagrange polynomial basis in mind. We do so by altering Formulation 2.2 to read as follows.

Formulation 2.6. Find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i, \rho_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i \times \mathcal{P}_0(\partial\mathcal{T}_h)$ such that the local equations

$$\mathrm{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h, \nabla\cdot\mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.35a}$$

$$-\left(\nabla\cdot\mathbf{L}_h, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right), \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h} = \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h}, \tag{2.35b}$$

$$-\left(\boldsymbol{u}_h, \nabla q\right)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q - \overline{q}\right\rangle_{\partial\mathcal{T}_h} + \left\langle p_h - \rho_h, \overline{q}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.35c}$$

the conservation equation and Neumann boundary condition

$$-\left\langle -\mathbf{L}_h \boldsymbol{n} + p_h \boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right), \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\right\rangle_{\partial\Omega_N}, \tag{2.35d}$$

and the constraint

$$\left\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \psi\right\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.35e}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}, \psi)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i \times \mathcal{P}_0(\partial\mathcal{T}_h)$, where $\widehat{\boldsymbol{u}}_h^D$ is defined by (2.22) and $\mathbf{S}$ is defined by (2.25). If $\partial\Omega_N = \emptyset$, we additionally require the zero mean pressure conditions for the uniqueness of the pressure, (2.26).

In the above, the notation $\overline{q}$ is defined by $\overline{q} := |\partial K|^{-1} \langle q, 1\rangle_{\partial K}$ as the ∂K-wise average of q, and $|\partial K|$ is the length of the perimeter of element K. The new unknowns ρ_h which are sought in $\mathcal{P}_0(\partial\mathcal{T}_h)$ represent the ∂K-wise average pressure. Indeed, taking q to be an elementwise constant in (2.35c), we recover $\overline{p}_h = \rho_h$.

We observe that Formulations 2.2 and 2.6 give the same solution. Indeed, we can show that (2.35c) and (2.35e) are equivalent to (2.24c). Given that we've already shown $\overline{p}_h = \rho_h$, we have $-\left(\boldsymbol{u}_h, \nabla q\right)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q - \overline{q}\right\rangle_{\partial\mathcal{T}_h} = 0$. Setting ψ in (2.35e) equal to $\overline{q}$ and adding the result to the previous expression, we recover (2.24c). Conversely, setting $\boldsymbol{q}$ in (2.24c) equal to any elementwise constant ψ, we recover (2.35e). Then setting $\psi = \overline{q}$ and subtracting (2.35e) from (2.24c), and *defining* $\rho_h := \overline{p}_h$ and therefore that $\langle \overline{p}_h, \overline{q}\rangle_{\partial K} = \langle p_h, \overline{q}\rangle_{\partial K} = \langle \rho_h, \overline{q}\rangle_{\partial K}$ for any q, we recover (2.35c).

As with the Augmented Lagrangian iterative approach, we can see that the modifications result in a well-posed local solver. Indeed, repeating the arguments in the proof for Theorem 2.4, now with ρ_h as a forcing function, instead of (2.29) we will have

$$\operatorname{Re}\left(\mathbf{L}_h, \mathbf{L}_h\right)_K + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{u}_h\rangle_{\partial K} + \langle \overline{p}_h, \overline{p}_h\rangle_K = 0, \tag{2.36}$$

which allows us to conclude $p_h = 0$ on ∂K. Then, following the same arguments as before, we conclude that p_h is elementwise constant, and therefore zero.

As shown in [14], the condensed global system takes the form of a saddle point problem,

$$\begin{bmatrix} A & B^\top \\ -B & 0 \end{bmatrix} \begin{Bmatrix} \widehat{U} \\ \rho \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}, \tag{2.37}$$

where A is symmetric and positive definite. See Appendix B for details.

2.4. HDG Schemes Using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ Flux. In this section, we define new HDG schemes for the Stokes equations. We do this by using the flux (2.18) on all skeleton faces $\mathcal{E}_h^o$. The justification of this choice will become evident when we analyze the well-posedness of the local solver associated with this scheme, where we verify that no special treatment is required for the uniqueness of the local pressure. Recall that for trace unknowns, this flux has the tangent velocity $\widehat{\boldsymbol{u}}_h^t$ and a scalar $\widehat{f}_h$ which approximates $-\frac{1}{\mathrm{Re}}\boldsymbol{n} \cdot [\nabla \boldsymbol{u} \cdot \boldsymbol{n}] + p$. The volume unknowns will still be sought from the discontinuous polynomial spaces (2.5). The discontinuous polynomial space in which we seek $\widehat{f}_h$ and $\widehat{\boldsymbol{u}}_h^t$, respectively, are

$$\widehat{F}_h := \left\{\widehat{g} \in L^2(\mathcal{E}_h) \ : \ \widehat{g}|_e \in \widehat{F}_h(e)\right\}, \tag{2.38}$$

$$\widehat{\boldsymbol{V}}_h^t := \left\{\widehat{\boldsymbol{v}}^t \in \left[L^2(\mathcal{E}_h)\right]^d \ : \ \widehat{\boldsymbol{v}}^t|_e \in \widehat{\boldsymbol{V}}_h^t(e)\right\}, \tag{2.39}$$

where $\widehat{F}_h(e)$ is a scalar polynomial space, and $\widehat{\boldsymbol{V}}^t_h(e)$ is a vector valued polynomial space with no normal component, defined by

$$\widehat{\boldsymbol{V}}^t_h(e) = \left\{ \sum_{i=1}^{d-1} \boldsymbol{t}^i \widehat{v}_{h,i} \ : \ \widehat{v}_{h,i} \in \widehat{V}_h(e) \right\}, \tag{2.40}$$

where $\widehat{V}_h(e)$ is a scalar polynomial space defined on e, and $\left\{\boldsymbol{t}^1, \dots, \boldsymbol{t}^{d-1}\right\}$ is a basis of the tangent space of e.

Realize that (2.18) defines $\boldsymbol{u}_h^*$ as

$$\boldsymbol{u}_h^* = \widehat{\boldsymbol{u}}_h^t + \mathbf{N}\boldsymbol{u}_h + \frac{1}{\tau_n}\left(-\boldsymbol{n} \cdot [\mathbf{L}_h \boldsymbol{n}] + p_h - \widehat{f}_h\right)\boldsymbol{n}. \tag{2.41}$$

The enforcement of the tangent component of the Dirichlet boundary condition (2.4g) then simplifies to an L^2 projection of the tangent part of the Dirichlet boundary data $\boldsymbol{u}_D$ to the trace unknown $\widehat{\boldsymbol{u}}_h^t$ on $\partial\Omega_D$, thereby decoupling $\widehat{\boldsymbol{u}}_h^t$ on $\partial\Omega_D$ from the rest of the unknowns. The normal part of the Dirichlet condition is enforced weakly as will be shown below.

Similarly, (2.18) defines

$$-\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n} = \widehat{f}_h \boldsymbol{n} + \mathbf{T}\left(-\mathbf{L}_h \boldsymbol{n}\right) + \tau_t \left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right), \tag{2.42}$$

so the enforcement of the normal component of the Neumann boundary condition (2.4h) simplifies to an L^2 projection of the normal part of the Neumann boundary data $\boldsymbol{f}_N$ to the trace unknown $\widehat{f}_h$ on $\partial\Omega_N$, thereby decoupling $\widehat{f}_h$ on $\partial\Omega_N$ from the rest of the unknowns. The tangent part of the Neumann condition is enforced weakly as will be shown below.

As before, we decompose the trace unknowns into the decoupled parts and the coupled parts of the trace unknowns. We decompose $\widehat{f}_h$ by

$$\widehat{f}_h = \widehat{f}_h^i + \widehat{f}_h^N \tag{2.43}$$

where $\widehat{f}_h^N$ is defined on $\partial\Omega_N$ as the L^2 projection of the normal component of the Neumann boundary data,

$$\left\langle \widehat{f}_h^N, \widehat{g} \right\rangle_{\partial\Omega_N} = \left\langle \boldsymbol{f}_N \cdot \boldsymbol{n}, \widehat{g} \right\rangle_{\partial\Omega_N} \quad \text{for all } \widehat{g} \in \widehat{F}_h(e) \text{ for all } e \in \partial\Omega_N, \tag{2.44}$$

and $\widehat{f}_h^i$ is the trace unknown $\widehat{f}_h$ restricted to $\mathcal{E}_h \backslash \partial\Omega_N$. Similarly, we decompose $\widehat{\boldsymbol{u}}_h^t$ by

$$\widehat{\boldsymbol{u}}_h^t = \widehat{\boldsymbol{u}}_h^{t,i} + \widehat{\boldsymbol{u}}_h^{t,D} \tag{2.45}$$

where $\widehat{\boldsymbol{u}}_h^{t,D}$ is defined on $\partial\Omega_D$ as the L^2 projection of the tangential component of the Dirichlet boundary data,

$$\left\langle \widehat{\boldsymbol{u}}_h^{t,D}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega_D} = \left\langle \boldsymbol{u}_D^t, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega_D} \quad \text{for all } \widehat{\boldsymbol{v}}^t \in \widehat{\boldsymbol{V}}^t_h(e) \text{ for all } e \in \partial\Omega_D, \tag{2.46}$$

and $\widehat{\boldsymbol{u}}_h^{t,i}$ is the trace unknown $\widehat{\boldsymbol{u}}_h^t$ restricted to $\mathcal{E}_h \backslash \partial\Omega_D$. Again, in writing (2.43) and (2.45) we identify $\widehat{f}_h^i$, $\widehat{f}_h^N$, $\widehat{\boldsymbol{u}}_h^{t,i}$, and $\widehat{\boldsymbol{u}}_h^{t,D}$ with their extensions by zero to $\mathcal{E}_h$.

We assume that all discrete spaces are of equal polynomial order. We also note that we have made a slight abuse of notation as the superscript "i" (for "interior") has a different meaning for $\widehat{f}_h^i$ and $\widehat{\boldsymbol{u}}_h^{t,i}$. Finally, we define the polynomial spaces

$$\widehat{F}_h^i := \left\{ \widehat{g} \in L^2(\mathcal{E}_h \backslash \partial\Omega_N) \ : \ \widehat{g}|_e \in \widehat{F}_h(e) \right\}, \tag{2.47}$$

$$\widehat{\boldsymbol{V}}_h^{t,i} := \left\{ \widehat{\boldsymbol{v}}^t \in \left[L^2(\mathcal{E}_h \backslash \partial\Omega_D) \right]^d \ : \ \widehat{\boldsymbol{v}}^t|_e \in \widehat{\boldsymbol{V}}_h^t(e) \right\}, \tag{2.48}$$

in which $\widehat{f}_h^i$ and $\widehat{\boldsymbol{u}}_h^{t,i}$, respectively, lie. With this in place, we write the HDG scheme as follows.

Formulation 2.7. Find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^{t,i} \times \widehat{F}_h^i$ such that the local equations

$$\begin{aligned} \operatorname{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} - \left(\nabla \boldsymbol{u}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} & \\ + \left\langle \frac{1}{\tau_n} \left(f_h - \widehat{f}_h \right), -\boldsymbol{n} \cdot [\mathbf{G}\boldsymbol{n}] \right\rangle_{\partial\mathcal{T}_h} &= 0, \end{aligned} \tag{2.49a}$$

$$\begin{aligned} \left(\mathbf{L}_h, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(p_h, \nabla\cdot\boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \widehat{f}_h, \boldsymbol{v}\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} & \\ - \left\langle \mathbf{L}_h\boldsymbol{n}, \boldsymbol{v}^t \right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t \left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t \right), \boldsymbol{v}^t \right\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \end{aligned} \tag{2.49b}$$

$$\left(\nabla\cdot\boldsymbol{u}_h, q\right)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} \left(f_h - \widehat{f}_h \right), q \right\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.49c}$$

and the conservation equations combined with the tangential part of the Neumann boundary condition and the normal part of the Dirichlet boundary condition

$$-\left\langle -\mathbf{L}_h\boldsymbol{n} + \tau_t \left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t \right), \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} = -\left\langle \boldsymbol{f}_N^t, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega_N}, \tag{2.49d}$$

$$-\left\langle \boldsymbol{u}_h\cdot\boldsymbol{n} + \frac{1}{\tau_n} \left(f_h - \widehat{f}_h \right), \widehat{g} \right\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_N} = -\left\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, \widehat{g} \right\rangle_{\partial\Omega_D} \tag{2.49e}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^{t,i} \times \widehat{F}_h^i$, where $f_h := -\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h$, $\widehat{\boldsymbol{u}}_h^{t,D}$ is defined by (2.46), and $\widehat{f}_h^N$ is defined by (2.44). In the case that $\partial\Omega_N = \emptyset$, we require the zero mean pressure condition for uniqueness of the pressure, (2.26).

Note that we have identified the scalar test function $\widehat{g}$ with $-\boldsymbol{n}\cdot\left[\widehat{\mathbf{G}}\boldsymbol{n}\right] + \widehat{q}$ on $\partial\mathcal{T}_h \backslash \partial\Omega$ and with $\widehat{\boldsymbol{w}}\cdot\boldsymbol{n}$ on $\partial\Omega$ in order to write (2.4d), (2.4f), and the normal part of (2.4g) in a combined manner as (2.49e). Similarly, the normal part of (2.4e) is automatically satisfied, and we identify $\mathbf{T}\widehat{\boldsymbol{w}}$ with $\widehat{\boldsymbol{v}}^t$ to write (2.4e) and the tangent part of (2.4h) in a combined manner as (2.49d). We are now ready to prove well-posedness of Formulation 2.7 and its local solver.

THEOREM 2.8. *(well-posedness of Formulation 2.7)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$. Then Formulation 2.7 is well-posed in the sense that given $\boldsymbol{f}$, $\boldsymbol{u}_D$, and $\boldsymbol{f}_N$, there exists a unique solution $\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^t, \widehat{f}_h\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^t \times \widehat{F}_h$.*

Proof. It is sufficient to prove that if $\boldsymbol{f} = \mathbf{0}$, $\boldsymbol{u}_D = \mathbf{0}$ and $\boldsymbol{f}_N = \mathbf{0}$, then the solution $\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^t, \widehat{f}_h\right)$ is zero. We can rewrite (2.49) as

$$
\begin{aligned}
&a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) \\
&\quad + a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) = l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)
\end{aligned}
$$

where, using for simplicity $g := -\boldsymbol{n} \cdot [\mathbf{G}\boldsymbol{n}] + q$,

$$
\begin{aligned}
&a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) := \\
&\quad \operatorname{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h, g \right\rangle_{\partial\Omega_N} + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h^i\right), g - \widehat{g} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_N} \\
&\quad + \left\langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{v}^t \right\rangle_{\partial\Omega_D} + \left\langle \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right), \boldsymbol{v}^t - \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D},
\end{aligned}
$$

$$
\begin{aligned}
a_{skew}&\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) := -\left(\nabla \boldsymbol{u}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\mathbf{L}_h, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} \\
&- \left(p_h, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla \cdot \boldsymbol{u}_h, q\right)_{\mathcal{T}_h} + \left\langle \widehat{f}_h^i, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_N} - \left\langle \boldsymbol{u}_h \cdot \boldsymbol{n}, \widehat{g} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_N} \\
&- \left\langle \widehat{\boldsymbol{u}}_h^{t,i}, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} + \left\langle \mathbf{L}_h \boldsymbol{n}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} + \left\langle \boldsymbol{u}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} - \left\langle \mathbf{L}_h \boldsymbol{n}, \boldsymbol{v}^t \right\rangle_{\partial\mathcal{T}_h},
\end{aligned}
$$

and

$$
\begin{aligned}
l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right) &:= \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left\langle \boldsymbol{f}_N^t, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega_N} - \left\langle \boldsymbol{u}_D \cdot \boldsymbol{n}, \widehat{g} \right\rangle_{\partial\Omega_D} + \left\langle \frac{1}{\tau_n} \widehat{f}_h^N, g \right\rangle_{\partial\Omega_N} \\
&+ \left\langle \tau_t \widehat{\boldsymbol{u}}_h^{t,D}, \boldsymbol{v}^t \right\rangle_{\partial\Omega_D} - \left\langle \widehat{f}_h^N, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega_N} + \left\langle \widehat{\boldsymbol{u}}_h^{t,D}, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\Omega_D}.
\end{aligned}
$$

Setting $\boldsymbol{f} = \mathbf{0}$, $\boldsymbol{u}_D = \mathbf{0}$ (and therefore $\widehat{\boldsymbol{u}}_h^{t,D} = 0$), and $\boldsymbol{f}_N = \mathbf{0}$ (and therefore $\widehat{f}_h^N = 0$), we have $l = 0$. Setting $\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right) = \left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i\right)$, we have $a_{skew} = 0$. What remains are the symmetric terms a_{sym}, giving

$$
\begin{aligned}
\operatorname{Re}\left(\mathbf{L}_h, \mathbf{L}_h\right)_{\mathcal{T}_h} &+ \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h^i\right), f_h - \widehat{f}_h^i \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_N} + \left\langle \frac{1}{\tau_n} f_h, f_h \right\rangle_{\partial\Omega_N} \\
&+ \left\langle \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right), \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} + \left\langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{u}_h^t \right\rangle_{\partial\Omega_D} = 0.
\end{aligned}
\tag{2.50}
$$

All the terms in the previous expression are nonnegative and therefore must be zero. Thus $\mathbf{L}_h = \mathbf{0}$ in $\mathcal{T}_h$, $\boldsymbol{u}_h^t = \widehat{\boldsymbol{u}}_h^{t,i}$ on $\mathcal{E}_h^o \cup \partial\Omega_N$, $\boldsymbol{u}_h^t = 0$ on $\partial\Omega_D$, $p_h = \widehat{f}_h$ on $\mathcal{E}_h^o \cup \partial\Omega_D$, and $p_h = 0$ on $\partial\Omega_N$.

Equation (2.49a) reduces to $(\nabla u_h, \mathbf{G})_{\mathcal{T}_h} = 0$, and since $\nabla \boldsymbol{V}_h \subset \mathbf{G}_h$ we can set $\mathbf{G} = \nabla u_h$ to conclude that u_h is elementwise constant. But since $\boldsymbol{u}_h^t = \widehat{\boldsymbol{u}}_h^{t,i}$ on $\mathcal{E}_h^o$ and $\widehat{\boldsymbol{u}}_h^t$ is single valued on $\mathcal{E}_h^o$, and since the remainder (2.49e) implies $\langle \boldsymbol{u}_h \cdot \boldsymbol{n}, \widehat{g} \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0$, the tangential and normal components of $\boldsymbol{u}_h$ are continuous across each internal interface, and therefore $\boldsymbol{u}_h$ is globally constant. Equation (2.49e) also implies the normal component of $\boldsymbol{u}_h$ is zero on $\partial\Omega_D$, and we already have that $\boldsymbol{u}_h^t$ is zero on $\partial\Omega_D$, we conclude that $\boldsymbol{u}_h$ and $\widehat{\boldsymbol{u}}_h^{t,i}$ are zero.

Integrating (2.49b) by parts gives $(\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} = 0$, and since $\nabla Q_h \subset \boldsymbol{V}_h$ we have p_h is elementwise constant. And since $p_h = \widehat{f}_h$ on $\mathcal{E}_h^o$, p_h is globally constant. In the case that $\partial\Omega_N \neq \emptyset$, since $p_h = 0$ on $\partial\Omega_N$ we can conclude $p_h = 0$ and $\widehat{f}_h = 0$. Otherwise, if $\partial\Omega_N = \emptyset$, then (2.26) implies p_h and $\widehat{f}_h$ are zero. ∎

THEOREM 2.9. *(well-posedness of the local solver of Formulation 2.7) Suppose that $\tau_t > 0$ and $\tau_n > 0$. Given $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ to the local equations (2.49a)–(2.49c) .*

Proof. It is sufficient to restrict our attention to a single element, and prove that if $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$ are zero, then the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ is zero. We can rewrite the local problem associated with Formulation 2.7 as: seek $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$ such that

(2.51)
$$\begin{aligned}
&\operatorname{Re}(\mathbf{L}_h, \mathbf{G})_K + \left\langle \frac{1}{\tau_n} f_h, g \right\rangle_{\partial K} + \langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{v}^t \rangle_{\partial K} - (\nabla \boldsymbol{u}_h, \mathbf{G})_K + (\mathbf{L}_h, \nabla \boldsymbol{v})_K \\
&\quad - (p_h, \nabla \cdot \boldsymbol{v})_K + (\nabla \cdot \boldsymbol{u}_h, q)_K + \langle \boldsymbol{u}_h^t, \mathbf{G}\boldsymbol{n} \rangle_{\partial K} - \langle \mathbf{L}_h \boldsymbol{n}, \boldsymbol{v}^t \rangle_{\partial K} \\
&\quad = (\boldsymbol{f}, \boldsymbol{v})_K + \left\langle \frac{1}{\tau_n} \widehat{f}_h, g \right\rangle_{\partial K} + \left\langle \tau_t \widehat{\boldsymbol{u}}_h^t, \boldsymbol{v}^t \right\rangle_{\partial K} + \left\langle \widehat{\boldsymbol{u}}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial K} - \left\langle \widehat{f}_h, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial K}
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$. Setting $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$ to zero, and setting $(\mathbf{G}, \boldsymbol{v}, q) = (\mathbf{L}_h, \boldsymbol{u}_h, p_h)$, we have

$$\operatorname{Re}(\mathbf{L}_h, \mathbf{L}_h)_K + \langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{u}_h^t \rangle_{\partial K} + \left\langle \frac{1}{\tau_n} f_h, f_h \right\rangle_{\partial K} = 0. \tag{2.52}$$

Thus $\mathbf{L}_h = \mathbf{0}$ in K, and $\boldsymbol{u}_h^t = \mathbf{0}$ and $p_h = 0$ on ∂K.

Integrating (2.49b) by parts gives that p_h is constant in K, and since $p_h = 0$ on ∂K, that $p_h = 0$ in K. What remains of (2.49a) gives that $\boldsymbol{u}_h$ is constant in K, and since $\boldsymbol{u}_h^t = \mathbf{0}$ on ∂K, that $\boldsymbol{u}_h = \mathbf{0}$ in K. ∎

Finally, we note that the condensed global system associated with Formulation 2.7 takes the form

$$\begin{bmatrix} A & B^\top \\ -B & D \end{bmatrix} \begin{bmatrix} \widehat{U}^t \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{2.53}$$

where A and D are symmetric and positive semi-definite. If $\partial\Omega_N$ is nonempty, then D is positive definite. Otherwise, constraining one degree of freedom associated with $\widehat{f}_h$ renders D positive definite (see the Discussion section at the end of this section). Details are in Appendix B.

2.5. Numerical Results. We consider as a numerical test problem an analytical solution by Kovasznay [12] to the two dimensional incompressible Navier-Stokes equations. The solution is given by

$$u_1 = 1 - \exp \lambda x_1 \cos 2\pi x_2, \tag{2.54}$$

$$u_2 = \frac{\lambda}{2\pi} \exp \lambda x_1 \sin 2\pi x_2, \tag{2.55}$$

$$p = -\frac{1}{2} \exp 2\lambda x_1. \tag{2.56}$$

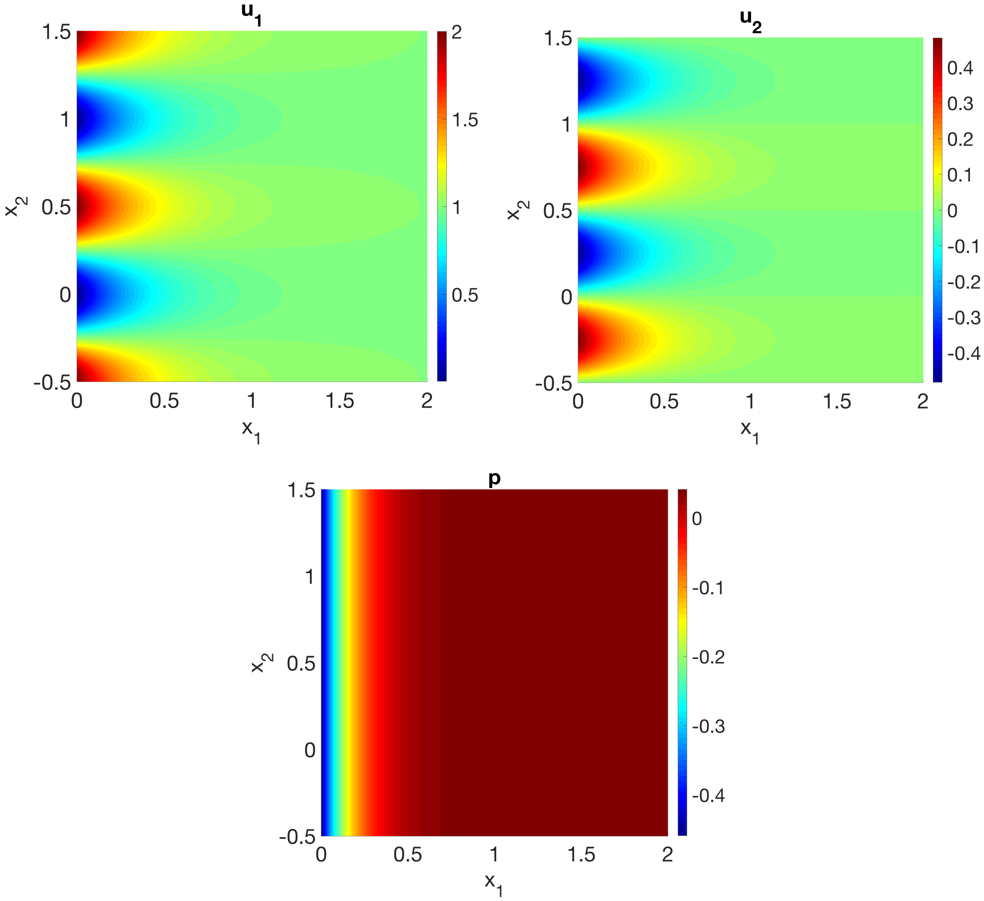

Fig. 1. *Stokes HDG schemes: Kovasznay flow problem solution -* $\boldsymbol{u}_{h1}$ *(top left),* $\boldsymbol{u}_{h2}$ *(top right), and* p_h *(bottom).*

For the Stokes equations, we apply the advection term of the exact solution as a forcing term, i.e., we set

$$\boldsymbol{f} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}. \tag{2.57}$$

A domain of $[0,2] \times [-0.5, 1.5]$ is considered, with the exact velocity solution prescribed as Dirichlet boundary conditions on all parts of the domain boundary. We compute on a mesh of $N \times N$ tensor product square elements, defining the element size $h := \frac{2}{N}$.

In Figure 1, the numerical solution $\boldsymbol{u}_h$ and p_h are plotted. In Figure 2, the $L^2(\Omega)$ error of the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ are plotted along with their convergence rates. The left column of plots shows the L^2 error obtained using the $\widehat{\boldsymbol{u}}_h$ flux (2.16) on all skeleton faces (i.e., Formulation 2.2), while the right column shows the L^2 error obtained using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (2.18) on the interior skeleton faces and the $\widehat{\boldsymbol{u}}_h$ flux (2.16) on the boundary skeleton faces. In both cases τ_t and τ_n are chosen as the upwind parameters τ_t^S and τ_n^S, respectively. As expected, the errors using the two versions of the Godunov flux are virtually identical. In both cases, the observed convergence rates are $k+1$ for $\boldsymbol{u}_h$, and close to $k+1$ for $\mathbf{L}_h$ and p_h.

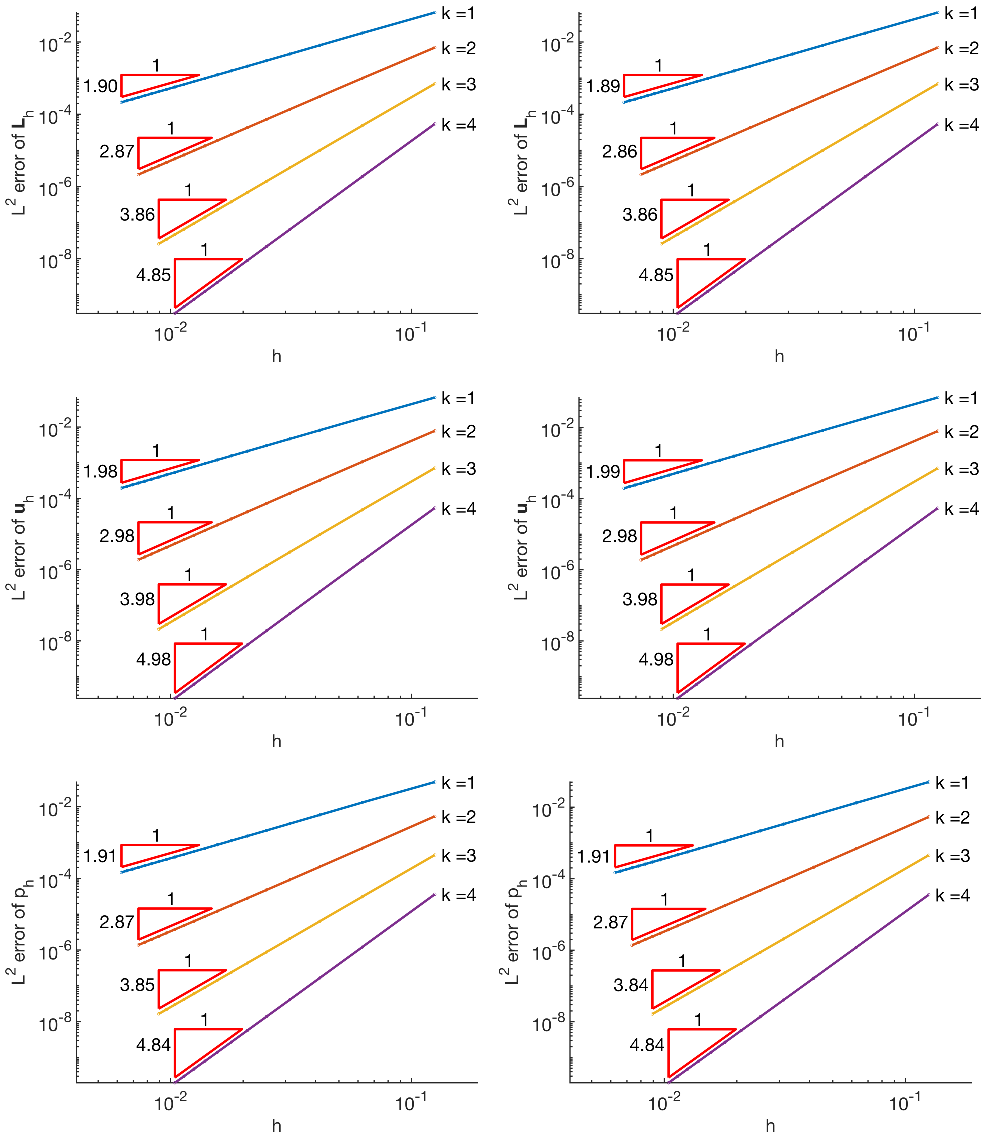

FIG. 2. *Stokes HDG schemes: Kovasznay flow problem L^2 convergence of volume unknowns using $\widehat{\boldsymbol{u}}_h$ flux (2.16) (left), using $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (2.18) (right).*

2.6. Discussion. We used the upwind HDG framework in [2] to derive an HDG scheme based on the $\widehat{\boldsymbol{u}}_h$ flux (2.16), rediscovering the existing HDG scheme in [14], and relating specific values for the stabilization tensor that result in the upwind flux. Additionally, through manipulation of the upwind flux, we have developed a new HDG scheme based on the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (2.18). The schemes based on the $\widehat{\boldsymbol{u}}_h$ flux require modifications in order for the HDG local solver to be well-posed. One modification involves solving a trace system iteratively (in addition to any iterative linear solver), while introducing multiple parameters related to the iterations. Another modification involves introducing an elementwise constant *global* unknown, rendering the global system a saddle point system. The global unknowns in the latter modified system are of a different nature; the $\widehat{\boldsymbol{u}}_h$ unknowns are discontinuous polynomials on the mesh skeleton, whereas the ρ_h unknowns are elementwise discontinuous constants. This presents challenges in the design of linear solvers and preconditioners. The new scheme based on the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux offers some advantages from both of these schemes. No iterations are needed, and all unknowns in the condensed global system are of the same nature: discontinuous polynomials on the mesh skeleton. Additionally, the trace system does not result in a traditional saddle point system; there are no zero blocks on the diagonal, which allows more flexibility in the types of preconditioners we can apply, including allowing for the application of the simple Jacobi/block Jacobi preconditioners.

When using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (2.18), it can be convenient to use that flux on the interior skeleton face only, and to use a different flux on the domain boundary. In addition to being potentially easier to implement, applying the boundary conditions in this way minimizes the number of globally coupled unknowns, since all of the boundary unknowns are decoupled from the interior ones. For example, if all of the boundary conditions are Dirichlet boundary conditions (2.2a), then we can use the $\widehat{\boldsymbol{u}}_h$ flux (2.16) on the domain boundary so that the application of the boundary conditions are simply the projection of the boundary data to the trace unknown, rather than the "mixed" way of applying them described in Formulation 2.7. It can be shown that the global system and the local solver remain well-posed, and that the condensed global matrix structure (2.53) does not change.

As pointed out in the definitions of the HDG schemes, an additional constraint is required when we have $\partial\Omega_N = \emptyset$ in order to uniquely define the pressure. Even though the zero mean pressure constraint (2.26) appears to be a global equation that couples volume variables across elements, the implementation can be handled in a way that does not break the locality of the local problems. In the case of Formulation 2.2, the analysis reveals that we must only constrain one degree of freedom associated with ρ_h in order to uniquely define ρ_h and therefore p_h. Depending on the linear solver, it may or may not be necessary to explicitly constrain that degree of freedom. Similarly for Formulation 2.7, we must only constrain one degree of freedom associated with $\widehat{f}_h$. Then we must only shift p_h in a postprocessing step in order to satisfy (2.26) (if desired).

3. Oseen Equations.

In this section, we employ the upwind HDG framework proposed in [2] in order to derive HDG schemes for the Oseen equations. Similar to the the previous section on the Stokes equations, we manipulate the upwind flux in order to express it in four different ways, each of which can be shown to lead to a well-posed HDG scheme. One of the schemes is related to the scheme in [5], whereas the other three are new contributions in this work. We present two of these schemes in detail and prove the aforementioned well-posedness. The two schemes are employed in numerical tests and their convergence is demonstrated. Additionally we define a Picard-type iterative method that can be used to solve the (nonlinear) incompressible Navier-Stokes equations, and we demonstrate the convergence of the scheme.

3.1. Construction of Upwind HDG Schemes.

For notation used in this section and throughout this work, see Appendix A. The Oseen equations in dimensionless form read

$$-\frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{3.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.1b}$$

where $\boldsymbol{w}$ is assumed to be divergence free and is assumed to reside in $H(div,\Omega)$. For simplicity, we consider only Dirichlet boundary conditions,

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \partial\Omega. \tag{3.2}$$

A compatibility condition on the Dirichlet boundary data $\int_{\partial\Omega} \boldsymbol{u}_D\cdot\boldsymbol{n} = 0$ should be satisfied, and we have to impose an additional constraint on the pressure. We choose this constraint to be $\int_\Omega p = 0$. Comments will be made later on generalizations to different types of boundary conditions.

Toward applying the upwind HDG framework [2], we first put (3.1) into first order form through the definition of an auxiliary variable. We define the auxiliary variable $\mathbf{L}$ through the velocity gradient, resulting in the first order system

$$\mathrm{Re}\mathbf{L} - \nabla\boldsymbol{u} = 0, \tag{3.3a}$$

$$-\nabla\cdot\mathbf{L} + \nabla p + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{w}) = \boldsymbol{f}, \tag{3.3b}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{3.3c}$$

In the above, we have used the divergence-free assumption on $\boldsymbol{w}$ to put the system into divergence form. To define a general HDG scheme for the Oseen equations, we multiply (3.3) by test functions, integrate over the computational domain, integrate by parts, and replace the boundary terms with yet-to-be-defined numerical flux terms, which we then enforce to be weakly continuous across element interfaces. HDG schemes derived in this manner for (3.3) will take a general form consisting of the local equations

$$\mathrm{Re}\,(\mathbf{L}_h,\mathbf{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h,\nabla\cdot\mathbf{G})_{\mathcal{T}_h} - \langle \boldsymbol{u}_h^*\otimes\boldsymbol{n},\mathbf{G}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.4a}$$

$$\begin{aligned}(\mathbf{L}_h,\nabla\boldsymbol{v})_{\mathcal{T}_h} - (p_h,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - (\boldsymbol{u}_h\otimes\boldsymbol{w},\nabla\boldsymbol{v})_{\mathcal{T}_h}&\\ + \langle -\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n} + (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h^*,\boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f},\boldsymbol{v})_{\mathcal{T}_h}\,,\end{aligned} \tag{3.4b}$$

$$-(\boldsymbol{u}_h,\nabla q)_{\mathcal{T}_h} + \langle \boldsymbol{u}_h^*\cdot\boldsymbol{n},q\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.4c}$$

the conservation equations

$$\left\langle \boldsymbol{u}_h^*\otimes\boldsymbol{n},\widehat{\mathbf{G}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{3.4d}$$

$$-\langle -\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n} + (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h^*,\widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{3.4e}$$

$$-\langle \boldsymbol{u}_h^*\cdot\boldsymbol{n},\widehat{q}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{3.4f}$$

and the Dirichlet boundary condition

$$\langle \boldsymbol{u}_h^*,\widehat{\boldsymbol{w}}\rangle_{\partial\Omega} = \langle \boldsymbol{u}_D,\widehat{\boldsymbol{w}}\rangle_{\partial\Omega}\,. \tag{3.4g}$$

The volume unknowns $(\mathbf{L}_h,\boldsymbol{u}_h,p_h)$ and the test functions $(\mathbf{G},\boldsymbol{v},q)$ will belong to the discontinuous polynomial spaces (2.5). The quantities $\boldsymbol{u}_h^*$ and $-\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n} + (\boldsymbol{w}\,\cdot$

$\boldsymbol{n})\boldsymbol{u}_h^*$ are yet-to-be-defined, not-necessarily-single-valued numerical fluxes, which are function of the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ and trace variables $\left(\widehat{\mathbf{L}}_h, \widehat{\boldsymbol{u}}_h, \widehat{p}_h\right)$. The trace variables reside in discontinuous polynomial spaces defined on the mesh skeleton, as do the interior test functions $\left(\widehat{\mathbf{G}}, \widehat{\boldsymbol{v}}, \widehat{q}\right)$, and boundary test function $\widehat{\boldsymbol{w}}$. In what follows, we derive different choices for the starred quantities and analyze schemes that result from some specific choices. The fluxes we derive will have a minimal number of trace unknowns (d scalar unknowns) so that not all of the trace unknowns $\left(\widehat{\mathbf{L}}_h, \widehat{\boldsymbol{u}}_h, \widehat{p}_h\right)$ (and their corresponding test functions) will exist as unknowns (and test functions). Related to this is the fact that not all of the conservation equations (3.4d)–(3.4f) must be explicitly enforced, as some will be automatically satisfied depending on the choice of the numerical flux. Additionally, the boundary test function $\widehat{\boldsymbol{w}}$ will have a natural association with the interior skeleton test functions among $\left(\widehat{\mathbf{G}}, \widehat{\boldsymbol{v}}, \widehat{q}\right)$ that do exist in the scheme. These points will be made clearer after we derive the HDG numerical fluxes.

To derive the numerical fluxes, we observe that the first order system (3.3) fits into the framework of (1.1) and is, in fact, a symmetric hyperbolic system. Choosing the ordering of unknowns as the column vector $\boldsymbol{U} := (\mathrm{vec}\,(\mathbf{L})\,; \boldsymbol{u}; p)$, and defining $m := \boldsymbol{w} \cdot \boldsymbol{n}$, we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{n}\otimes_K \mathbf{I} & \mathbf{0} \\ -\boldsymbol{n}^\top \otimes_K \mathbf{I} & m\mathbf{I} & \boldsymbol{n} \\ \mathbf{0} & \boldsymbol{n}^\top & 0 \end{bmatrix}. \tag{3.5}$$

We perform the eigendecomposition $\mathbf{A} = \mathbf{R}\mathbf{D}\mathbf{R}^{-1}$, where $\mathbf{D}$ is a diagonal matrix comprising the eigenvalues of $\mathbf{A}$, and $\mathbf{R}$ is a matrix whose columns are the eigenvectors corresponding those eigenvalues. Defining $|\mathbf{D}|$ by taking the absolute value of each eigenvalue in $\mathbf{D}$, we can define $|\mathbf{A}| := \mathbf{R}\,|\mathbf{D}|\,\mathbf{R}^{-1}$. It can be shown that for the Oseen system we have

(3.6)
$$|\mathbf{A}| = \begin{bmatrix} \mathbf{N}\otimes_K \left(\frac{1}{\tau_t^O}\mathbf{T} + \frac{1}{\tau_n^O}\mathbf{N}\right) & -\frac{m}{2}\boldsymbol{n}\otimes_K \left(\frac{1}{\tau_t^O}\mathbf{T} + \frac{1}{\tau_n^O}\mathbf{N}\right) & -\frac{1}{\tau_n^O}\boldsymbol{n}\otimes_K \boldsymbol{n} \\ -\frac{m}{2}\boldsymbol{n}^\top \otimes_K \left(\frac{1}{\tau_t^O}\mathbf{T} + \frac{1}{\tau_n^O}\mathbf{N}\right) & \begin{pmatrix} \left(\frac{m}{2}\right)^2 \left(\frac{1}{\tau_t^O}\mathbf{T} + \frac{1}{\tau_n^O}\mathbf{N}\right) \\ + \left(\tau_t^O \mathbf{T} + \tau_n^O \mathbf{N}\right) \end{pmatrix} & \frac{m}{2}\frac{1}{\tau_n^O}\boldsymbol{n} \\ -\frac{1}{\tau_n^O}\boldsymbol{n}^\top \otimes_K \boldsymbol{n}^\top & \frac{m}{2}\frac{1}{\tau_n^O}\boldsymbol{n}^\top & \frac{1}{\tau_n^O} \end{bmatrix},$$

where $\tau_t^O := \frac{1}{2}\sqrt{4+m^2}$ and $\tau_n^O := \frac{1}{2}\sqrt{8+m^2}$. Later we will allow for the generalization $\tau_t^O \to \tau_t$, $\tau_n^O \to \tau_n$, where τ_t and τ_n are freely chosen positive parameters, allowing us to define simpler fluxes and relate the upwind schemes to existing schemes. We define the normal upwind flux $\boldsymbol{F}_n^*$ as a column vector $\boldsymbol{F}_n^* := (\mathrm{vec}\,(-\boldsymbol{u}^* \otimes \boldsymbol{n})\,; -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} + m\boldsymbol{u}^*; \boldsymbol{u}^* \cdot \boldsymbol{n})$. Since there is a one-to-one correspondence between $\mathrm{vec}\,(-\boldsymbol{u}^* \otimes \boldsymbol{n})$ and $-\boldsymbol{u}^* \otimes \boldsymbol{n}$, we also identify $\boldsymbol{F}_n^*$ with the triple

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\boldsymbol{u}^* \otimes \boldsymbol{n} \\ -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} + m\boldsymbol{u}^* \\ \boldsymbol{u}^* \cdot \boldsymbol{n} \end{bmatrix}. \tag{3.7}$$

In this way, we can write the exact upwind flux $\boldsymbol{F}_n^* = \mathbf{A}\boldsymbol{U} + |\mathbf{A}|\left(\boldsymbol{U} - \boldsymbol{U}^*\right)$ as

$$\boldsymbol{F}_n^* = \left[\begin{array}{c} -\left(\boldsymbol{u} + \mathbf{S}_O^{-1}\left(-\left(\mathbf{L}-\mathbf{L}^*\right)\boldsymbol{n} + \left(p-p^*\right)\boldsymbol{n} + \frac{m}{2}\left(\boldsymbol{u}-\boldsymbol{u}^*\right)\right)\right)\otimes\boldsymbol{n} \\ -\mathbf{L}\boldsymbol{n} + p\boldsymbol{n} + m\boldsymbol{u} + \mathbf{S}_O\left(\boldsymbol{u}-\boldsymbol{u}^*\right) \\ +\frac{m}{2}\mathbf{S}_O^{-1}\left(-\left(\mathbf{L}-\mathbf{L}^*\right)\boldsymbol{n} + \left(p-p^*\right)\boldsymbol{n} + \frac{m}{2}\left(\boldsymbol{u}-\boldsymbol{u}^*\right)\right) \\ \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\tau_n^O}\left(-\boldsymbol{n}\cdot\left[\mathbf{L}-\mathbf{L}^*\right]\boldsymbol{n} + \left(p-p^*\right) + \frac{m}{2}\left(\boldsymbol{u}-\boldsymbol{u}^*\right)\cdot\boldsymbol{n}\right) \end{array}\right], \tag{3.8}$$

where

$$\mathbf{S}_O := \tau_t^O\mathbf{T} + \tau_n^O\mathbf{N}, \quad \mathbf{S}_O^{-1} = \frac{1}{\tau_t^O}\mathbf{T} + \frac{1}{\tau_n^O}\mathbf{N}. \tag{3.9}$$

At this point, we can eliminate "starred quantities" with the aim of defining an HDG flux with minimal trace unknowns. As we did the Stokes equations, we manipulate the flux (3.8) in several different ways leading to fluxes that are suitable for use in HDG schemes. We begin with a lemma that gives key relationship between the upwind states.

Lemma 3.1. *The following relationships between the upwind states hold:*

$$\tau_t^O\mathbf{T}\left(\boldsymbol{u}-\boldsymbol{u}^*\right) = -\mathbf{T}\left[-\left(\mathbf{L}-\mathbf{L}^*\right)\boldsymbol{n} + \frac{m}{2}\left(\boldsymbol{u}-\boldsymbol{u}^*\right)\right], \tag{3.10a}$$

$$\tau_n^O\mathbf{N}\left(\boldsymbol{u}-\boldsymbol{u}^*\right) = -\mathbf{N}\left[-\left(\mathbf{L}-\mathbf{L}^*\right)\boldsymbol{n} + \left(p-p^*\right)\boldsymbol{n} + \frac{m}{2}\left(\boldsymbol{u}-\boldsymbol{u}^*\right)\right]. \tag{3.10b}$$

Proof. We arrive at the result by equating the normal components of the left and right side of the first component of flux (3.8), and doing the same for the tangent components. ∎

Note that (3.10) can be arrived at by equating the second component of (3.8), and (3.10b) can be arrived at by equating the third component of (3.8). That is to say that (3.10a) and (3.10b) are the only two relations we can discover from (3.8).

Next, we use (3.10) to reduce the number of upwind quantities on the right hand side of (3.8) to d scalar unknowns in different ways. The presence of the advection term in the Navier-Stokes momentum equations opens up the possibility of expressing the upwind flux in more ways than we could for the Stokes equations. First, we explore different forms of the flux based on choosing the normal component of either $\boldsymbol{u}^*$ or $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}^*$, and choosing the tangential component of either $\boldsymbol{u}^*$ or $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}^*$. Essentially, we can choose either the left or right side of (3.10a) and either the left or right side of (3.10b). It turns out that these fluxes, when discretized, lead to well-posed HDG schemes. These fluxes are listed below.

The $\boldsymbol{u}_h^*$ flux: The quantities $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$ can be eliminated from (3.8) so that (3.8) can be written as

$$\boldsymbol{F}_n^* = \left[\begin{array}{c} -\boldsymbol{u}^*\otimes\boldsymbol{n} \\ -\mathbf{L}\boldsymbol{n} + p\boldsymbol{n} + \frac{m}{2}\boldsymbol{u} + \frac{m}{2}\boldsymbol{u}^* + \mathbf{S}_O\left(\boldsymbol{u}-\boldsymbol{u}^*\right) \\ \boldsymbol{u}^*\cdot\boldsymbol{n} \end{array}\right]. \tag{3.11}$$

The $\mathbf{F}^*\boldsymbol{n}$ flux: Defining

$$\mathbf{F} := -\mathbf{L} + p\mathbf{I} + \frac{1}{2}\boldsymbol{u}\otimes\boldsymbol{w}, \quad \mathbf{F}^* := -\mathbf{L}^* + p^*\mathbf{I} + \frac{1}{2}\boldsymbol{u}^*\otimes\boldsymbol{w}, \tag{3.12}$$

the flux (3.8) can be written with $\mathbf{F}^*\boldsymbol{n}$ as the only starred quantities,

$$\boldsymbol{F}_n^* = \left[\begin{array}{c} -\left(\boldsymbol{u} + \mathbf{S}_O^{-1}\left(\mathbf{F}-\mathbf{F}^*\right)\boldsymbol{n}\right)\otimes\boldsymbol{n} \\ \mathbf{F}^*\boldsymbol{n} + \frac{m}{2}\boldsymbol{u} + \frac{m}{2}\mathbf{S}_O^{-1}\left(\mathbf{F}-\mathbf{F}^*\right)\boldsymbol{n} \\ \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\tau_n^O}\boldsymbol{n}\cdot\left[\left(\mathbf{F}-\mathbf{F}^*\right)\boldsymbol{n}\right] \end{array}\right]. \tag{3.13}$$

The $(\mathbf{T}\boldsymbol{u}^*, f^*)$ flux: Defining

$$f := -\boldsymbol{n}\cdot[\mathbf{F}\boldsymbol{n}], \quad f^* := -\boldsymbol{n}\cdot[\mathbf{F}^*\boldsymbol{n}], \tag{3.14}$$

the flux (3.8) can be written with f^* and $\mathbf{T}\boldsymbol{u}^*$ as the only starred quantities,

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\left(\mathbf{T}\boldsymbol{u}^* + \mathbf{N}\boldsymbol{u} + \frac{1}{\tau_n^O}(f-f^*)\,\boldsymbol{n}\right)\otimes\boldsymbol{n} \\ f^*\boldsymbol{n} + \frac{m}{2}\mathbf{T}\boldsymbol{u}^* + \frac{m}{2}\boldsymbol{u} - \mathbf{TL}\boldsymbol{n} + \frac{m}{2}\frac{1}{\tau_n^O}(f-f^*)\,\boldsymbol{n} + \tau_t^O\mathbf{T}(\boldsymbol{u}-\boldsymbol{u}^*) \\ \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\tau_n^O}(f-f^*) \end{bmatrix}. \tag{3.15}$$

The $(\mathbf{N}\boldsymbol{u}^*, \mathbf{T}\mathbf{F}^*\boldsymbol{n})$ flux: The flux (3.8) can be written with $\mathbf{N}\boldsymbol{u}^*$ and $\mathbf{T}\mathbf{F}^*\boldsymbol{n}$ as the only starred quantities,

$$\boldsymbol{F}_n^* = \begin{bmatrix} -\left(\mathbf{N}\boldsymbol{u}^* + \mathbf{T}\boldsymbol{u} + \frac{1}{\tau_t^O}\mathbf{T}(\mathbf{F}-\mathbf{F}^*)\,\boldsymbol{n}\right)\otimes\boldsymbol{n} \\ \mathbf{T}\mathbf{F}^*\boldsymbol{n} + \mathbf{N}\mathbf{F}\boldsymbol{n} + \frac{m}{2}\mathbf{N}\boldsymbol{u}^* + \frac{m}{2}\mathbf{T}\boldsymbol{u} + \frac{m}{2}\frac{1}{\tau_t^O}\mathbf{T}(\mathbf{F}-\mathbf{F}^*)\,\boldsymbol{n} + \tau_n^O\mathbf{N}(\boldsymbol{u}-\boldsymbol{u}^*) \\ \boldsymbol{u}^*\cdot\boldsymbol{n} \end{bmatrix}. \tag{3.16}$$

It is not obvious that the above forms of the upwind flux will lead to well-posed HDG schemes, and they are in fact not the only ways that we can express the upwind flux. The relations (3.10) between the upwind states can be re-expressed as

$$\left(\tau_t^O + \frac{m}{2}\right)\mathbf{T}(\boldsymbol{u}-\boldsymbol{u}^*) = -\mathbf{T}\left[-(\mathbf{L}-\mathbf{L}^*)\,\boldsymbol{n}\right], \tag{3.17a}$$

$$\left(\tau_n^O + \frac{m}{2}\right)\mathbf{N}(\boldsymbol{u}-\boldsymbol{u}^*) = -\mathbf{N}\left[-(\mathbf{L}-\mathbf{L}^*)\,\boldsymbol{n} + (p-p^*)\,\boldsymbol{n}\right]. \tag{3.17b}$$

Then, we can write the upwind flux in terms of the normal component of either $\boldsymbol{u}^*$ and $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$ and the tangential component of either $\boldsymbol{u}^*$ and $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$. That is, we can choose either the left or right side of (3.17a) and either the left or right side of (3.17b). We have already considered the case where we write the upwind flux in terms of $\boldsymbol{u}^*$ only, giving (3.11). The three remaining forms, as it turns out, do not lead to well-posed HDG schemes when used on all skeleton faces, but it is possible that they could serve a purpose by being used on the domain boundary in order to decouple as many unknowns as possible. For the sake of readability, these additional forms of the flux, and their discrete counterparts, are given in Appendix C.

In order to define numerical fluxes

$$\boldsymbol{F}_{n,h}^* = \begin{bmatrix} -\boldsymbol{u}_h^*\otimes\boldsymbol{n} \\ -\mathbf{L}_h^*\boldsymbol{n} + p_h^*\boldsymbol{n} + (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h^* \\ \boldsymbol{u}_h^*\cdot\boldsymbol{n} \end{bmatrix} \tag{3.18}$$

to be used in the HDG scheme (3.4), we append a subscript h to the terms in (3.11), (3.13), (3.15), and (3.16) and replace the starred quantities on the right side of the different forms of the upwind flux with hatted unknown quantities residing on the mesh skeleton. Additionally we replace τ_t^O and τ_n^O with τ_t and τ_n, which, from the well-posedness analysis, can be freely chosen positive values. It is sometimes convenient to use the following notation for the normal and tangential stabilization terms,

$$\mathbf{S} := \tau_t\mathbf{T} + \tau_n\mathbf{N}, \quad \mathbf{S}^{-1} = \frac{1}{\tau_t}\mathbf{T} + \frac{1}{\tau_n}\mathbf{N}. \tag{3.19}$$

This gives the following numerical fluxes.

The $\widehat{\boldsymbol{u}}_h$ flux:

$$\boldsymbol{F}^*_{n,h} := \begin{bmatrix} -\widehat{\boldsymbol{u}}_h \otimes \boldsymbol{n} \\ -\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \tfrac{m}{2}\boldsymbol{u}_h + \tfrac{m}{2}\widehat{\boldsymbol{u}}_h + \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right) \\ \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n} \end{bmatrix}. \tag{3.20}$$

The $\widehat{\boldsymbol{f}}_h$ flux (where $\widehat{\boldsymbol{f}}_h$ approximates $-\mathbf{L}^*\tilde{\boldsymbol{n}} + p^*\tilde{\boldsymbol{n}} + \mathrm{sgn}\tfrac{m}{2}\boldsymbol{u}^*$):

$$\boldsymbol{F}^*_{n,h} = \begin{bmatrix} -\left(\boldsymbol{u}_h + \mathbf{S}^{-1}\left(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \tfrac{m}{2}\boldsymbol{u}_h - \mathrm{sgn}\widehat{\boldsymbol{f}}_h\right)\right) \otimes \boldsymbol{n} \\ \mathrm{sgn}\widehat{\boldsymbol{f}}_h + \tfrac{m}{2}\boldsymbol{u}_h + \tfrac{m}{2}\mathbf{S}^{-1}\left(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \tfrac{m}{2}\boldsymbol{u}_h - \mathrm{sgn}\widehat{\boldsymbol{f}}_h\right) \\ \boldsymbol{u}_h\cdot\boldsymbol{n} + \tfrac{1}{\tau_n}\left(-\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h + \tfrac{m}{2}\boldsymbol{u}_h\cdot\boldsymbol{n} - \widehat{\boldsymbol{f}}_h\cdot\tilde{\boldsymbol{n}}\right) \end{bmatrix}. \tag{3.21}$$

The $(\widehat{\boldsymbol{u}}^t_h, \widehat{f}_h)$ flux (where $\widehat{f}_h$ approximates $-\boldsymbol{n}\cdot[\mathbf{L}^*\boldsymbol{n}] + p^* + \tfrac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}^*\cdot\boldsymbol{n}$):

$$\boldsymbol{F}^*_{n,h} := \begin{bmatrix} -\left(\widehat{\boldsymbol{u}}^t_h + \mathbf{N}\boldsymbol{u}_h + \tfrac{1}{\tau_n}\left(f_h - \widehat{f}_h\right)\boldsymbol{n}\right)\otimes\boldsymbol{n}, \\ \widehat{f}_h\boldsymbol{n} + \tfrac{m}{2}\widehat{\boldsymbol{u}}^t_h + \tfrac{m}{2}\boldsymbol{u}_h - \mathbf{T}\mathbf{L}_h\boldsymbol{n} + \tfrac{m}{2}\tfrac{1}{\tau_n}\left(f_h - \widehat{f}_h\right)\boldsymbol{n} + \tau_t\left(\boldsymbol{u}^t_h - \widehat{\boldsymbol{u}}^t_h\right), \\ \boldsymbol{u}_h\cdot\boldsymbol{n} + \tfrac{1}{\tau_n}\left(f_h - \widehat{f}_h\right) \end{bmatrix}, \tag{3.22}$$

where

$$f_h := -\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{u}_h\cdot\boldsymbol{n}). \tag{3.23}$$

The $\left(\widehat{u}^{\tilde{n}}_h, \widehat{\boldsymbol{f}}^t_h\right)$ flux (where $\widehat{\boldsymbol{f}}^t_h$ approximates $\mathbf{T}\left(-\mathbf{L}^*\tilde{\boldsymbol{n}} + \mathrm{sgn}\tfrac{m}{2}\boldsymbol{u}^*\right)$ and $\widehat{u}^{\tilde{n}}_h$ approximates $\boldsymbol{u}^*\cdot\tilde{\boldsymbol{n}}$):

$$\boldsymbol{F}^*_{n} = \begin{bmatrix} -\left(\widehat{u}^{\tilde{n}}_h\tilde{\boldsymbol{n}} + \boldsymbol{u}^t_h + \tfrac{1}{\tau_t}\left(\mathbf{T}\mathbf{F}_h\boldsymbol{n} - \mathrm{sgn}\widehat{\boldsymbol{f}}^t_h\right)\right)\otimes\boldsymbol{n} \\ \mathrm{sgn}\widehat{\boldsymbol{f}}^t_h + \mathbf{N}\mathbf{F}_h\boldsymbol{n} + \tfrac{m}{2}\widehat{u}^{\tilde{n}}_h\tilde{\boldsymbol{n}} + \tfrac{m}{2}\mathbf{T}\boldsymbol{u}_h + \tfrac{m}{2}\tfrac{1}{\tau_t}\left(\mathbf{T}\mathbf{F}_h\boldsymbol{n} - \mathrm{sgn}\widehat{\boldsymbol{f}}^t_h\right) + \tau_n\left(\mathbf{N}\boldsymbol{u} - \widehat{u}^{\tilde{n}}_h\tilde{\boldsymbol{n}}\right) \\ \mathrm{sgn}\widehat{u}^{\tilde{n}}_h \end{bmatrix}, \tag{3.24}$$

where

$$\mathbf{F}_h := -\mathbf{L}_h + p_h\mathbf{I} + \frac{1}{2}\boldsymbol{u}_h\otimes\boldsymbol{w}. \tag{3.25}$$

It can be shown that the use of fluxes (3.20) through (3.24) lead to well-posed HDG schemes, but some of the fluxes are more practical than others. Using (3.20) or (3.24) results in a scheme that requires modifications in order to uniquely define the pressure p_h in the local solver, similar to some of the fluxes discussed in section 2 for the Stokes equations. The flux (3.21) results in a scheme where the velocity $\widehat{\boldsymbol{u}}_h$ is not uniquely defined by the local solver if $\boldsymbol{w}\cdot\boldsymbol{n} = 0$ on a set of nonzero measure on $\partial\mathcal{T}_h$ (unless we consider the time-dependent version of the Oseen equations with implicit time stepping, in which case it is well-posed without modifications). The flux (3.22) results in a scheme that is in any case well-posed without modifications. In what follows, we concretely define and prove the well-posedness of HDG schemes based on the fluxes (3.20) and (3.22).

3.2. HDG Schemes Using the $\widehat{\boldsymbol{u}}_h$ Flux. In this section, we define an HDG scheme based on (3.11), which is the "familiar" form that can be related to the scheme proposed in the work by Cesmelioglu et al. [5], and can be related to the fluid subsystem of the incompressible MHD scheme [13]. As before, we consider polynomial spaces of equal order $k \geq 1$ for all volume and trace unknowns. The discontinuous polynomial spaces in which we seek the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ and to which their corresponding test functions $(\mathbf{G}, \boldsymbol{v}, q)$ belong are (2.5), the same as for the Stokes HDG schemes. The discontinuous polynomial space in which we seek the trace unknowns $\widehat{\boldsymbol{u}}_h$ is

$$\widehat{\boldsymbol{V}}_h := \left\{ \widehat{\boldsymbol{v}} \in \left[L^2(\mathcal{E}_h)\right]^d \ : \ \widehat{\boldsymbol{v}}|_e \in \widehat{\boldsymbol{V}}_h(e) \right\}, \tag{3.26}$$

where $\widehat{\boldsymbol{V}}_h(e)$ is a polynomial space defined on e.

With the numerical flux (3.20), the enforcement of the Dirichlet boundary condition (3.4g) simplifies to an L^2 projection of the Dirichlet boundary data to the trace unknown on $\partial\Omega$, thereby decoupling the trace unknowns on $\partial\Omega$ from the rest of the unknowns. Then we can decompose the trace unknown

$$\widehat{\boldsymbol{u}}_h = \widehat{\boldsymbol{u}}_h^i + \widehat{\boldsymbol{u}}_h^D \tag{3.27}$$

where $\widehat{\boldsymbol{u}}_h^D$ is defined on $\partial\Omega$ as the L^2 projection of the boundary data,

$$\left\langle \widehat{\boldsymbol{u}}_h^D, \widehat{\boldsymbol{v}} \right\rangle_{\partial\Omega} = \langle \boldsymbol{u}_D, \widehat{\boldsymbol{v}} \rangle_{\partial\Omega} \quad \text{for all } \widehat{\boldsymbol{v}} \in \widehat{\boldsymbol{V}}_h(e) \text{ for all } e \in \partial\Omega, \tag{3.28}$$

and $\widehat{\boldsymbol{u}}_h^i$ is the trace unknown $\widehat{\boldsymbol{u}}_h$ restricted to the interior skeleton faces $\mathcal{E}_h^o$. Note that in writing (3.27) we identify $\widehat{\boldsymbol{u}}_h^i$ and $\widehat{\boldsymbol{u}}_h^D$ with their extensions by zero to the whole skeleton $\mathcal{E}_h$. Then $\widehat{\boldsymbol{u}}_h^i$ resides in the polynomial space

$$\widehat{\boldsymbol{V}}_h^i := \left\{ \widehat{\boldsymbol{v}} \in \left[L^2(\mathcal{E}_h^o)\right]^d \ : \ \widehat{\boldsymbol{v}}|_e \in \widehat{\boldsymbol{V}}_h(e) \right\}. \tag{3.29}$$

With this in place, we write the HDG scheme as follows.

Formulation 3.2. Find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i$ such that the local equations

$$\operatorname{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h^i, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.30a}$$

$$\begin{aligned} -\left(\nabla \cdot \mathbf{L}_h, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h, \boldsymbol{v}\right)_{\mathcal{T}_h} - \frac{1}{2}\left(\boldsymbol{u}_h \otimes \boldsymbol{w}, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} & \\ + \frac{1}{2}\left(\nabla \boldsymbol{u}_h, \boldsymbol{v} \otimes \boldsymbol{w}\right)_{\mathcal{T}_h} + \left\langle \frac{1}{2}\left(\boldsymbol{w} \cdot \boldsymbol{n}\right)\widehat{\boldsymbol{u}}_h + \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right), \boldsymbol{v} \right\rangle_{\partial\mathcal{T}_h} &= \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h}, \end{aligned} \tag{3.30b}$$

$$-\left(\boldsymbol{u}_h, \nabla q\right)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q \right\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.30c}$$

and the conservation equation

$$-\left\langle -\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \frac{1}{2}\left(\boldsymbol{w} \cdot \boldsymbol{n}\right)\boldsymbol{u}_h + \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right), \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0 \tag{3.30d}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^i$, where $\mathbf{S}$ is defined as in (3.19), $\widehat{\boldsymbol{u}}_h^D$ is defined as in (3.28), and with the zero mean pressure conditions for the uniqueness of the pressure,

$$\left(p_h, 1\right)_{\partial\mathcal{T}_h} = 0. \tag{3.31}$$

To come to the above formulation from (3.4), realize that use of the flux (3.20) implies that the conservation conditions (3.4d) and (3.4f) are automatically satisfied, and so we do not need to explicitly include these equations in the formulation. We have integrated by parts terms in (2.4e) in order to write the scheme in a concise manner that reveals the symmetric and skew symmetric terms, and have used the divergence-free assumption on $\boldsymbol{w}$. Also, we have used the fact that $\boldsymbol{w} \in H(div, \Omega)$ to conclude $-\left\langle \frac{1}{2} (\boldsymbol{w} \cdot \boldsymbol{n}) \widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{v}} \right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} = 0$ and have removed this term from (3.30d).

In the following, we discuss the well-posedness of Formulation 3.2.

THEOREM 3.3. *(well-posedness of Formulation 3.2)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$ (which is always true for $\tau_t = \tau_t^O$ and $\tau_n = \tau_n^O$). Then Formulation 3.2 is well-posed in the sense that given $\boldsymbol{f}$ and $\boldsymbol{u}_D$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h$.*

Proof. It is sufficient to prove that setting $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_D = \mathbf{0}$ implies that the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h)$ is zero. We can rewrite (3.30) as

$$
\begin{aligned}
& a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h^i\right), (\mathbf{G}, \boldsymbol{v}, \widehat{\boldsymbol{v}})\right) \\
&\quad + a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i\right), (\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})\right) = l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h^i\right), (\mathbf{G}, \boldsymbol{v}, \widehat{\boldsymbol{v}})\right) &= \operatorname{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{v}\right\rangle_{\partial\Omega} \\
&+ \left\langle \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h^i\right), \boldsymbol{v} - \widehat{\boldsymbol{v}}\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega},
\end{aligned}
$$

$$
\begin{aligned}
& a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i\right), (\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}})\right) = \left(\boldsymbol{u}_h, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left(\nabla \cdot \mathbf{L}_h, \boldsymbol{v}\right)_{\mathcal{T}_h} \\
&\quad + \left(\nabla p_h, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h, \nabla q\right)_{\mathcal{T}_h} - \frac{1}{2}\left(\boldsymbol{u}_h \otimes \boldsymbol{w}, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} + \frac{1}{2}\left(\nabla \boldsymbol{u}_h, \boldsymbol{v} \otimes \boldsymbol{w}\right)_{\mathcal{T}_h} \\
&\quad - \left\langle \widehat{\boldsymbol{u}}_h^i, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} + \left\langle \mathbf{L}_h \boldsymbol{n}, \widehat{\boldsymbol{v}}\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} + \left\langle \widehat{\boldsymbol{u}}_h^i \cdot \boldsymbol{n}, q\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} - \left\langle p_h, \widehat{\boldsymbol{v}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} \\
&\quad + \frac{1}{2}\left\langle (\boldsymbol{w} \cdot \boldsymbol{n})\widehat{\boldsymbol{u}}_h^i, v\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} - \frac{1}{2}\left\langle (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{u}_h, \widehat{\boldsymbol{v}}\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega},
\end{aligned}
$$

and

$$
l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h^D, -\mathbf{G}\boldsymbol{n} + q\boldsymbol{n} + \frac{1}{2}(\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{v} - \mathbf{S}\boldsymbol{v}\right\rangle_{\partial\Omega}.
$$

Setting $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_D = \mathbf{0}$ (and therefore $\widehat{\boldsymbol{u}}_h^D = \mathbf{0}$ on $\partial\Omega$), we have $l = 0$. Setting $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}) = \left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^i\right)$, then $a_{skew} = 0$, and the only remaining terms are a_{sym}, giving

$$
\operatorname{Re}\left(\mathbf{L}_h, \mathbf{L}_h\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h^i\right), \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h^i\right\rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} + \left\langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{u}_h\right\rangle_{\partial\Omega} = 0. \tag{3.32}
$$

Thus $\mathbf{L}_h = \mathbf{0}$ in $\mathcal{T}_h$, $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h^i$ on $\mathcal{E}_h^o$, and $\boldsymbol{u}_h = \mathbf{0}$ on $\partial\Omega$.

Equation (3.30a) reduces to $\left(\nabla u_h, \mathbf{G}\right)_{\mathcal{T}_h} = 0$, and since $\nabla \boldsymbol{V}_h \subset \mathbf{G}_h$, we set $\mathbf{G} = \nabla u_h$ to conclude that u_h is elementwise constant. But since $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h$ on $\mathcal{E}_h^o$ and

$\widehat{\boldsymbol{u}}_h$ is single valued on $\mathcal{E}_h^o$, $\boldsymbol{u}_h$ is continuous across each internal interface, and therefore $\boldsymbol{u}_h$ is globally constant. With the zero boundary condition we conclude $\boldsymbol{u}_h = \boldsymbol{0}$ and $\widehat{\boldsymbol{u}}_h = \boldsymbol{0}$.

Integrating what remains of (3.30b) by parts gives $(\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} = 0$, and since $\nabla Q_h \subset \boldsymbol{V}_h$ we conclude that p_h is elementwise constant. Since (3.30d) reduces to $\langle p_h \boldsymbol{n}, \widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega}$, then p_h is globally continuous and globally constant. Then (3.31) implies p_h is zero. ∎

We next prove that the local solver, (3.30a)–(3.30c), in Formulation 3.2 determines the local pressure p_h only up to an elementwise constant.

THEOREM 3.4. *(well-posedness of the local solver of Formulation 3.2)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$. Given $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h/\mathcal{P}_0(\mathcal{T}_h)$ to the local equations (3.30a)–(3.30c).*

Proof. It is sufficient to restrict our attention to a single element, and prove that if $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$ are zero, then the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ is zero. We can rewrite the local problem associated with Formulation 3.2 as find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$ such that

$$
\begin{aligned}
\text{(3.33)}\qquad & \operatorname{Re}(\mathbf{L}_h, \mathbf{G})_K + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{v}\rangle_{\partial K} + (\boldsymbol{u}_h, \nabla\cdot\mathbf{G})_K - (\nabla\cdot\mathbf{L}_h, \boldsymbol{v})_K \\
&+ (\nabla p_h, \boldsymbol{v})_K - (\boldsymbol{u}_h, \nabla q)_K - \frac{1}{2}(\boldsymbol{u}_h \otimes \boldsymbol{w}, \nabla\boldsymbol{v})_K + \frac{1}{2}(\nabla\boldsymbol{u}_h, \boldsymbol{v}\otimes\boldsymbol{w})_K \\
&= (\boldsymbol{f}, \boldsymbol{v})_K - \left\langle \widehat{\boldsymbol{u}}_h, -\mathbf{G}\boldsymbol{n} + q\boldsymbol{n} + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{v} - \mathbf{S}\boldsymbol{v} \right\rangle_{\partial K}
\end{aligned}
$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$. Setting $\boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h$ to zero, and setting $(\mathbf{G}, \boldsymbol{v}, q) = (\mathbf{L}_h, \boldsymbol{u}_h, p_h)$, we have

$$
\text{(3.34)}\qquad \operatorname{Re}(\mathbf{L}_h, \mathbf{L}_h)_K + \langle \mathbf{S}\boldsymbol{u}_h, \boldsymbol{u}_h\rangle_{\partial K} = 0.
$$

Thus $\mathbf{L}_h = \mathbf{0}$ in K and $\boldsymbol{u}_h = \boldsymbol{0}$ on ∂K.

What remains of (3.30a) gives that $\boldsymbol{u}_h$ is constant in K, and since $\boldsymbol{u}_h = \boldsymbol{0}$ on ∂K, that $\boldsymbol{u}_h = \boldsymbol{0}$ in K. Integrating (3.30b) by parts gives that p_h is constant in K. ∎

Formulation 3.2 can be modified in the same way that Formulation 2.2 that the Stokes equations can be modified in order to attain a unique pressure p_h in Q_h, and therefore well-posedness of the local solver. See subsection 2.3.1 for a discussion on the augmented Lagrangian (iterative) method of modifying Formulation 3.2. The matrix system (which must be solved multiple times) associated with the Formulation 3.2 altered by the augmented Lagrangian method looks like

$$
\text{(3.35)}\qquad A\widehat{U}^k = F^{k-1},
$$

where A^k is positive definite. See subsection 2.3.2 for a discussion on a direct method involving an elementwise edge-average pressure as a global variable. The matrix system associated with the Formulation 3.2 altered by the average edge-pressure method looks like

$$
\text{(3.36)}\qquad \begin{bmatrix} A & B^\top \\ -B & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \rho \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},
$$

where A is positive definite.

3.3. HDG Schemes Using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ Flux. In this section, we define new HDG schemes for the Oseen equations. We do this by using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (3.22) on all skeleton faces $\mathcal{E}_h^o$. The justification of this choice will become evident when we analyze the well-posedness of the local solver associated with this scheme, where we verify that no special treatment is required for uniqueness of the local pressure. Recall that for trace unknowns, this flux has the tangent velocity $\widehat{\boldsymbol{u}}_h^t$ and a scalar $\widehat{f}_h$ which approximates $-\frac{1}{\mathrm{Re}}\boldsymbol{n} \cdot [\nabla \boldsymbol{u} \cdot \boldsymbol{n}] + p + \frac{1}{2}(\boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{n})$. The volume unknowns will still be sought from the discontinuous polynomial spaces (2.5). The discontinuous polynomial space in which we seek $\widehat{f}_h$ and $\widehat{\boldsymbol{u}}_h^t$, respectively, are

$$\widehat{F}_h := \left\{ \widehat{g} \in L^2(\mathcal{E}_h) \ : \ \widehat{g}|_e \in \widehat{F}_h(e) \right\}, \tag{3.37}$$

$$\widehat{\boldsymbol{V}}_h^t := \left\{ \widehat{\boldsymbol{v}}^t \in \left[L^2(\mathcal{E}_h)\right]^d \ : \ \widehat{\boldsymbol{v}}^t|_e \in \widehat{\boldsymbol{V}}_h^t(e) \right\}, \tag{3.38}$$

where $\widehat{F}_h(e)$ is a scalar polynomial space, and $\widehat{\boldsymbol{V}}_h^t(e)$ is a vector valued polynomial space with no normal component, defined by

$$\widehat{\boldsymbol{V}}_h^t(e) = \left\{ \sum_{i=1}^{d-1} \boldsymbol{t}^i \widehat{v}_{h,i} \ : \ \widehat{v}_{h,i} \in \widehat{V}_h(e) \right\}, \tag{3.39}$$

where $\widehat{V}_h(e)$ is a scalar polynomial space defined on e, and $\left\{\boldsymbol{t}^1, \ldots, \boldsymbol{t}^{d-1}\right\}$ is a basis of the tangent space of e.

Realize that (3.22) defines $\boldsymbol{u}_h^*$ as

$$\boldsymbol{u}_h^* = \widehat{\boldsymbol{u}}_h^t + \mathbf{N}\boldsymbol{u}_h + \frac{1}{\tau_n}\left(-\boldsymbol{n} \cdot [\mathbf{L}_h \boldsymbol{n}] + p_h + \frac{1}{2}(\boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{u}_h \cdot \boldsymbol{n}) - \widehat{f}_h \right) \boldsymbol{n}. \tag{3.40}$$

The enforcement of the tangent component of the Dirichlet boundary condition (3.4g) then simplifies to an L^2 projection of the tangent part of the Dirichlet boundary data $\boldsymbol{u}_D$ to the trace unknown $\widehat{\boldsymbol{u}}_h^t$ on $\partial\Omega$, thereby decoupling $\widehat{\boldsymbol{u}}_h^t$ on $\partial\Omega$ from the rest of the unknowns. The normal part of the Dirichlet condition is enforced weakly as will be shown below.

Also (3.22) defines

$$-\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n} + \frac{m}{2}\boldsymbol{u}_h^* = \widehat{f}_h \boldsymbol{n} + \mathbf{T}\left(-\mathbf{L}_h \boldsymbol{n} + \frac{1}{2}(\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{u}_h \right) + \tau_t \left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t \right). \tag{3.41}$$

In contrast to Formulation 2.7 for the Stokes equations, this does not correspond to any known boundary condition, so the $\widehat{f}_h$ unknowns on $\partial\Omega$ will remain coupled to the rest of the unknowns, even if we consider boundary conditions beyond pure Dirichlet conditions.

As before, we decompose the velocity trace unknowns into the decoupled parts and the coupled parts of the trace unknowns,

$$\widehat{\boldsymbol{u}}_h^t = \widehat{\boldsymbol{u}}_h^{t,i} + \widehat{\boldsymbol{u}}_h^{t,D}, \tag{3.42}$$

where $\widehat{\boldsymbol{u}}_h^{t,D}$ is defined on $\partial\Omega$ as the L^2 projection of the tangential components of the boundary data,

$$\left\langle \widehat{\boldsymbol{u}}_h^{t,D}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega} = \left\langle \boldsymbol{u}_D^t, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega} \quad \text{for all } \widehat{\boldsymbol{v}}^t \in \widehat{\boldsymbol{V}}_h^t(e) \text{ for all } e \in \partial\Omega, \tag{3.43}$$

and $\widehat{\boldsymbol{u}}_h^{t,i}$ is the trace unknown $\widehat{\boldsymbol{u}}_h^t$ restricted to $\mathcal{E}_h^o$. Again, in writing (3.42) we identify $\widehat{\boldsymbol{u}}_h^{t,i}$, and $\widehat{\boldsymbol{u}}_h^{t,D}$ with their extensions by zero to $\mathcal{E}_h$. We assume that all discrete spaces are of equal polynomial order. Finally, we define the polynomial space

$$\widehat{\boldsymbol{V}}_h^{t,i} := \left\{ \widehat{\boldsymbol{v}}^t \in \left[L^2(\mathcal{E}_h^o)\right]^d \;:\; \widehat{\boldsymbol{v}}^t|_e \in \widehat{\boldsymbol{V}}_h^t(e) \right\}, \tag{3.44}$$

in which $\widehat{\boldsymbol{u}}_h^{t,i}$ lies. With this in place, we write the HDG scheme as follows.

Formulation 3.5. Find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^{t,i} \times \widehat{F}_h$ such that the local equations

$$\begin{aligned} &\mathrm{Re}\,(\mathbf{L}_h, \mathbf{G})_{\mathcal{T}_h} - (\nabla \boldsymbol{u}_h, \mathbf{G})_{\mathcal{T}_h} + \left\langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} \\ &\qquad + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h\right), -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}] \right\rangle_{\partial\mathcal{T}_h} = 0, \end{aligned} \tag{3.45a}$$

$$\begin{aligned} &(\mathbf{L}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (p_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \frac{1}{2}(\boldsymbol{u}_h\otimes\boldsymbol{w}, \nabla\boldsymbol{v})_{\mathcal{T}_h} + \frac{1}{2}(\nabla\boldsymbol{u}_h, \boldsymbol{v}\otimes\boldsymbol{w})_{\mathcal{T}_h} \\ &\qquad + \left\langle \widehat{f}_h, \boldsymbol{v}\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} - \langle \mathbf{L}_h\boldsymbol{n}, \boldsymbol{v}^t\rangle_{\partial\mathcal{T}_h} + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h\right), \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\boldsymbol{v}\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} \\ &\qquad + \left\langle \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\widehat{\boldsymbol{u}}_h^{t,i} + \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right), \boldsymbol{v}^t \right\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \end{aligned} \tag{3.45b}$$

$$(\nabla\cdot\boldsymbol{u}_h, q)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h\right), q \right\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.45c}$$

and the conservation equations combined with the normal part of the boundary condition

$$-\left\langle -\mathbf{L}_h\boldsymbol{n} + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\boldsymbol{u}_h^t + \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right), \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{3.45d}$$

$$-\left\langle \boldsymbol{u}_h\cdot\boldsymbol{n} + \frac{1}{\tau_n}\left(f_h - \widehat{f}_h\right), \widehat{g} \right\rangle_{\partial\mathcal{T}_h} = -\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, \widehat{g}\rangle_{\partial\Omega} \tag{3.45e}$$

hold for all $(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^{t,i} \times \widehat{F}_h$, where f_h is defined as in (3.23), where $\widehat{\boldsymbol{u}}_h^{t,D}$ is defined as in (3.43), and with the zero mean pressure conditions for the uniqueness of the pressure, (3.31).

Note that we have identified the scalar test function $\widehat{g}$ with $-\boldsymbol{n}\cdot\left[\widehat{\mathbf{G}}\boldsymbol{n}\right] + \widehat{q} + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(\widehat{\boldsymbol{v}}\cdot\boldsymbol{n})$ on $\partial\mathcal{T}_h\setminus\partial\Omega$ and with $\widehat{\boldsymbol{w}}\cdot\boldsymbol{n}$ on $\partial\Omega$ in order to write (3.4d), (3.4f), the normal part of (3.4e), and the normal part of (3.4g) in a combined manner as (3.45e). Similarly, we identify $\mathbf{T}\widehat{\boldsymbol{w}}$ with $\widehat{\boldsymbol{v}}^t$ to write the tangent part of (3.4e) as (3.45d). Also note that we have integrated by parts the terms in (3.45a) and (3.45c) and half of the advection term in (3.45b) in order to put the scheme into the form as the above formulation, which readily reveals the symmetric and skew-symmetric terms. Also, we have used the fact that $\boldsymbol{w}\in H(div,\Omega)$ to conclude $-\left\langle \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0$ and have removed this term from (3.45d). We are now ready to prove well-posedness of Formulation 3.5 and its local solver.

THEOREM 3.6. *(well-posedness of Formulation 3.5)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$ (which is always true for $\tau_t = \tau_t^O$ and $\tau_n = \tau_n^O$).*

Then Formulation 3.5 is well-posed in the sense that given $\boldsymbol{f}$ *and* $\boldsymbol{u}_D$*, there exists a unique solution* $\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^t, \widehat{f}_h\right)$ *in* $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h \times \widehat{\boldsymbol{V}}_h^t \times \widehat{F}_h$.

Proof. It is sufficient to prove that if $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_D = \mathbf{0}$, then $\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^t, \widehat{f}_h\right)$ is zero. We can rewrite (3.45) as

$$\begin{aligned} &a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) \\ &\quad + a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) = l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right) \end{aligned}$$

where

$$\begin{aligned} &a_{sym}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) := \operatorname{Re}\left(\mathbf{L}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{v}^t\right\rangle_{\partial\Omega} \\ &\quad + \left\langle \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right), \boldsymbol{v}^t - \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h^i\right), g - \widehat{g}\right\rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

$$\begin{aligned} &a_{skew}\left(\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h\right), \left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right)\right) := -\left(\nabla \boldsymbol{u}_h, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\mathbf{L}_h, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} \\ &\quad - \left(p_h, \nabla\cdot\boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla\cdot\boldsymbol{u}_h, q\right)_{\mathcal{T}_h} + \left\langle \widehat{f}_h^i, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} - \left\langle \boldsymbol{u}_h\cdot\boldsymbol{n}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \boldsymbol{u}_h^t, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\ &\quad - \left\langle \mathbf{L}_h\boldsymbol{n}, \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{u}}_h^{t,i}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} + \left\langle \mathbf{L}_h\boldsymbol{n}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} - \frac{1}{2}\left(\boldsymbol{u}_h\otimes\boldsymbol{w}, \nabla\boldsymbol{v}\right)_{\mathcal{T}_h} \\ &\quad + \frac{1}{2}\left(\nabla\boldsymbol{u}_h, \boldsymbol{v}\otimes\boldsymbol{w}\right)_{\mathcal{T}_h} + \frac{1}{2}\left\langle (\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h^{t,i}, \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} - \frac{1}{2}\left\langle (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h^t, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}, \end{aligned}$$

and

$$\begin{aligned} l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right) &:= \left(\boldsymbol{f}, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, \widehat{g}\right\rangle_{\partial\Omega} \\ &\quad - \left\langle \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h^{t,i} - \tau_t\widehat{\boldsymbol{u}}_h^{t,D}, \boldsymbol{v}^t\right\rangle_{\partial\Omega} + \left\langle \widehat{\boldsymbol{u}}_h^{t,D}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\Omega}, \end{aligned}$$

where we have have written for simplicity the combination of test functions

$$g := -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}] + q + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n}). \tag{3.46}$$

Setting $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_D = \mathbf{0}$ (and therefore $\widehat{\boldsymbol{u}}_h^{t,D} = 0$) gives $l = 0$, and setting $\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{v}}^t, \widehat{g}\right) = \left(\mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h\right)$ gives $a_{skew} = 0$. All that remains is the a_{sym} terms, giving

$$\begin{aligned} &\operatorname{Re}\left(\mathbf{L}_h, \mathbf{L}_h\right)_{\mathcal{T}_h} + \left\langle \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right), \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^{t,i}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} \\ &\quad + \left\langle \tau_t\boldsymbol{u}_h^t, \boldsymbol{u}_h^t\right\rangle_{\partial\Omega} + \left\langle \frac{1}{\tau_n}\left(f_h - \widehat{f}_h\right), f_h - \widehat{f}_h\right\rangle_{\partial\mathcal{T}_h} = 0. \end{aligned} \tag{3.47}$$

All the terms on the left side of the preceding expression are nonnegative and therefore must each be zero. Thus $\mathbf{L}_h = \mathbf{0}$ in $\mathcal{T}_h$, $\boldsymbol{u}_h^t = \widehat{\boldsymbol{u}}_h^{t,i}$ on $\mathcal{E}_h^o$, $\boldsymbol{u}_h^t = 0$ on $\partial\Omega$, and $p_h + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{u}_h\cdot\boldsymbol{n}) = \widehat{f}_h$ on $\mathcal{E}_h$.

Equation (3.45a) reduces to $(\nabla u_h, \mathbf{G})_{\mathcal{T}_h} = 0$, and since $\nabla\boldsymbol{V}_h \subset \mathbf{G}_h$ we can set $\mathbf{G} = \nabla u_h$ to conclude that u_h is elementwise constant. But since $\boldsymbol{u}_h^t = \widehat{\boldsymbol{u}}_h^{t,i}$ on

$\mathcal{E}_h^o$ and $\widehat{\boldsymbol{u}}_h^t$ is single valued on $\mathcal{E}_h^o$, and since (3.45e) reduces to $\langle \boldsymbol{u}_h \cdot \boldsymbol{n}, \widehat{g} \rangle_{\partial \mathcal{T}_h} = 0$, the tangential and normal components of $\boldsymbol{u}_h$ are continuous across each internal interface, and therefore $\boldsymbol{u}_h$ and is globally constant. Since we already have concluded that $\boldsymbol{u}_h^t$ is zero on $\partial\Omega$ (and additionally (3.45e) implies the normal component of $\boldsymbol{u}_h$ is zero on $\partial\Omega$), we can conclude that $\boldsymbol{u}_h$ and $\widehat{\boldsymbol{u}}_h^t$ are zero.

Integrating (3.45b) by parts gives $(\nabla p_h, \boldsymbol{v})_{\mathcal{T}_h} = 0$, and since $\nabla Q_h \subset \boldsymbol{V}_h$ we can set $\boldsymbol{v}$ to ∇p_h to conclude that p_h is elementwise constant. Because $p_h = \widehat{f}_h$ on $\mathcal{E}_h$, p_h is globally constant. Then (3.31) implies p_h and $\widehat{f}_h$ are zero. ∎

THEOREM 3.7. *(well-posedness of the local solver of Formulation 3.5)* *Suppose that $\tau_t > 0$ and $\tau_n > 0$. Given $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$, there exists a unique solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ to the local equations (3.45a)–(3.45c).*

Proof. It is sufficient to restrict our attention to a single element, and prove that if $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$ are zero, then the solution $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ is zero. We can rewrite the local problem associated with Formulation 3.5 as find $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$ such that

$$
\begin{aligned}
(3.48)\quad & \operatorname{Re}(\mathbf{L}_h, \mathbf{G})_K + \left\langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{v}^t \right\rangle_{\partial K} + \left\langle \frac{1}{\tau_n} f_h, g \right\rangle_{\partial K} \\
& - (\nabla \boldsymbol{u}_h, \mathbf{G})_K + (\mathbf{L}_h, \nabla \boldsymbol{v})_K - (p_h, \nabla\cdot\boldsymbol{v})_K + (\nabla\cdot\boldsymbol{u}_h, q)_K \\
& - \frac{1}{2}(\boldsymbol{u}_h \otimes \boldsymbol{w}, \nabla\boldsymbol{v})_K + \frac{1}{2}(\nabla\boldsymbol{u}_h, \boldsymbol{v}\otimes\boldsymbol{w})_K + \left\langle \boldsymbol{u}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial K} - \left\langle \mathbf{L}_h\boldsymbol{n}, \boldsymbol{v}^t \right\rangle_{\partial K} \\
& = (\boldsymbol{f}, \boldsymbol{v})_K + \left\langle \widehat{\boldsymbol{u}}_h^t, \mathbf{G}\boldsymbol{n} \right\rangle_{\partial K} - \left\langle \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\widehat{\boldsymbol{u}}_h^t - \tau_t \widehat{\boldsymbol{u}}_h^t, \boldsymbol{v}^t \right\rangle_{\partial K} \\
& - \left\langle \widehat{f}_h, \boldsymbol{v}\cdot\boldsymbol{n} \right\rangle_{\partial K} + \left\langle \frac{1}{\tau_n}\widehat{f}_h, g \right\rangle_{\partial K}
\end{aligned}
$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h(K) \times \boldsymbol{V}_h(K) \times Q_h(K)$, where f_h is defined as in (3.23) and g is defined as in (3.46). Setting $\boldsymbol{f}$, $\widehat{\boldsymbol{u}}_h^t$, and $\widehat{f}_h$ to zero, and setting $(\mathbf{G}, \boldsymbol{v}, q) = (\mathbf{L}_h, \boldsymbol{u}_h, p_h)$, we have

$$
(3.49)\qquad \operatorname{Re}(\mathbf{L}_h, \mathbf{L}_h)_K + \left\langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{u}_h^t \right\rangle_{\partial K} + \left\langle \frac{1}{\tau_n} f_h, f_h \right\rangle_{\partial K} = 0.
$$

Thus $\mathbf{L}_h = \mathbf{0}$ in K, and $\boldsymbol{u}_h^t = \mathbf{0}$ and $p_h + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\,\boldsymbol{u}_h\cdot\boldsymbol{n} = 0$ on ∂K.

What remains of (3.45a) gives that $\boldsymbol{u}_h$ is constant in K, and since $\boldsymbol{u}_h^t = \mathbf{0}$ on ∂K, that $\boldsymbol{u}_h = \mathbf{0}$ in K. Integrating (3.45b) by parts gives that p_h is constant in K, and since $p_h + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{u}_h\cdot\boldsymbol{n}) = p_h = 0$ on ∂K, that $p_h = 0$ in K. ∎

Finally, we note that the condensed global system associated with Formulation 3.5 takes the form

$$
(3.50)\qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \widehat{U}^t \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},
$$

where A and D are positive semi-definite and constraining one degree of freedom associated with $\widehat{f}_h$ (which is done to enforce (3.31)) renders D positive definite.

3.4. Numerical Results. We consider as a numerical test problem the same problems as considered in the previous section on the Stokes equations. The problem is an analytical solution by Kovasznay [12] to the two dimensional incompressible Navier-Stokes equations. The solution is given by

$$u_1 = 1 - \exp \lambda x_1 \cos 2\pi x_2, \tag{3.51}$$

$$u_2 = \frac{\lambda}{2\pi} \exp \lambda x_1 \sin 2\pi x_2, \tag{3.52}$$

$$p = -\frac{1}{2} \exp 2\lambda x_1. \tag{3.53}$$

A domain of $[0,2]\times[-0.5,1.5]$ is considered, with the exact velocity solution prescribed as Dirichlet boundary conditions on all parts of the domain boundary. Setting $\boldsymbol{f} = 0$, $\boldsymbol{w} = \boldsymbol{u}$, and $\boldsymbol{u}_D = \boldsymbol{u}$, we compute on a mesh of $N \times N$ tensor product square elements, defining the element size $h := \frac{2}{N}$.

In Figure 3, the numerical solution $\boldsymbol{u}_h$ and p_h are plotted. In Figure 4, the $L^2(\Omega)$ error of the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ are plotted along with their convergence rates. The left column of plots shows the L^2 error obtained using the $\widehat{\boldsymbol{u}}_h$ flux (3.20) on all skeleton faces (i.e., Formulation 3.2), while the right column shows the L^2 error obtained using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (3.22) on the interior skeleton faces and the $\widehat{\boldsymbol{u}}_h$ flux (3.20) on the boundary skeleton faces. In both cases τ_t and τ_n are chosen as the upwind parameters τ_t^O and τ_n^O, respectively. As expected, the errors using the two versions of the Godunov flux are virtually identical. In both cases, the observed convergence rates are $k+1$ for $\boldsymbol{u}_h$, and close to $k+1$ for $\mathbf{L}_h$ and p_h.

Next we demonstrate the utility of the HDG schemes for the Oseen equations for solving the (nonlinear) incompressible Navier-Stokes equations. If we consider the Oseen equations (3.1) to be a linear map $\boldsymbol{w} \mapsto \boldsymbol{u}$, then any fixed point of that mapping is a solution to the steady state incompressible Navier-Stokes equations. With this in mind, we can use the general Oseen HDG scheme (3.4) in an iterative manner to numerically solve the incompressible Navier-Stokes equations. Omitting the specification of trial/test spaces for simplicity, we can express the Oseen HDG schemes as solving

$$a\left(\boldsymbol{w}; \mathbf{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{U}}_h; \mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{V}}\right) = l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{V}}\right), \tag{3.54}$$

where $\widehat{\boldsymbol{U}}_h$ and $\widehat{\boldsymbol{V}}$ represent the global unknowns and test functions, respectively. For example, for Formulation 3.2 with the average edge-pressure modification, $\widehat{\boldsymbol{U}}_h$ represents $(\widehat{\boldsymbol{u}}_h^i, \rho_h)$ and $\widehat{\boldsymbol{V}}$ represents $(\widehat{\boldsymbol{v}}, \psi)$, and for Formulation 3.5, $\widehat{\boldsymbol{U}}_h$ represents $(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{f}_h^i)$ and $\widehat{\boldsymbol{V}}$ represents $(\widehat{\boldsymbol{v}}^t, \widehat{g})$. Then, we can define one step of the Picard iteration as solving for $\left(\mathbf{L}_h^m, \boldsymbol{u}_h^m, p_h^m, \widehat{\boldsymbol{U}}_h^m\right)$ using

$$a\left(\boldsymbol{u}_h^{m-1}; \mathbf{L}_h^m, \boldsymbol{u}_h^m, p_h^m, \widehat{\boldsymbol{U}}_h^m; \mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{V}}\right) = l\left(\mathbf{G}, \boldsymbol{v}, q, \widehat{\boldsymbol{V}}\right). \tag{3.55}$$

It remains to define stopping criteria for the nonlinear iteration. One possible stopping criterion involves using a residual $\boldsymbol{r}^m \in \boldsymbol{V}_h$ to the discretized momentum equation that we define by

$$(\boldsymbol{r}^m, \boldsymbol{v})_{\mathcal{T}_h} = a\left(\boldsymbol{u}_h^m; \mathbf{L}_h^m, \boldsymbol{u}_h^m, p_h^m, \widehat{\boldsymbol{U}}_h^m; \mathbf{0}, \boldsymbol{v}, 0, \mathbf{0}\right) - l\left(\mathbf{0}, \boldsymbol{v}, 0, \mathbf{0}\right) \tag{3.56}$$

for all $\boldsymbol{v}$ in $\boldsymbol{V}_h$ and stopping when

$$\|\boldsymbol{r}^m\|_{L^2(\Omega)} < \delta \tag{3.57}$$

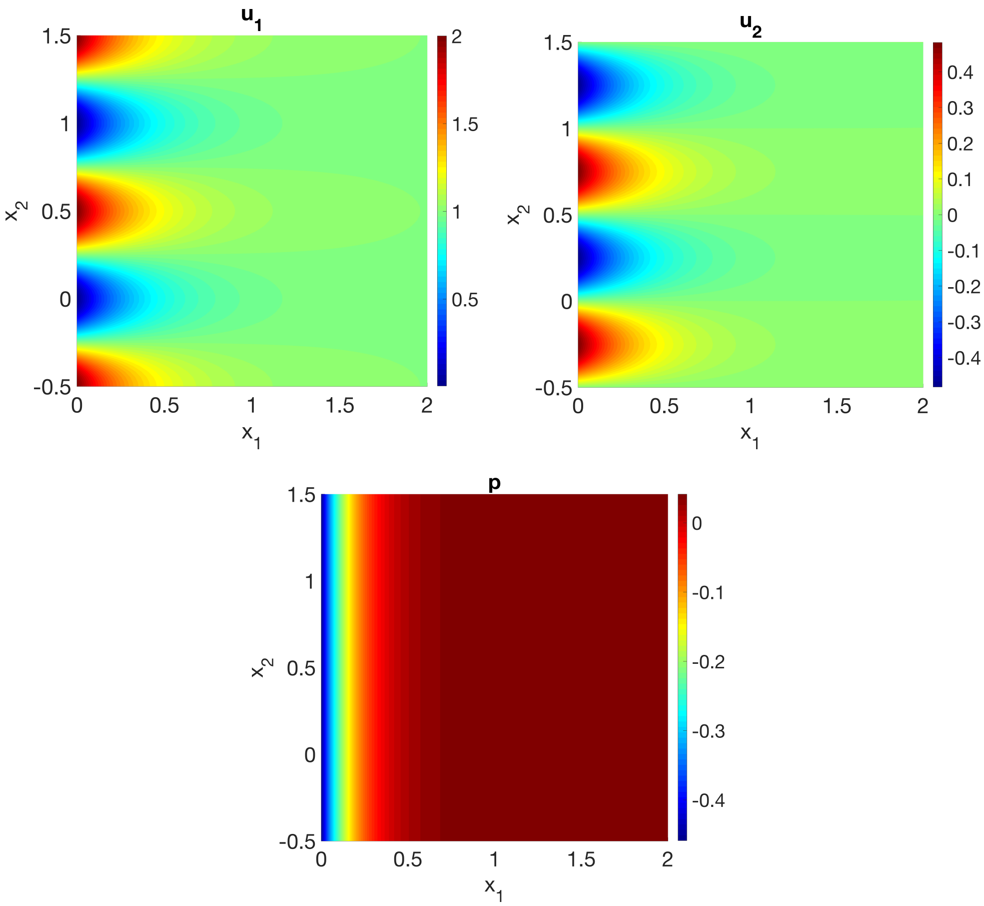

FIG. 3. *Oseen HDG schemes: Kovasznay flow problem solution -* $\boldsymbol{u}_{h1}$ *(top left),* $\boldsymbol{u}_{h2}$ *(top right), and* p_h *(bottom).*

Algorithm 3.1 Picard Iteration for Steady Incompressible Navier-Stokes HDG Schemes.

set initial guess $\boldsymbol{u}_h^0$, choose stopping tolerance δ, and set $m = 1$
while true **do**
  solve for $\left(\mathbf{L}_h^m, \boldsymbol{u}_h^m, p_h^m, \widehat{\boldsymbol{U}}_h^m\right)$ using (3.55)
  if (3.57) is true **then**
    break
  end if
  $m \leftarrow m + 1$
end while

for some $\delta > 0$. The Picard iteration is outlined in Algorithm 3.1

Using the Picard iteration, we can solve the Kovasznay problem by applying the boundary conditions $\boldsymbol{u}_D$ as the exact solution $\boldsymbol{u}$ and applying zero forcing. In Figure 5, the $L^2(\Omega)$ error of the volume unknowns $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ are plotted along with their convergence rates. The left column of plots shows the L^2 error obtained using

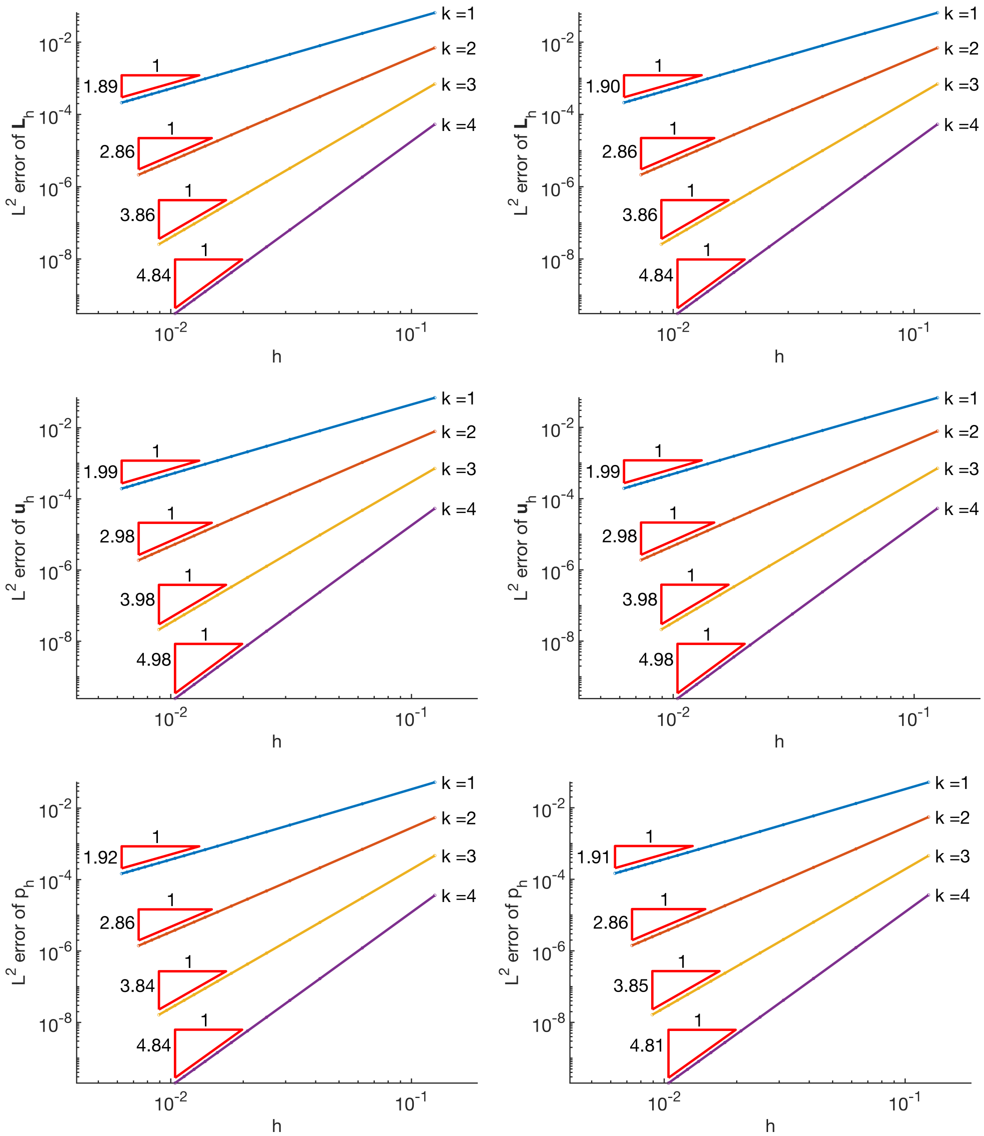

FIG. 4. *Oseen HDG schemes: Kovasznay flow problem L^2 convergence of volume unknowns using $\widehat{\boldsymbol{u}}_h$ flux* (3.20) *(left), using $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux* (3.22) *(right).*

the $\widehat{\boldsymbol{u}}_h$ flux (3.20) on all skeleton faces (i.e., Formulation 3.2), while the right column shows the L^2 error obtained using the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (3.22) on the interior skeleton faces and the $\widehat{\boldsymbol{u}}_h$ flux (3.20) on the boundary skeleton faces. In both cases τ_t and τ_n are chosen as the upwind parameters τ_t^O and τ_n^O, respectively. In both cases, the tolerance for the stopping criterion (3.57) was taken as $\delta = 10^{-10}$ in order to avoid that the error plots level out. For the $\widehat{\boldsymbol{u}}_h$ flux, 10-11 iterations were needed in order to reach the stopping criterion regardless of polynomial order or mesh refinement level. For the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux, it took 11-12 iterations regardless of polynomial order or mesh refinement level. In both cases, an initial guess of zero was used. Again, the errors

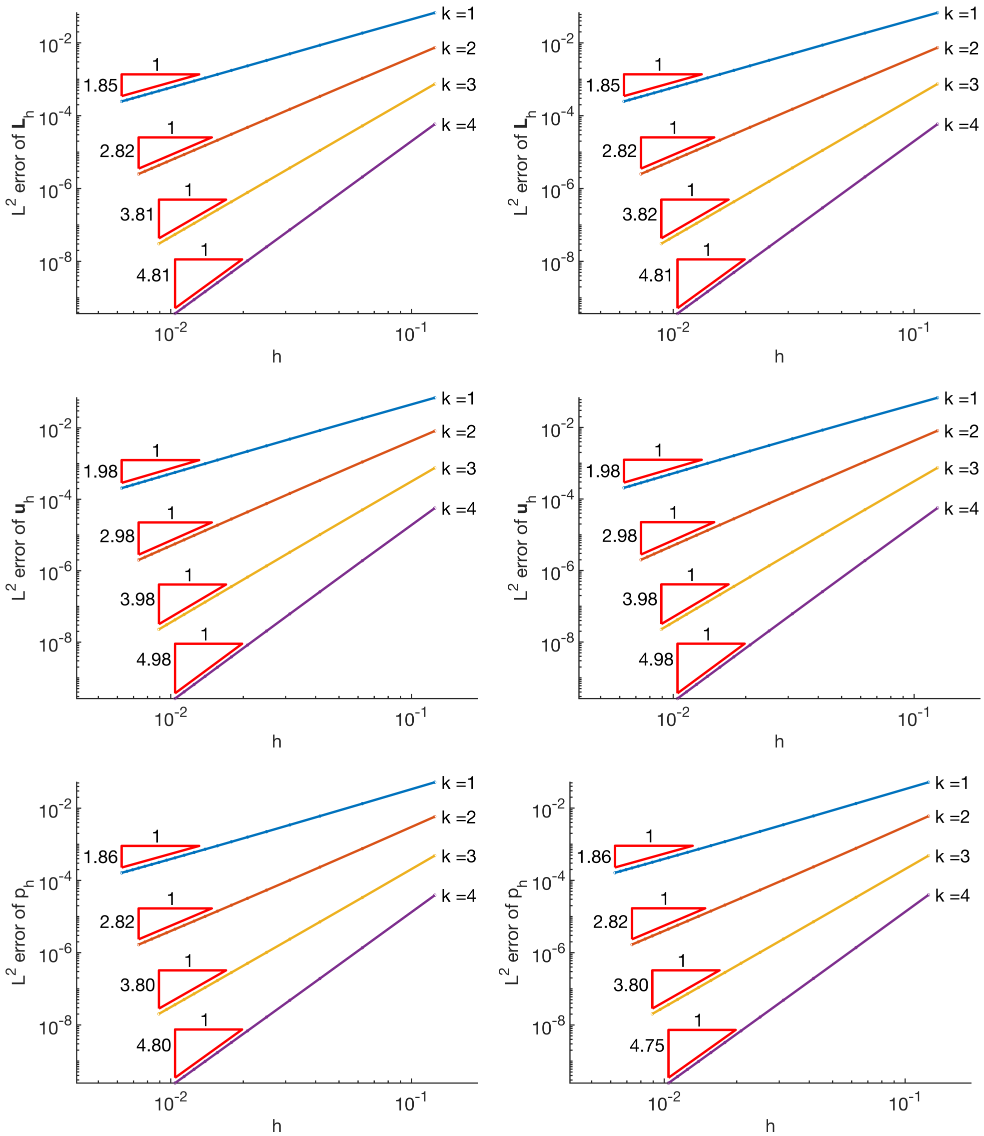

Fig. 5. *Oseen HDG schemes: Kovasznay flow problem nonlinear solution with Picard iteration - L^2 convergence of volume unknowns using $\widehat{\boldsymbol{u}}_h$ flux* (3.20) *(left), using $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux* (3.22) *(right).*

using the two versions of the Godunov flux are virtually identical. In both cases, the observed convergence rates are $k+1$ for $\boldsymbol{u}_h$, and close to $k+1$ for $\mathbf{L}_h$ and p_h, which are the same convergence rates as for the linear Oseen scheme.

3.5. Discussion. Through the upwind HDG methodology [2], we have derived two families of HDG schemes for the Oseen equations. One scheme is based on the $\widehat{\boldsymbol{u}}_h$ flux, and can be related to the scheme analyzed by Cesmelioglu et. al [5]. Rearranging the second term of (3.20), we can write

$$-\mathbf{L}_h^* \boldsymbol{n} + p_h^* \boldsymbol{n} + (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h^* = -\mathbf{L}_h \boldsymbol{n} + p_h \boldsymbol{n} + (\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h + \left(\left[\tau_t + \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}\right]\mathbf{T} + \left[\tau_n + \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}\right]\mathbf{N}\right)(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h).$$

If we denote the stabilization tensor used in [5] by $\mathbf{S}^C := \frac{1}{\mathrm{Re}}\tau_n^C\mathbf{N} + \frac{1}{\mathrm{Re}}\tau_t^C\mathbf{T}$, then we can recover the scheme from [5] by choosing $\tau_n = \frac{1}{\mathrm{Re}}\tau_n^C - \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}$ and $\tau_t = \frac{1}{\mathrm{Re}}\tau_t^C - \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}$ in Formulation 3.2.

Some comments are in order regarding the difference between these similar fluxes. First, we have already shown in the well-posedness for Formulation 3.2 that we must only choose $\tau_t > 0$ and $\tau_n > 0$ for well-posedness, which is always true in particular for the upwind flux parameters τ_t^O and τ_n^O. So, if we would like to define a scheme with ∂K-wise constant, skeleton face-wise constant, or globally constant stability parameters τ_t and τ_n, the only restriction on those stability parameters is that they are positive. On the other hand, using the scheme analyzed in [5], if we would like to define a scheme with ∂K-wise constant, skeleton face-wise constant, or globally constant stability parameters τ_t^C and τ_n^C, we must ensure that $\min\left(\frac{1}{\mathrm{Re}}\tau_t^C - \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}\right) > 0$ ∂K-wise, skeleton face-wise, or globally.

Second, it may appear that the form of the flux in [5] with $(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h$ is a simpler form of the flux than the one in (3.20) which has the terms $\frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h + \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h$. But as we put the advection term in Formulation 3.2 into a form which ensures the skew symmetry of the volume terms upon discretization,

$$-\left(\boldsymbol{u}_h\otimes\boldsymbol{w}, \nabla\boldsymbol{v}\right)_{\mathcal{T}_h} = -\frac{1}{2}\left(\boldsymbol{u}_h\otimes\boldsymbol{w}, \nabla\boldsymbol{v}\right)_{\mathcal{T}_h} + \frac{1}{2}\left(\nabla\boldsymbol{u}_h, \boldsymbol{v}\otimes\boldsymbol{w}\right)_{\mathcal{T}_h} - \frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h},$$

the only advection boundary term remaining in Formulation 3.2 is $\frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h}$, whereas putting the formulation analyzed in [5] into a similar form gives advection boundary terms as $\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h - \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}_h, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h}$. Because of this and the discussion in the previous paragraph, we favor defining the stabilization parameters as in Formulation 3.2 for the Oseen HDG scheme based on the $\widehat{\boldsymbol{u}}_h$ flux.

Third, the formulation in [5] with constant stability parameters (satisfying the conditions already discussed) was proven to converge at order $k+1$ for equal order total degree (simplicial) elements for sufficiently smooth solutions. Here, we have numerically demonstrated the convergence of Formulation 3.2 for 2D tensor product elements, but have made no theoretical claims. This is reserved for future work.

The second family of schemes that we have derived is based on the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux. These schemes are new schemes that are published only in this work (at the time of writing). As opposed to the HDG schemes based on the $\widehat{\boldsymbol{u}}_h$ flux, these HDG schemes do not require special modifications to achieve well-posedness of the local solver. Thus we avoid the iterative nature of the augmented Lagrangian method, and we avoid the introduction additional unknowns of a different nature and the saddle point system that arises from the average edge-pressure method.

It should be reiterated that we have assumed $\nabla\cdot\boldsymbol{w} = 0$ throughout this section by setting $((\nabla\cdot\boldsymbol{w})\boldsymbol{u}_h, \boldsymbol{v}) = 0$ upon integration by parts of half the advection term in (3.4b) to write (3.30b) and (3.45b). When using these schemes iteratively to solve the incompressible Navier-Stokes equations using the Picard iteration outlined in the previous section, we take $\boldsymbol{w}$ to be $\boldsymbol{u}_h^{m-1}$ when solving the mth iterate. It can be seen from (3.30c) and (3.45c) that $\boldsymbol{u}_h$ is only weakly divergence free, and not exactly divergence free. It is an option to perform a postprocessing on the velocity in order to obtain a postprocessed velocity which is exactly divergence free and lies in $H(div,\Omega)$ [8], and then to use the postprocessed velocity as $\boldsymbol{w}$ in the next iteration. Postprocessing is not explored in this work, however, and we simply use the previous iterate of $\boldsymbol{u}_h$. However, we still use Formulations 3.2 and 3.5 as they are written. With this in mind, it can be interpreted that we have added $-\frac{1}{2}(\nabla\cdot\boldsymbol{w})\boldsymbol{u}$ to the left side of the momentum equation (3.1a) and therefore have *added the source term* $-\frac{1}{2}\left((\nabla\cdot\boldsymbol{w})\boldsymbol{u}_h,\boldsymbol{v}\right)_{\mathcal{T}_h}$ to the left side of (3.4b). This term will then cancel the term of opposite sign arising from integration by parts that we have up to this point assumed to be zero on the basis of $\boldsymbol{w}$ being divergence free.

A similar idea applies to the conservation conditions (3.30d) and (3.45d), where we have assumed $\boldsymbol{w}\in H(div,\Omega)$ in order to exclude the $-\frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}$ and $-\frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h^{t,i},\widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}$ terms in Formulations 3.2 and 3.5, respectively. When $\boldsymbol{w}$ is taken as the previous iterate of $\boldsymbol{u}_h$, these terms would no longer be exactly zero, so their omission is interpreted as an approximate enforcement of conservation, or as *adding the stabilization terms* $\frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}$ and $\frac{1}{2}\left\langle(\boldsymbol{w}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_h^{t,i},\widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega}$ to the conservation conditions of Formulations 3.2 and 3.5, respectively. It is interesting to note that using the $\widehat{\boldsymbol{f}}_h$ flux (3.21) avoids this issue altogether.

4. Conclusions.

Through the upwind HDG framework, we have introduced three new HDG schemes for the Stokes equations and three new HDG schemes for the Oseen equations. One Stokes scheme and one Oseen scheme uses a numerical flux based on the tangent velocity trace unknown and an additional scalar trace unknown. The well-posedness analysis reveals that the local solvers associated with these schemes are well-posed without modifications. This is in contrast to the HDG schemes based on the full trace velocity, which require modifications that either require an iterative solution procedure, or introduce additional unknowns and result in a saddle point system. Numerical studies show that the different fluxes give solutions that are nearly identical.

Appendix A. Notation.

In this appendix we review common notation and conventions that apply to the entirety of this work. The spatial dimension of the problem under consideration is denoted by d. Let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be a bounded domain and its boundary $\partial\Omega$ is a Lipschitz manifold. We partition Ω into disjoint elements K (simplices or quadrilaterals/hexahedra), and define $\mathcal{T}_h := \{K\}$ as the collection of elements. We define $\partial\mathcal{T} := \{\partial K : K\in\mathcal{T}\}$ as the collection of element faces (where we use the term "face" regardless of the spatial dimension). For any K, $e=\partial K\cap\partial\Omega$ is a ($d-1$ dimensional) boundary face if e has a nonzero $d-1$ Lebesgue measure. For any two distinct elements K^- and K^+, $e=\partial K^-\cap\partial K^+$ is an interior face if e has a nonzero $d-1$ Lebesgue measure. The collection of all interior faces is denoted by $\mathcal{E}_h^o$ and the collection of all boundary faces is denoted by $\mathcal{E}_h^\partial$. The mesh skeleton $\mathcal{E}_h := \mathcal{E}_h^o\cup\mathcal{E}_h^\partial$ is the collection of all faces, boundary and interior.

We use $(\cdot,\cdot)_D$ or $\langle\cdot,\cdot\rangle_D$ to denote the L^2-inner product on D if D is a d or $(d-1)$ dimensional domain, respectively. For vector (first order tensor) valued functions or second order tensor valued functions, these notations are naturally extended with a component-wise inner product. We define the gradient of a vector (first order tensor), the divergence of a second order tensor, and the outer product symbol $\otimes$ as

$$(\nabla \boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}, \quad (\nabla \cdot \mathbf{L})_i = \sum_{j=1}^{d} \frac{\partial \mathbf{L}_{ij}}{\partial x_j}, \quad (\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j = \left(\boldsymbol{a}\boldsymbol{b}^\top\right)_{ij}. \tag{A.1}$$

In general, we denote vectors by bold, italicized symbols, and we denote matrices and tensors by non-italicized, bold, uppercase letters. When relevant, vectors are to be interpreted as column vectors, and $\mathbf{A}^\top$ denotes the vector or matrix transpose.

In this work $\boldsymbol{n}$ denotes a unit normal vector field on a face of ∂K, and it points outward relative to the element K with which ∂K is associated. If $\partial K^- \cap \partial K^+ \in \mathcal{E}_h$ for two distinct simplices K^-, K^+, then $\boldsymbol{n}^-$ and $\boldsymbol{n}^+$ denote the outward unit normal vector fields on ∂K^- and ∂K^+, respectively, and $\boldsymbol{n}^- = -\boldsymbol{n}^+$ on $\partial K^- \cap \partial K^+$. We simply use $\boldsymbol{n}$ to denote either $\boldsymbol{n}^-$ or $\boldsymbol{n}^+$ in an expression that is valid for both cases, and this convention is also used for other quantities restricted to a face $e \in \mathcal{E}_h$. We use $\tilde{\boldsymbol{n}}$ to define a unique normal vector associated with the face $\partial K^- \cap \partial K^+$. That is, $\tilde{\boldsymbol{n}}$ is chosen arbitrarily as either $\boldsymbol{n}^-$ or $\boldsymbol{n}^+$, so that either $\tilde{\boldsymbol{n}} = \boldsymbol{n}^- = -\boldsymbol{n}^+$ or $\tilde{\boldsymbol{n}} = -\boldsymbol{n}^- = \boldsymbol{n}^+$. Associated with each skeleton face, we define the double valued sgn by

$$\mathrm{sgn} := \mathrm{sgn}(\boldsymbol{n}) = \begin{cases} 1, & \text{if } \boldsymbol{n} = \tilde{\boldsymbol{n}}, \\ -1, & \text{if } \boldsymbol{n} = -\tilde{\boldsymbol{n}} \end{cases}$$

which is either positive or negative one. We define $\mathbf{N} := \boldsymbol{n} \otimes \boldsymbol{n}$ so that the normal component of some vector $\boldsymbol{b}$ can be written as $\boldsymbol{b}^n := (\boldsymbol{b} \cdot \boldsymbol{n})\, \boldsymbol{n} = \mathbf{N}\boldsymbol{b}$. Similarly, we define $\mathbf{T} := \mathbf{I} - \mathbf{N} = -\boldsymbol{n} \times (\boldsymbol{n} \times \cdot)$, where $\mathbf{I}$ is the identity matrix, so that the tangential component of some vector $\boldsymbol{b}$ can be written as $\boldsymbol{b}^t := -\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{b}) = \mathbf{T}\boldsymbol{b}$.

Finally, in the derivation of numerical fluxes for HDG schemes with second order tensor valued auxiliary variables, for conciseness and convenience we will use the Kronecker product and vectorization operator [11, 17]. The Kronecker product is typically denoted by the same symbol ($\otimes$) as the tensor product. Because we use both the tensor product and Kronecker product in this work, in order to avoid confusion we will denote the Kronecker product by $\otimes_K$ (where the subscript refers to "Kronecker"). For an arbitrary $m \times n$ matrix $\mathbf{A}$ and $p \times q$ matrix $\mathbf{B}$, the Kronecker product $\mathbf{A} \otimes_K \mathbf{B}$ is defined by

$$\mathbf{A} \otimes_K \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}, \tag{A.2}$$

or, more concisely, $(\mathbf{A} \otimes_K \mathbf{B})_{p(i-1)+k,q(j-1)+l} = \mathbf{A}_{ij}\mathbf{B}_{kl}$. Among the useful properties of the Kronecker product are the following:

$$(\mathbf{A} \otimes_K \mathbf{B})^\top = \mathbf{A}^\top \otimes_K \mathbf{B}^\top, \tag{A.3}$$

$$(\mathbf{A} \otimes_K \mathbf{B})(\mathbf{C} \otimes_K \mathbf{D}) = (\mathbf{AC}) \otimes_K (\mathbf{BD}). \tag{A.4}$$

The vectorization operator, vec, maps a matrix to a vector that is composed of the columns of the matrix "stacked" on top of each other. For example a 3×3 matrix $\mathbf{L}$ is mapped to the column vector $\mathrm{vec}\,(\mathbf{L}) = (L_{11}; L_{21}; L_{31}; L_{12}; L_{22}; L_{32}; L_{13}; L_{23}; L_{33})$. A convenient relationship between the Kronecker product and the vectorization operator is

$$\mathrm{vec}\,(\mathbf{ABC}) = \left(\mathbf{C}^\top \otimes_K \mathbf{A}\right) \mathrm{vec}\,(\mathbf{B}). \tag{A.5}$$

Appendix B. Characterization of HDG Schemes for the Stokes Equations.

For conforming finite element methods, it is a relatively easy task to determine the form that the matrix structure will take. For the Stokes equations with homogeneous Dirichlet boundary conditions, a conforming finite element method looks like: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h \subset H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\frac{1}{\mathrm{Re}} \left(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}\right)_\Omega - \left(p_h, \nabla \cdot \boldsymbol{v}\right)_\Omega = \left(\boldsymbol{f}, \boldsymbol{v}\right)_\Omega, \tag{B.1}$$

$$-\left(\nabla \cdot \boldsymbol{u}_h, q\right)_\Omega = 0, \tag{B.2}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$ for some stable finite element space pair $(\boldsymbol{V}_h, Q_h)$. Here the letters $\boldsymbol{V}_h$ and Q_h are reused and are not meant to refer to (2.5), and $L_0^2(\Omega)$ refers to functions in $L^2(\Omega)$ with zero average. It is clear that the matrix associated with (B.1) will take the form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{Bmatrix} U \\ P \end{Bmatrix} = F. \tag{B.3}$$

For the HDG schemes for the Stokes equations in section 2, it is not clear what form the condensed global system will take just by looking at the weak form of the HDG scheme. In this appendix, we prove the properties of the condensed global matrices for the Stokes HDG schemes discussed in section 2.

B.1. Characterization of Formulation 2.5. In the following, we characterize the statically condensed global system of the Stokes HDG scheme Formulation 2.5, which uses the $\widehat{\boldsymbol{u}}_h$ flux (2.16) and the augmented Lagrangian modification for well-posedness of the local solver. The following characterization sheds light on the matrix system associated with this formulation. Toward this goal, we define the following local solvers, where $\mathbf{S}$ is a stabilization tensor defined in (2.25).

For $\boldsymbol{\mu} \in \widehat{\boldsymbol{V}}_h^i$, we define $(\mathbf{L}_h^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{\mu}}, p_h^{\boldsymbol{\mu}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\mathrm{Re}\left(\mathbf{L}_h^{\boldsymbol{\mu}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \boldsymbol{\mu}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = 0, \tag{B.4a}$$

$$-\left(\nabla \cdot \mathbf{L}_h^{\boldsymbol{\mu}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{\mu}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu}\right), \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{\mu}}, \boldsymbol{v}\right\rangle_{\partial\Omega_D} = 0, \tag{B.4b}$$

$$\frac{1}{\Delta\tau}\left(p_h^{\boldsymbol{\mu}}, q\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla q\right)_{\mathcal{T}_h} + \left\langle \boldsymbol{\mu}\cdot\boldsymbol{n}, q\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = 0, \tag{B.4c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{U} \in \mathcal{P}_k(\partial\Omega_D)^d$, we define $\left(\mathbf{L}_h^{\boldsymbol{U}}, \boldsymbol{u}_h^{\boldsymbol{U}}, p_h^{\boldsymbol{U}}\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\mathrm{Re}\left(\mathbf{L}_h^{\boldsymbol{U}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{U}}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \left\langle \boldsymbol{U}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\Omega_D} = 0, \tag{B.5a}$$

$$\begin{aligned} -\left(\nabla \cdot \mathbf{L}_h^{\boldsymbol{U}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{U}}, \boldsymbol{v}\right)_{\mathcal{T}_h} &+ \left\langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{U}}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} \\ &+ \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\boldsymbol{U}} - \boldsymbol{U}\right), \boldsymbol{v}\right\rangle_{\partial\Omega_D} = 0, \end{aligned} \tag{B.5b}$$

$$\frac{1}{\Delta\tau}\left(p_h^{\boldsymbol{U}}, q\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h^{\boldsymbol{U}}, \nabla q\right)_{\mathcal{T}_h} + \left\langle \boldsymbol{U}\cdot\boldsymbol{n}, q\right\rangle_{\partial\Omega_D} = 0, \tag{B.5c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{g} \in L^2(\Omega)$, we define $(\mathbf{L}_h^{\boldsymbol{g}}, \boldsymbol{u}_h^{\boldsymbol{g}}, p_h^{\boldsymbol{g}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{\boldsymbol{g}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{g}}, \nabla\cdot\mathbf{G}\right)_{\mathcal{T}_h} = 0, \tag{B.6a}$$

$$-\left(\nabla\cdot\mathbf{L}_h^{\boldsymbol{g}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{g}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{g}}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h} = \left(\boldsymbol{g}, \boldsymbol{v}\right)_{\mathcal{T}_h}, \tag{B.6b}$$

$$\frac{1}{\Delta\tau}\left(p_h^{\boldsymbol{g}}, q\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h^{\boldsymbol{g}}, \nabla q\right)_{\mathcal{T}_h} = 0, \tag{B.6c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $r \in Q_h$, we define $(\mathbf{L}_h^{r}, \boldsymbol{u}_h^{r}, p_h^{r})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{r}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{r}, \nabla\cdot\mathbf{G}\right)_{\mathcal{T}_h} = 0, \tag{B.7a}$$

$$-\left(\nabla\cdot\mathbf{L}_h^{r}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{r}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{r}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.7b}$$

$$\frac{1}{\Delta\tau}\left(p_h^{r}, q\right)_{\mathcal{T}_h} - \left(\boldsymbol{u}_h^{r}, \nabla q\right)_{\mathcal{T}_h} = \frac{1}{\Delta\tau}\left(r, q\right)_{\mathcal{T}_h}, \tag{B.7c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

The local solvers (B.4)–(B.7) can be shown to be well-posed in an identical manner to how the well-posedness of the local solver of Formulation 2.5 is shown in section 2.

At this point, we are in a position to state the main result.

THEOREM B.1. *(characterization of condensed global system for Formulation 2.5) The combined jump condition and Neumann boundary condition (2.31d) can be written as*

$$a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{v}}\right) = l\left(\widehat{\boldsymbol{v}}\right), \tag{B.8}$$

where

$$\begin{aligned} a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{v}}\right) := {} & \left(\operatorname{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \frac{1}{\Delta\tau}\left(p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, p_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}\right\rangle_{\partial\Omega_D} \\ & + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} - \widehat{\boldsymbol{u}}_h^{i,k}\right), \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}} - \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} l_1\left(\widehat{\boldsymbol{v}}\right) := {} & -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\right\rangle_{\partial\Omega_N} + \left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} \\ & + \left\langle -\mathbf{L}_h^{\boldsymbol{f}}\boldsymbol{n} + p_h^{\boldsymbol{f}}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} \\ & + \left\langle -\mathbf{L}_h^{\frac{1}{\Delta\tau}p_h^{k-1}}\boldsymbol{n} + p_h^{\frac{1}{\Delta\tau}p_h^{k-1}}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\frac{1}{\Delta\tau}p_h^{k-1}}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D}. \end{aligned} \tag{B.10}$$

Proof. Due to the linearity of the local solver (2.31a)–(2.31c), we can decompose the volume solution to (2.31a)–(2.31c) as

$$\begin{aligned} \left(\mathbf{L}_h^k, \boldsymbol{u}_h^k, p_h^k\right) = {} & \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}\right) + \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, p_h^{\widehat{\boldsymbol{u}}_h^D}\right) \\ & + \left(\mathbf{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{f}}\right) + \left(\mathbf{L}_h^{\frac{1}{\Delta\tau}p_h^{k-1}}, \boldsymbol{u}_h^{\frac{1}{\Delta\tau}p_h^{k-1}}, p_h^{\frac{1}{\Delta\tau}p_h^{k-1}}\right). \end{aligned}$$

That is, $\left(\mathbf{L}_h^k, \boldsymbol{u}_h^k, p_h^k\right)$ is the sum of the solutions to (B.4)–(B.7) with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^{i,k}$, $\boldsymbol{U} = \widehat{\boldsymbol{u}}_h^D$, $\boldsymbol{g} = \boldsymbol{f}$, and $\boldsymbol{r} = \frac{1}{\Delta\tau} p_h^{k-1}$. Then, the combined jump and Neumann boundary condition (2.31d) can be written as

$$
\begin{aligned}
&-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} - \widehat{\boldsymbol{u}}_h^{i,k}\right), \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} \\
&\quad - \left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D} \boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^D} \boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} - \left\langle -\mathbf{L}_h^{\boldsymbol{f}} \boldsymbol{n} + p_h^{\boldsymbol{f}} \boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} \\
&\quad - \left\langle -\mathbf{L}_h^{\frac{1}{\Delta\tau} p_h^{k-1}} \boldsymbol{n} + p_h^{\frac{1}{\Delta\tau} p_h^{k-1}} \boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\frac{1}{\Delta\tau} p_h^{k-1}}, \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}} \right\rangle_{\partial\Omega_N}.
\end{aligned}
$$

It remains to show $-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} - \widehat{\boldsymbol{u}}_h^{i,k}\right), \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{v}}\right)$ as defined by (B.9). In (B.4a) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}$ and $\mathbf{G} = \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}$, in (B.4b) take $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^{i,k}$ and $\boldsymbol{v} = \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}$, and in (B.4c) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}$ and $q = p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}$. Summing the result, we have

$$
\begin{aligned}
&\left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \frac{1}{\Delta\tau}\left(p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, p_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} - \widehat{\boldsymbol{u}}_h^{i,k}\right), \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} \\
&\qquad + \left\langle \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}} \right\rangle_{\partial\Omega_D} - \left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n}, \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} + \left\langle p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \widehat{\boldsymbol{v}} \cdot \boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} \boldsymbol{n}, \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} - \left\langle p_h^{\widehat{\boldsymbol{u}}_h^{i,k}}, \widehat{\boldsymbol{v}} \cdot \boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} \\
&\qquad - \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} - \widehat{\boldsymbol{u}}_h^{i,k}\right), \widehat{\boldsymbol{v}} \right\rangle_{\partial\mathcal{T}_h \setminus \partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{v}}\right). \qquad \square
\end{aligned}
$$

We can conclude from Theorem B.1 that the condensed global system will take the form

$$
A\widehat{U}^k = F^{k-1}.
$$

Inspecting (B.9), we can see that the block matrix A is symmetric and positive semi-definite. We can further claim that A is positive definite. To support this claim we must show $a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{u}}_h^{i,k}\right) = 0 \Rightarrow \widehat{\boldsymbol{u}}_h^{i,k} = \mathbf{0}$. Indeed, $a\left(\widehat{\boldsymbol{u}}_h^{i,k}, \widehat{\boldsymbol{u}}_h^{i,k}\right) = 0$ implies $\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = \mathbf{0}$, $p_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = 0$, $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = 0$ on $\partial\Omega_D$, and $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = \widehat{\boldsymbol{u}}_h^{i,k}$ on $\mathcal{E}_h \setminus \partial\Omega_D$. Then, with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^{i,k}$ in (B.4a), integrating by parts reveals that $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}}$ is elementwise constant, and therefore globally constant since $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = \widehat{\boldsymbol{u}}_h^{i,k}$ on $\mathcal{E}_h \setminus \partial\Omega_D$. Since $\partial\Omega_D \neq \emptyset$ then $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = 0$ and therefore $\widehat{\boldsymbol{u}}_h^{i,k} = 0$.

B.2. Characterization of Formulation 2.6. In the following, we characterize the statically condensed global system of the Stokes HDG scheme Formulation 2.6, which uses the $\widehat{\boldsymbol{u}}_h$ flux (2.16) and the average edge-pressure modification for well-posedness of the local solver. The following characterization sheds light on the matrix system associated with this formulation. Toward this goal, we define the following local solvers, where $\mathbf{S}$ is a stabilization tensor defined in (2.25).

For $\boldsymbol{\mu} \in \widehat{\boldsymbol{V}}_h^i$, we define $(\mathbf{L}_h^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{\mu}}, p_h^{\boldsymbol{\mu}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{\boldsymbol{\mu}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \langle \boldsymbol{\mu}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} = 0, \tag{B.11a}$$

$$-\left(\nabla \cdot \mathbf{L}_h^{\boldsymbol{\mu}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{\mu}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{\mu}}, \boldsymbol{v}\rangle_{\partial\Omega_D} + \langle \mathbf{S}\left(\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu}\right), \boldsymbol{v}\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} = 0, \tag{B.11b}$$

$$-\left(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla q\right)_{\mathcal{T}_h} + \langle \boldsymbol{\mu} \cdot \boldsymbol{n}, q - \overline{q}\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} + \left\langle \overline{p_h^{\boldsymbol{\mu}}}, \overline{q}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.11c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\beta \in \mathcal{P}_0(\partial\mathcal{T}_h)$, we define $\left(\mathbf{L}_h^{\beta}, \boldsymbol{u}_h^{\beta}, p_h^{\beta}\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{\beta}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\beta}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} = 0, \tag{B.12a}$$

$$-\left(\nabla \cdot \mathbf{L}_h^{\beta}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\beta}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\beta}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.12b}$$

$$-\left(\boldsymbol{u}_h^{\beta}, \nabla q\right)_{\mathcal{T}_h} + \left\langle \overline{p_h^{\beta}}, \overline{q}\right\rangle_{\partial\mathcal{T}_h} - \langle \beta, \overline{q}\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.12c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{U} \in \mathcal{P}_k(\partial\Omega_D)^d$, we define $\left(\mathbf{L}_h^{\boldsymbol{U}}, \boldsymbol{u}_h^{\boldsymbol{U}}, p_h^{\boldsymbol{U}}\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{\boldsymbol{U}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{U}}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} - \langle \boldsymbol{U}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\Omega_D} = 0, \tag{B.13a}$$

$$\begin{aligned} -\left(\nabla \cdot \mathbf{L}_h^{\boldsymbol{U}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{U}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{U}}, \boldsymbol{v}\right\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega_D} & \\ + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\boldsymbol{U}} - \boldsymbol{U}\right), \boldsymbol{v}\right\rangle_{\partial\Omega_D} &= 0, \end{aligned} \tag{B.13b}$$

$$-\left(\boldsymbol{u}_h^{\boldsymbol{U}}, \nabla q\right)_{\mathcal{T}_h} + \langle \boldsymbol{U} \cdot \boldsymbol{n}, q\rangle_{\partial\Omega_D} + \left\langle \overline{p_h^{\boldsymbol{U}}}, \overline{q}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.13c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{g} \in L^2(\Omega)$, we define $(\mathbf{L}_h^{\boldsymbol{g}}, \boldsymbol{u}_h^{\boldsymbol{g}}, p_h^{\boldsymbol{g}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\operatorname{Re}\left(\mathbf{L}_h^{\boldsymbol{g}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left(\boldsymbol{u}_h^{\boldsymbol{g}}, \nabla \cdot \mathbf{G}\right)_{\mathcal{T}_h} = 0, \tag{B.14a}$$

$$-\left(\nabla \cdot \mathbf{L}_h^{\boldsymbol{g}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \left(\nabla p_h^{\boldsymbol{g}}, \boldsymbol{v}\right)_{\mathcal{T}_h} + \langle \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{g}}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{g}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{B.14b}$$

$$-\left(\boldsymbol{u}_h^{\boldsymbol{g}}, \nabla q\right)_{\mathcal{T}_h} + \left\langle \overline{p_h^{\boldsymbol{g}}}, \overline{q}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.14c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

The local solvers (B.11)–(B.14) can be shown to be well-posed in an identical manner to how the well-posedness of the local solver of Formulation 2.6 is shown in section 2.

At this point, we are in a position to state the main result.

Theorem B.2. *(characterization of condensed global system for Formulation 2.6) The combined jump condition and Neumann boundary condition (2.35d) with the additional condition (2.35e) can be written as*

$$a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{v}}\right) + b\left(\widehat{\boldsymbol{v}}, \rho_h\right) = l_1\left(\widehat{\boldsymbol{v}}\right), \tag{B.15a}$$

$$-b\left(\widehat{\boldsymbol{u}}_h^i, \psi\right) = l_2\left(\psi\right), \tag{B.15b}$$

where

$$a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{v}}\right) := \left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i}, \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}\right\rangle_{\partial\Omega_D} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} - \widehat{\boldsymbol{u}}_h^i\right), \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}} - \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D}, \tag{B.16}$$

$$b\left(\widehat{\boldsymbol{v}}, \psi\right) := -\left\langle \widehat{\boldsymbol{v}}\cdot\boldsymbol{n}, \psi\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D}, \tag{B.17}$$

$$l_1\left(\widehat{\boldsymbol{v}}\right) := -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\right\rangle_{\partial\Omega_N}, + \left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} + \left\langle -\mathbf{L}_h^{\boldsymbol{f}}\boldsymbol{n} + p_h^{\boldsymbol{f}}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D}, \tag{B.18}$$

and

$$l_2\left(\psi\right) := -\left\langle \psi, \widehat{\boldsymbol{u}}_h^D\cdot\boldsymbol{n}\right\rangle_{\partial\Omega_D}. \tag{B.19}$$

Proof. Due to the linearity of the local solver (2.35a)–(2.35c), we can decompose the volume solution to (2.35a)–(2.35c) as $(\mathbf{L}_h, \boldsymbol{u}_h, p_h) = \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i}, p_h^{\widehat{\boldsymbol{u}}_h^i}\right) + (\mathbf{L}_h^{\rho_h}, \boldsymbol{u}_h^{\rho_h}, p_h^{\rho_h}) + \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, p_h^{\widehat{\boldsymbol{u}}_h^D}\right) + \left(\mathbf{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{f}}\right)$. That is, $(\mathbf{L}_h, \boldsymbol{u}_h, p_h)$ is the sum of the solutions to (B.11)–(B.14) with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^i$, $\beta = \rho_h$, $\boldsymbol{U} = \widehat{\boldsymbol{u}}_h^D$, and $\boldsymbol{g} = \boldsymbol{f}$. Then, the combined jump and Neumann boundary condition (2.35d) can be written as

$$\begin{aligned} &-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} - \widehat{\boldsymbol{u}}_h^i\right), \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\ &\quad - \left\langle -\mathbf{L}_h^{\rho_h}\boldsymbol{n} + p_h^{\rho_h}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\rho_h}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\ &\quad - \left\langle -\mathbf{L}_h^{\boldsymbol{f}}\boldsymbol{n} + p_h^{\boldsymbol{f}}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = -\left\langle \boldsymbol{f}_N, \widehat{\boldsymbol{v}}\right\rangle_{\partial\Omega_N}. \end{aligned} \tag{B.20}$$

It remains to show that $-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n} + p_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n} + \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} - \widehat{\boldsymbol{u}}_h^i\right), \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{v}}\right)$ as defined by (B.16) and that $-\left\langle -\mathbf{L}_h^{\rho_h}\boldsymbol{n} + p_h^{\rho_h}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\rho_h}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = b\left(\widehat{\boldsymbol{v}}, \rho_h\right)$ as defined by (B.17).

Step 1: Taking q equal to a (nonzero) elementwise constant in (B.12c) gives

$$\overline{p_h^{\beta}} = \beta \tag{B.21}$$

and

$$-\left(\boldsymbol{u}_h^{\beta}, \nabla q\right)_{\mathcal{T}_h} = 0. \tag{B.22}$$

Then setting $(\mathbf{G}, \boldsymbol{v}, q) = \left(\mathbf{L}_h^{\beta}, \boldsymbol{u}_h^{\beta}, p_h^{\beta}\right)$ in (B.12a), (B.12b), and (B.22), we conclude by summing the results that

$$\left(\mathrm{Re}\mathbf{L}_h^{\beta}, \mathbf{L}_h^{\beta}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\beta}, \boldsymbol{u}_h^{\beta}\right\rangle_{\partial\mathcal{T}_h} = 0$$

and therefore that $\mathbf{L}_h^{\beta} = \mathbf{0}$, and $\boldsymbol{u}_h^{\beta} = \mathbf{0}$ on $\partial\mathcal{T}_h$. Integrating what remains of (B.12a) by parts, we conclude that $\boldsymbol{u}_h^{\beta}$ is elementwise constant and therefore zero. Then what remains of (B.12b) implies that p_h^{β} is elementwise constant, and therefore $p_h^{\beta} = \beta$. Summarizing, we have that for any β in $\mathcal{P}_0(\partial\mathcal{T}_h)$, that $\left(\mathbf{L}_h^{\beta}, \boldsymbol{u}_h^{\beta}, p_h^{\beta}\right) = (\mathbf{0}, \mathbf{0}, \beta)$. Therefore $-\langle -\mathbf{L}_h^{\rho_h}\boldsymbol{n} + p_h^{\rho_h}\boldsymbol{n} + \mathbf{S}\boldsymbol{u}_h^{\rho_h}, \widehat{\boldsymbol{v}}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = b\left(\rho_h, \widehat{\boldsymbol{v}}\right)$.

Step 2: Taking q equal to a (nonzero) constant in (B.11c) gives

$$\bar{p}_h^{\boldsymbol{\mu}} = 0 \tag{B.23}$$

and

$$-\left(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla q\right)_{\mathcal{T}_h} + \langle \boldsymbol{\mu}\cdot\boldsymbol{n}, q - \overline{q}\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = 0. \tag{B.24}$$

In (B.11a) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}$ and $\mathbf{G} = \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}$, in (B.11b) take $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^i$ and $\boldsymbol{v} = \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}$, and in (B.24) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}$ and $q = p_h^{\widehat{\boldsymbol{u}}_h^i}$. Summing the result, and recalling (B.23), we have

$$\begin{aligned}\left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}}\right)_{\mathcal{T}_h} + \left\langle \mathbf{S}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i}, \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}\right\rangle_{\partial\Omega_D} + \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} - \widehat{\boldsymbol{u}}_h^i\right), \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} \\ - \left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \left\langle p_h^{\widehat{\boldsymbol{u}}_h^i}, \widehat{\boldsymbol{v}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = 0.\end{aligned} \tag{B.25}$$

Therefore,

$$\left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i}\boldsymbol{n}, \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} - \left\langle p_h^{\widehat{\boldsymbol{u}}_h^i}, \widehat{\boldsymbol{v}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} - \left\langle \mathbf{S}\left(\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} - \widehat{\boldsymbol{u}}_h^i\right), \widehat{\boldsymbol{v}}\right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{v}}\right). \blacksquare$$

We can conclude from Theorem B.2 that the condensed global system will take the form

$$\begin{bmatrix} A & B^{\top} \\ -B & 0 \end{bmatrix} \begin{Bmatrix} \widehat{U} \\ \rho \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}.$$

Inspecting (B.16), we can see that the block matrix A is symmetric and positive semi-definite. We can further claim that A is positive definite. To claim this we must show $a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{u}}_h^i\right) = 0 \Rightarrow \widehat{\boldsymbol{u}}_h^i = \mathbf{0}$. Indeed, $a\left(\widehat{\boldsymbol{u}}_h^i, \widehat{\boldsymbol{u}}_h^i\right) = 0$ implies $\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^i} = \mathbf{0}$, $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} = 0$ on $\partial\Omega_D$, and $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} = \widehat{\boldsymbol{u}}_h^i$ on $\mathcal{E}_h\backslash\partial\Omega_D$. Then, with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^i$ in (B.11a), integrating by parts reveals that $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i}$ is elementwise constant, and therefore globally constant since $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^i} = \widehat{\boldsymbol{u}}_h^i$ on $\mathcal{E}_h\backslash\partial\Omega_D$. Since $\partial\Omega_D \neq \emptyset$, then $\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{i,k}} = 0$ and therefore $\widehat{\boldsymbol{u}}_h^i = 0$.

B.3. Characterization of Formulation 2.7. In the following, we characterize the statically condensed global system of the Stokes HDG scheme Formulation 2.7, which uses the $(\widehat{\boldsymbol{u}}_h^t, \widehat{f}_h)$ flux (2.18). The following characterization sheds light on the matrix system associated with this formulation. Toward this goal, we define the following local solvers, where

$$\begin{aligned} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}} &:= -\boldsymbol{n}\cdot\left[\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n}\right] + p_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n}, \\ f_h^{\boldsymbol{\mu}} &:= -\boldsymbol{n}\cdot\left[\mathbf{L}_h^{\boldsymbol{\mu}}\boldsymbol{n}\right] + p_h^{\boldsymbol{\mu}}\boldsymbol{n}, \end{aligned}$$

etc.

For $\boldsymbol{\mu} \in \widehat{\boldsymbol{V}}_h^{t,i}$, we define $(\mathbf{L}_h^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{\mu}}, p_h^{\boldsymbol{\mu}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\begin{aligned}
&\text{(B.26a)} && \operatorname{Re}(\mathbf{L}_h^{\boldsymbol{\mu}}, \mathbf{G})_{\mathcal{T}_h} - (\nabla \boldsymbol{u}_h^{\boldsymbol{\mu}}, \mathbf{G})_{\mathcal{T}_h} + \langle \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{\mu}}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\Omega_D} \\
&&& + \langle \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{\mu}}, -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\mathcal{T}_h} = 0, \\
&\text{(B.26b)} && (\mathbf{L}_h^{\boldsymbol{\mu}}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (p_h^{\boldsymbol{\mu}}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \left\langle -\mathbf{L}_h^{\boldsymbol{\mu}}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{\mu}}, \boldsymbol{v}^t\right\rangle_{\partial\Omega_D} \\
&&& + \left\langle -\mathbf{L}_h^{\boldsymbol{\mu}}\boldsymbol{n} + \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu}\right), \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = 0, \\
&\text{(B.26c)} && (\nabla\cdot\boldsymbol{u}_h^{\boldsymbol{\mu}}, q)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{\mu}}, q\right\rangle_{\partial\mathcal{T}_h} = 0,
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\gamma \in \widehat{F}_h^i$, we define $(\mathbf{L}_h^{\gamma}, \boldsymbol{u}_h^{\gamma}, p_h^{\gamma})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\begin{aligned}
&\text{(B.27a)} && \operatorname{Re}(\mathbf{L}_h^{\gamma}, \mathbf{G})_{\mathcal{T}_h} - (\nabla \boldsymbol{u}_h^{\gamma}, \mathbf{G})_{\mathcal{T}_h} + \langle \mathbf{T}\boldsymbol{u}_h^{\gamma}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&&& + \left\langle \frac{1}{\tau_n}\left(f_h^{\gamma} - \gamma\right), -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \frac{1}{\tau_n} f_h^{\gamma}, -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\Omega_N} = 0, \\
&\text{(B.27b)} && (\mathbf{L}_h^{\gamma}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (p_h^{\gamma}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} \\
&&& + \left\langle -\mathbf{L}_h^{\gamma}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\gamma}, \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h} + \langle \gamma, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} = 0, \\
&\text{(B.27c)} && (\nabla\cdot\boldsymbol{u}_h^{\gamma}, q)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n}\left(f_h^{\gamma} - \gamma\right), q\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \frac{1}{\tau_n} f_h^{\gamma}, q\right\rangle_{\partial\Omega_N} = 0,
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{U} \in \widehat{\boldsymbol{V}}_h^t(\partial\Omega_D)$, we define $\left(\mathbf{L}_h^{\boldsymbol{U}}, \boldsymbol{u}_h^{\boldsymbol{U}}, p_h^{\boldsymbol{U}}\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\begin{aligned}
&\text{(B.28a)} && \operatorname{Re}\left(\mathbf{L}_h^{\boldsymbol{U}}, \mathbf{G}\right)_{\mathcal{T}_h} - \left(\nabla \boldsymbol{u}_h^{\boldsymbol{U}}, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{U}}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\
&&& + \left\langle \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{U}} - \boldsymbol{U}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\Omega_D} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{U}}, -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\mathcal{T}_h} = 0, \\
&\text{(B.28b)} && \left(\mathbf{L}_h^{\boldsymbol{U}}, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(p_h^{\boldsymbol{U}}, \nabla\cdot\boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle -\mathbf{L}_h^{\boldsymbol{U}}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{U}}, \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\
&&& + \left\langle -\mathbf{L}_h^{\boldsymbol{U}}\boldsymbol{n} + \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\boldsymbol{U}} - \boldsymbol{U}\right), \boldsymbol{v}^t\right\rangle_{\partial\Omega_D} = 0, \\
&\text{(B.28c)} && \left(\nabla\cdot\boldsymbol{u}_h^{\boldsymbol{U}}, q\right)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{U}}, q\right\rangle_{\partial\mathcal{T}_h} = 0,
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $F \in \widehat{F}_h(\partial\Omega_N)$, we define $\left(\mathbf{L}_h^{F}, \boldsymbol{u}_h^{F}, p_h^{F}\right)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\begin{aligned}
&\text{(B.29a)} && \operatorname{Re}\left(\mathbf{L}_h^{F}, \mathbf{G}\right)_{\mathcal{T}_h} - \left(\nabla \boldsymbol{u}_h^{F}, \mathbf{G}\right)_{\mathcal{T}_h} + \left\langle \mathbf{T}\boldsymbol{u}_h^{F}, \mathbf{G}\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} \\
&&& + \left\langle \frac{1}{\tau_n} f_h^{F}, -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \frac{1}{\tau_n}\left(f_h^{F} - F\right), -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}]\right\rangle_{\partial\Omega_N} = 0, \\
&\text{(B.29b)} && \left(\mathbf{L}_h^{F}, \nabla \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(p_h^{F}, \nabla\cdot\boldsymbol{v}\right)_{\mathcal{T}_h} \\
&&& + \left\langle -\mathbf{L}_h^{F}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{F}, \boldsymbol{v}^t\right\rangle_{\partial\mathcal{T}_h} + \langle F, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_N} = 0, \\
&\text{(B.29c)} && \left(\nabla\cdot\boldsymbol{u}_h^{F}, q\right)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{F}, q\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \frac{1}{\tau_n}\left(f_h^{F} - F\right), q\right\rangle_{\partial\Omega_N} = 0,
\end{aligned}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

For $\boldsymbol{g} \in L^2(\Omega)$, we define $(\mathbf{L}_h^{\boldsymbol{g}}, \boldsymbol{u}_h^{\boldsymbol{g}}, p_h^{\boldsymbol{g}})$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$ as the solution to

$$\mathrm{Re}\,(\mathbf{L}_h^{\boldsymbol{g}}, \mathbf{G})_{\mathcal{T}_h} - (\nabla \boldsymbol{u}_h^{\boldsymbol{g}}, \mathbf{G})_{\mathcal{T}_h} + \langle \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{g}}, \mathbf{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{g}}, -\boldsymbol{n}\cdot[\mathbf{G}\boldsymbol{n}] \right\rangle_{\partial\mathcal{T}_h} = 0 \tag{B.30a}$$

$$(\mathbf{L}_h^{\boldsymbol{g}}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (p_h^{\boldsymbol{g}}, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \left\langle -\mathbf{L}_h^{\boldsymbol{g}}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{g}}, \boldsymbol{v}^t \right\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{g}, \boldsymbol{v})_{\mathcal{T}_h} \tag{B.30b}$$

$$(\nabla\cdot\boldsymbol{u}_h^{\boldsymbol{g}}, q)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\boldsymbol{g}}, q \right\rangle_{\partial\mathcal{T}_h} = 0, \tag{B.30c}$$

for all $(\mathbf{G}, \boldsymbol{v}, q)$ in $\mathbf{G}_h \times \boldsymbol{V}_h \times Q_h$.

The local solvers (B.26)–(B.30) can be shown to be well-posed in an identical manner to how the well-posedness of the local solver of Formulation 2.7 is shown in section 2.

At this point, we are in a position to state the main result.

THEOREM B.3. *(characterization of condensed global system for Formulation 2.7) The jump conditions (2.49d) and (2.49e) can be written as*

$$a\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{\boldsymbol{v}}^t\right) + b\left(\widehat{\boldsymbol{v}}^t, \widehat{f}_h^i\right) = l_1\left(\widehat{\boldsymbol{v}}^t\right), \tag{B.31a}$$

$$-b\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{g}\right) + d\left(\widehat{f}_h^i, \widehat{g}\right) = l_2\left(\widehat{g}\right), \tag{B.31b}$$

where

$$\begin{aligned} a\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{\boldsymbol{v}}^t\right) := {} & \left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} + \left\langle \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} - \widehat{\boldsymbol{u}}_h^{t,i}\right), \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t} - \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\ & + \left\langle \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, f_h^{\widehat{\boldsymbol{v}}^t} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t} \right\rangle_{\partial\Omega_D}, \end{aligned} \tag{B.32}$$

$$\begin{aligned} d\left(\widehat{f}_h^i, \widehat{g}\right) := {} & \left(\mathrm{Re}\mathbf{L}_h^{\widehat{f}_h^i}, \mathbf{L}_h^{\widehat{g}}\right)_{\mathcal{T}_h} + \left\langle \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \mathbf{T}\boldsymbol{u}_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h} \\ & + \left\langle \frac{1}{\tau_n}\left(f_h^{\widehat{f}_h^i} - \widehat{f}_h^i\right), f_h^{\widehat{g}} - \widehat{g} \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \frac{1}{\tau_n} f_h^{\widehat{f}_h^i}, f_h^{\widehat{g}} \right\rangle_{\partial\Omega_N}, \end{aligned} \tag{B.33}$$

$$\begin{aligned} b\left(\widehat{\boldsymbol{v}}^t, \widehat{g}\right) := {} & \left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}, \mathbf{L}_h^{\widehat{g}}\right)_{\mathcal{T}_h} - \left(\nabla \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}, \mathbf{L}_h^{\widehat{g}}\right)_{\mathcal{T}_h} - \left(\mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}, \nabla \boldsymbol{u}_h^{\widehat{g}}\right)_{\mathcal{T}_h} + \left(p_h^{\widehat{\boldsymbol{v}}^t}, \nabla\cdot\boldsymbol{u}_h^{\widehat{g}}\right)_{\mathcal{T}_h} \\ & + \left(\nabla\cdot\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}, p_h^{\widehat{g}}\right)_{\mathcal{T}_h} + \left\langle \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}, \mathbf{L}_h^{\widehat{g}}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\boldsymbol{n}, \mathbf{T}\boldsymbol{u}_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h} \\ & - \left\langle \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}, \mathbf{T}\boldsymbol{u}_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{v}}^t}, f_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h}, \end{aligned} \tag{B.34}$$

$$\begin{aligned} l_1\left(\widehat{\boldsymbol{v}}^t\right) := {} & -\left\langle \mathbf{T}\boldsymbol{f}_N, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\Omega_N} + \left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} \\ & + \left\langle -\mathbf{L}_h^{\widehat{f}_h^N}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^N}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} + \left\langle -\mathbf{L}_h^{\boldsymbol{f}}\boldsymbol{n} + \tau_t \mathbf{T}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D}, \end{aligned} \tag{B.35}$$

and

$$l_2\left(\widehat{g}\right) := -\left\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, \widehat{g}\right\rangle_{\partial\Omega_D} + \left\langle \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \boldsymbol{u}_h^{\widehat{f}_h^N}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{f}_h^N}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} + \left\langle \boldsymbol{u}_h^{\boldsymbol{f}}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\boldsymbol{f}}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}. \tag{B.36}$$

Proof. Due to the linearity of the local solver (2.49a)–(2.49c), we can decompose the volume solution to (2.49a)–(2.49c) as

$$\left(\mathbf{L}_h, \boldsymbol{u}_h, p_h\right) = \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, p_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right) + \left(\mathbf{L}_h^{\widehat{f}_h^i}, \boldsymbol{u}_h^{\widehat{f}_h^i}, p_h^{\widehat{f}_h^i}\right) + \left(\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, p_h^{\widehat{\boldsymbol{u}}_h^D}\right) + \left(\mathbf{L}_h^{\widehat{f}_h^N}, \boldsymbol{u}_h^{\widehat{f}_h^N}, p_h^{\widehat{f}_h^N}\right) + \left(\mathbf{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{f}}\right).$$

That is, it is the sum of the solutions to (B.26)–(B.30) with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^{t,i}$, $\gamma = \widehat{f}_h^i$, $\boldsymbol{U} = \widehat{\boldsymbol{u}}_h^{t,D}$, $F = \widehat{f}_h^N$, and $\boldsymbol{g} = \boldsymbol{f}$. Then, the jump conditions and partial boundary condition imposition (2.49d) and (2.49e) can be written as

$$\begin{aligned}
&-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n} + \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} - \widehat{\boldsymbol{u}}_h^{t,i}\right), \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}\\
&-\left\langle -\mathbf{L}_h^{\widehat{f}_h^i}\boldsymbol{n} + \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \boldsymbol{u}_h^{\widehat{f}_h^i}\cdot\boldsymbol{n} + \frac{1}{\tau_n}\left(f_h^{\widehat{f}_h^i} - \widehat{f}_h^i\right), \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}\\
&-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^D}\boldsymbol{n} + \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^D}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^D}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}\\
&-\left\langle -\mathbf{L}_h^{\widehat{f}_h^N}\boldsymbol{n} + \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^N}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \boldsymbol{u}_h^{\widehat{f}_h^N}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{f}_h^N}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}\\
&-\left\langle -\mathbf{L}_h^{\boldsymbol{f}}\boldsymbol{n} + \tau_t\mathbf{T}\boldsymbol{u}_h^{\boldsymbol{f}}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \boldsymbol{u}_h^{\boldsymbol{f}}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\boldsymbol{f}}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N}\\
&= -\left\langle \mathbf{T}\boldsymbol{f}_N, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\Omega_N} - \left\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, \widehat{g}\right\rangle_{\partial\Omega_D}.
\end{aligned}$$

It remains to show that $-\left\langle -\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n} + \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} - \widehat{\boldsymbol{u}}_h^{t,i}\right), \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{\boldsymbol{v}}^t\right)$ as defined by (B.32), that $-\left\langle \boldsymbol{u}_h^{\widehat{f}_h^i}\cdot\boldsymbol{n} + \frac{1}{\tau_n}\left(f_h^{\widehat{f}_h^i} - \widehat{f}_h^i\right), \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} = d\left(\widehat{f}_h^i, \widehat{g}\right)$ as defined by (B.33), that $-\left\langle \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\cdot\boldsymbol{n} + \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \widehat{g}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_N} = -b\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{g}\right)$ as defined by (B.34), and that $-\left\langle -\mathbf{L}_h^{\widehat{f}_h^i}\boldsymbol{n} + \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = b\left(\widehat{\boldsymbol{v}}^t, \widehat{f}_h^i\right)$ as defined by (B.34).

Step 1: In (B.26a) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}^t$ and $\mathbf{G} = \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}$, in (B.26b) take $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h^{t,i}$ and $\boldsymbol{v} = \boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}$, and in (B.26c) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}^t$ and $q = p_h^{\widehat{\boldsymbol{u}}_h^{t,i}}$. Summing the result, we have

$$\left(\mathrm{Re}\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, f_h^{\widehat{\boldsymbol{v}}^t}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}\right\rangle_{\partial\Omega_D} + \left\langle \tau_t\left(\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} - \widehat{\boldsymbol{u}}_h^{t,i}\right), \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} - \left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n}, \widehat{\boldsymbol{v}}^t\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega_D} = 0. \tag{B.37}$$

Therefore, $\left\langle \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} \boldsymbol{n}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} - \left\langle \tau_t \left(\mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} - \widehat{\boldsymbol{u}}_h^{t,i}\right), \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = a\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{\boldsymbol{v}}^t\right)$.

Step 2: In (B.27a) take $\gamma = \widehat{f}_h^i$ and $\mathbf{G} = \mathbf{L}_h^{\widehat{g}}$, in (B.27b) take $\gamma = \widehat{g}$ and $\boldsymbol{v} = \boldsymbol{u}_h^{\widehat{f}_h^i}$, and in (B.27c) take $\gamma = \widehat{f}_h^i$ and $q = p_h^{\widehat{g}}$. Summing the result, we have

$$\begin{aligned}
&\left(\mathrm{Re}\mathbf{L}_h^{\widehat{f}_h^i}, \mathbf{L}_h^{\widehat{g}}\right)_{\mathcal{T}_h} + \left\langle \tau_t \mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \mathbf{T}\boldsymbol{u}_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\widehat{f}_h^i}, f_h^{\widehat{g}} \right\rangle_{\partial\Omega_N} \\
&\quad + \left\langle \frac{1}{\tau_n}\left(f_h^{\widehat{f}_h^i} - \widehat{f}_h^i\right), f_h^{\widehat{g}} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} + \left\langle \boldsymbol{u}_h^{\widehat{f}_h^i}\cdot\boldsymbol{n}, \widehat{g} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} = 0.
\end{aligned} \tag{B.38}$$

Therefore, $-\left\langle \boldsymbol{u}_h^{\widehat{f}_h^i}\cdot\boldsymbol{n}, \widehat{g} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} - \left\langle \frac{1}{\tau_n}\left(f_h^{\widehat{f}_h^i} - \widehat{f}_h^i\right), \widehat{g} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = d\left(\widehat{f}_h^i, \widehat{g}\right)$.

Step 3: In (B.27) take $\gamma = \widehat{g}$ and $(\mathbf{G}, \boldsymbol{v}, q) = \left(-\mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, -p_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)$. Summing the result, we have

$$\begin{aligned}
&-\left(\mathbf{L}_h^{\widehat{g}}, \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)_{\mathcal{T}_h} + \left(\mathbf{L}_h^{\widehat{g}}, \nabla\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)_{\mathcal{T}_h} + \left(\nabla\boldsymbol{u}_h^{\widehat{g}}, \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)_{\mathcal{T}_h} \\
&\quad - \left(\nabla\cdot\boldsymbol{u}_h^{\widehat{g}}, p_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)_{\mathcal{T}_h} - \left(p_h^{\widehat{g}}, \nabla\cdot\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\right)_{\mathcal{T}_h} - \left\langle \mathbf{L}_h^{\widehat{g}}\boldsymbol{n}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle \mathbf{T}\boldsymbol{u}_h^{\widehat{g}}, \mathbf{L}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{g}}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}} \right\rangle_{\partial\mathcal{T}_h} - \left\langle \frac{1}{\tau_n} f_h^{\widehat{g}}, f_h^{\widehat{\boldsymbol{u}}_h^{t,i}} \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle \frac{1}{\tau_n}\widehat{g}, f_h^{\widehat{\boldsymbol{u}}_h^{t,i}} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} + \left\langle \widehat{g}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} = 0.
\end{aligned} \tag{B.39}$$

Therefore, $-\left\langle \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}_h^{t,i}}\cdot\boldsymbol{n}, \widehat{g} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} - \left\langle \frac{1}{\tau_n} f_h^{\widehat{\boldsymbol{u}}_h^{t,i}}, \widehat{g} \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_N} = -b\left(\widehat{\boldsymbol{u}}_h^{t,i}, \widehat{g}\right)$.

Step 4: In (B.26) take $\boldsymbol{\mu} = \widehat{\boldsymbol{v}}^t$ and $(\mathbf{G}, \boldsymbol{v}, q) = \left(\mathbf{L}_h^{\widehat{f}_h^i}, -\boldsymbol{u}_h^{\widehat{f}_h^i}, p_h^{\widehat{f}_h^i}\right)$. Summing the result, we have

$$\begin{aligned}
&\left(\mathbf{L}_h^{\widehat{f}_h^i}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} - \left(\mathbf{L}_h^{\widehat{f}_h^i}, \nabla\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} - \left(\nabla\boldsymbol{u}_h^{\widehat{f}_h^i}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} \\
&\quad + \left(\nabla\cdot\boldsymbol{u}_h^{\widehat{f}_h^i}, p_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} + \left(p_h^{\widehat{f}_h^i}, \nabla\cdot\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t}\right)_{\mathcal{T}_h} + \left\langle \mathbf{L}_h^{\widehat{f}_h^i}\boldsymbol{n}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t} \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle \mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \mathbf{L}_h^{\widehat{\boldsymbol{v}}^t}\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h} - \left\langle \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \mathbf{T}\boldsymbol{u}_h^{\widehat{\boldsymbol{v}}^t} \right\rangle_{\partial\mathcal{T}_h} + \left\langle \frac{1}{\tau_n} f_h^{\widehat{f}_h^i}, f_h^{\widehat{\boldsymbol{v}}^t} \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle \mathbf{L}_h^{\widehat{f}_h^i}\boldsymbol{n}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} + \left\langle \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = 0.
\end{aligned} \tag{B.40}$$

Therefore, $\left\langle \mathbf{L}_h^{\widehat{f}_h^i}\boldsymbol{n}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} - \left\langle \tau_t\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i}, \widehat{\boldsymbol{v}}^t \right\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega_D} = b\left(\widehat{\boldsymbol{v}}^t, \widehat{f}_h^i\right)$. ∎

We can conclude from Theorem B.3 that the condensed global system will take the form

$$\begin{bmatrix} A & B^\top \\ -B & D \end{bmatrix} \begin{bmatrix} \widehat{U}^t \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

Inspecting (B.32) and (B.33), we can see that the block matrices A and D are symmetric and positive semi-definite. We can further claim that the matrix D is positive definite. To claim this we must show $d\left(\widehat{f}_h^i, \widehat{f}_h^i\right) = 0 \Rightarrow \widehat{f}_h^i = 0$. Indeed, $d\left(\widehat{f}_h^i, \widehat{f}_h^i\right) = 0$ implies $\mathbf{L}_h^{\widehat{f}_h^i} = \mathbf{0}$, $p_h^{\widehat{f}_h^i} = \widehat{f}_h^i$ on $\mathcal{E}_h \backslash \partial\Omega_N$, $p_h^{\widehat{f}_h^i} = 0$ on $\partial\Omega_N$, and $\mathbf{T}\boldsymbol{u}_h^{\widehat{f}_h^i} = \mathbf{0}$ on $\mathcal{E}_h$. Then, with $\gamma = \widehat{f}_h^i$ in (B.27b), integrating by parts reveals that $p_h^{\widehat{f}_h^i}$ is elementwise constant, and therefore globally constant since $p_h^{\widehat{f}_h^i} = \widehat{f}_h^i$ on $\mathcal{E}_h \backslash \partial\Omega_N$. If $\partial\Omega_N \neq \emptyset$, then $p_h^{\widehat{f}_h^i} = 0$ and therefore $\widehat{f}_h^i = 0$. Otherwise, constraining one value of $\widehat{f}_h^i$ to zero gives that $p_h = \widehat{f}_h^i = 0$. In this case, we can only claim positive definiteness for the D matrix that results from reducing the matrix by the one constrained degree of freedom.

Appendix C. Additional Fluxes for the Oseen Equations.

In section 3, we derived HDG schemes for the Oseen equations, where four different fluxes can be used. These four fluxes are based on four different forms of the upwind flux. These four forms of the upwind flux are not the only ways we can express the upwind flux, but they are the four that we know lead to well-posed HDG schemes when used on all faces of the mesh skeleton. When the problem being solved has boundary conditions on $-\frac{1}{\mathrm{Re}}\left[\nabla \boldsymbol{u}\right]\boldsymbol{n} + p\boldsymbol{n}$, or its normal or tangential components, it could be feasible to use an HDG flux that directly approximates these quantities so that the boundary conditions can be directly prescribed to the hatted trace variables. We present three numerical fluxes in this appendix that can serve such a purpose. First we rewrite the numerical flux (3.8) using the identities (3.17).

The $-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}$ flux: The quantity $\boldsymbol{u}^*$ can be eliminated from (3.8) so that (3.8) can be written as

$$\text{(C.1)}\quad \boldsymbol{F}_n^* = \begin{pmatrix} -\left(\boldsymbol{u} + \left(\frac{1}{\tau_t^O + \frac{m}{2}}\mathbf{T} + \frac{1}{\tau_n^O + \frac{m}{2}}\mathbf{N}\right)\left[-\left(\mathbf{L} - \mathbf{L}^*\right)\boldsymbol{n} + \left(p - p^*\right)\boldsymbol{n}\right]\right) \otimes \boldsymbol{n}, \\ -\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n} + m\boldsymbol{u} \\ +m\left(\frac{1}{\tau_t^O + \frac{m}{2}}\mathbf{T} + \frac{1}{\tau_n^O + \frac{m}{2}}\mathbf{N}\right)\left(-\left(\mathbf{L} - \mathbf{L}^*\right)\boldsymbol{n} + \left(p - p^*\right)\boldsymbol{n}\right), \\ \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\tau_n^O + \frac{m}{2}}\left[-\boldsymbol{n}\cdot\left[\left(\mathbf{L} - \mathbf{L}^*\right)\boldsymbol{n}\right] + \left(p - p^*\right)\right] \end{pmatrix}.$$

The $(\mathbf{T}\boldsymbol{u}^*, h^*)$ flux: The quantities $\mathbf{T}\mathbf{L}^*\boldsymbol{n}$ and $\mathbf{N}\boldsymbol{u}^*$ can be eliminated from (3.8) so that (3.8) can be written as

$$\text{(C.2)}\qquad \boldsymbol{F}_n^* = \begin{pmatrix} -\left(\mathbf{T}\boldsymbol{u}^* + \mathbf{N}\boldsymbol{u} + \frac{1}{\tau_n^O + \frac{m}{2}}\left(-\boldsymbol{n}\cdot\left[\mathbf{L}\boldsymbol{n}\right] + p - h^*\right)\boldsymbol{n}\right) \otimes \boldsymbol{n}, \\ h^*\boldsymbol{n} - \mathbf{T}\mathbf{L}\boldsymbol{n} + m\mathbf{N}\boldsymbol{u} + \frac{m}{2}\mathbf{T}\boldsymbol{u}^* + \frac{m}{2}\mathbf{T}\boldsymbol{u} \\ +\tau_t^O\mathbf{T}\left(\boldsymbol{u} - \boldsymbol{u}^*\right) + m\frac{1}{\tau_n^O + \frac{m}{2}}\left(-\boldsymbol{n}\cdot\left[\mathbf{L}\boldsymbol{n}\right] + p - h^*\right)\boldsymbol{n}, \\ \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\tau_n^O + \frac{m}{2}}\left(-\boldsymbol{n}\cdot\left[\mathbf{L}\boldsymbol{n}\right] + p - h^*\right) \end{pmatrix},$$

where $h^* := -\boldsymbol{n}\cdot\left[\mathbf{L}^*\boldsymbol{n}\right] + p^*$.

The $(\mathbf{N}\boldsymbol{u}^*, \mathbf{T}\mathbf{L}^*)$ flux: The quantities $\mathbf{N}\left(-\mathbf{L}^*\boldsymbol{n} + p^*\boldsymbol{n}\right)$ and $\mathbf{T}\boldsymbol{u}^*$ can be eliminated from (3.8) so that (3.8) can be written as

$$\text{(C.3)}\qquad \boldsymbol{F}_n^* = \begin{pmatrix} -\left(\mathbf{N}\boldsymbol{u}^* + \mathbf{T}\boldsymbol{u} - \frac{1}{\tau_t^O + \frac{m}{2}}\left(\mathbf{L} - \mathbf{L}^*\right)\boldsymbol{n}\right) \otimes \boldsymbol{n}, \\ -\mathbf{N}\mathbf{L}\boldsymbol{n} + p\boldsymbol{n} - \mathbf{T}\mathbf{L}^*\boldsymbol{n} + \frac{m}{2}\mathbf{N}\boldsymbol{u}^* + \frac{m}{2}\mathbf{N}\boldsymbol{u} + m\mathbf{T}\boldsymbol{u} \\ +\tau_n^O\mathbf{N}\left(\boldsymbol{u} - \boldsymbol{u}^*\right) - m\frac{1}{\tau_t^O + \frac{m}{2}}\mathbf{T}\left(\mathbf{L} - \mathbf{L}^*\right)\boldsymbol{n}, \\ \boldsymbol{u}^*\cdot\boldsymbol{n} \end{pmatrix}.$$

As before, in order to define the numerical flux (3.18) we append a subscript h to the terms in (C.1)–(C.3), replace the starred quantities on the right side of (C.1)–

(C.3) with hatted unknown quantities residing on the mesh skeleton, and replace τ_t^O and τ_n^O with τ_t and τ_n. The following numerical fluxes are the result.

The $\widehat{\boldsymbol{h}}_h$ flux (where $\widehat{\boldsymbol{h}}_h$ approximates $-\mathbf{L}^*\tilde{\boldsymbol{n}} + p^*\tilde{\boldsymbol{n}}$):

$$\text{(C.4)}\quad \boldsymbol{F}^*_{n,h} := \begin{pmatrix} -\left(\boldsymbol{u}_h + \left(\frac{1}{\tau_t+\frac{m}{2}}\mathbf{T} + \frac{1}{\tau_n+\frac{m}{2}}\mathbf{N}\right)\left(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} - \operatorname{sgn}\widehat{\boldsymbol{h}}_h\right)\right)\otimes\boldsymbol{n}, \\ -\operatorname{sgn}\widehat{\boldsymbol{h}}_h + m\boldsymbol{u} \\ +m\left(\frac{1}{\tau_t+\frac{m}{2}}\mathbf{T} + \frac{1}{\tau_n+\frac{m}{2}}\mathbf{N}\right)\left(-\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} - \operatorname{sgn}\widehat{\boldsymbol{h}}_h\right), \\ \boldsymbol{u}_h\cdot\boldsymbol{n} + \frac{1}{\tau_n+\frac{m}{2}}\left[-\boldsymbol{n}\cdot(\mathbf{L}_h\boldsymbol{n}) + p_h - \widehat{\boldsymbol{h}}_h\cdot\tilde{\boldsymbol{n}}\right] \end{pmatrix}.$$

The $(\widehat{\boldsymbol{u}}_h^t, \widehat{h}_h)$ flux (where $\widehat{h}_h$ approximates $-\boldsymbol{n}\cdot[\mathbf{L}^*\boldsymbol{n}] + p^*$):

$$\text{(C.5)}\qquad \boldsymbol{F}^*_{n,h} = \begin{pmatrix} -\left(\widehat{\boldsymbol{u}}_h^t + \mathbf{N}\boldsymbol{u}_h + \frac{1}{\tau_n+\frac{m}{2}}\left(-\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{h}_h\right)\boldsymbol{n}\right)\otimes\boldsymbol{n}, \\ \widehat{h}_h\boldsymbol{n} - \mathbf{T}\mathbf{L}_h\boldsymbol{n} + m\mathbf{N}\boldsymbol{u} + \frac{m}{2}\widehat{\boldsymbol{u}}_h^t + \frac{m}{2}\boldsymbol{u}_h^t \\ +\tau_t\mathbf{T}\left(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\right) + m\frac{1}{\tau_n+\frac{m}{2}}\left(-\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{h}_h\right)\boldsymbol{n}, \\ \boldsymbol{u}_h\cdot\boldsymbol{n} + \frac{1}{\tau_n+\frac{m}{2}}\left(-\boldsymbol{n}\cdot[\mathbf{L}_h\boldsymbol{n}] + p_h - \widehat{h}_h\right) \end{pmatrix}.$$

The $(\widehat{u}_h^{\tilde{n}}, \widehat{\boldsymbol{h}}_h^t)$ flux (where $\widehat{u}_h^{\tilde{n}}$ approximates $\boldsymbol{u}^*\cdot\tilde{\boldsymbol{n}}$ and $\widehat{\boldsymbol{h}}_h^t$ approximates $-\mathbf{T}\mathbf{L}^*\tilde{\boldsymbol{n}}$):

$$\text{(C.6)}\qquad \boldsymbol{F}^*_{n,h} = \begin{pmatrix} -\left(\widehat{u}_h^{\tilde{n}}\tilde{\boldsymbol{n}} + \boldsymbol{u}_h^t + \frac{1}{\tau_t+\frac{m}{2}}\left(-\mathbf{L}_h\boldsymbol{n} - \operatorname{sgn}\widehat{\boldsymbol{h}}_h^t\right)\right)\otimes\boldsymbol{n}, \\ -\mathbf{N}\mathbf{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \operatorname{sgn}\widehat{\boldsymbol{h}}_h^t + \frac{m}{2}\widehat{u}_h^{\tilde{n}}\tilde{\boldsymbol{n}} + \frac{m}{2}\mathbf{N}\boldsymbol{u}_h + m\mathbf{T}\boldsymbol{u}_h \\ +\tau_n\left(\mathbf{N}\boldsymbol{u}_h - \widehat{u}_h^{\tilde{n}}\tilde{\boldsymbol{n}}\right) + m\frac{1}{\tau_t+\frac{m}{2}}\left(-\mathbf{T}\mathbf{L}_h\boldsymbol{n} - \operatorname{sgn}\widehat{\boldsymbol{h}}_h^t\right), \\ \operatorname{sgn}\widehat{u}_h^{\tilde{n}} \end{pmatrix}.$$

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