CAM AREA A QUALIFYING EXAM
May 30, 2001, 9:00 a.m.–12:00 noon

Work any 6 of the following 7 problems.

1. Let \( V \) denote a cone given in cylindrical coordinates,
\[
V = \{(r, \theta, z) : 0 < z < a, \ 0 < r < z\},
\]
where \( a > 0 \). Let \( u \) be a velocity field given in the cylindrical coordinates:
\[
u = u_r e_r + u_\theta e_\theta + u_z e_z,
\]
where \( u_r = 1, u_\theta = 0, \) and \( u_z = 1 \). Compute the net flow through the surface of the cone
\[
\int_{\partial V} u \cdot n \, dS
\]
(\( n \) is the outward normal unit vector to the cone surfaces \( \partial V \)).

2. Consider the sequence of polynomial spaces in \( \mathbb{R}^2 \) and the corresponding linear operators,
\[
\mathbb{R} \xrightarrow{i} \mathcal{P}^{p+1} \xrightarrow{\nabla} \mathcal{P}^p \xrightarrow{\nabla \times} \mathcal{P}^{p-1} \xrightarrow{0} \{0\}.
\]
Here \( \mathcal{P}^p \) denotes \textit{scalar-valued} polynomials of order \( p \), \( \mathcal{P}^p \) denotes \textit{vector-valued} polynomials of order \( p \), and \( \nabla \times E = \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \). Demonstrate that this is an \textit{exact sequence}, i.e., that the null space of each of the operators (starting with gradient \( \nabla \)) coincides with the range of the previous operator in the sequence. Hint: Recall the fundamental relation between the nullity and rank of a linear operator defined on a finite-dimensional space.

3. Let \( X \) and \( Y \) be two Banach spaces, and \( D \) a dense subset of \( X \). Prove that every continuous linear operator \( A \) from \( D \) into \( Y \) has a \textit{unique} continuous and linear extension \( \hat{A} \) taking \( X \) into \( Y \). Show that the norms of \( A \) and \( \hat{A} \) are identical.

   (a) State the Closed Graph Theorem.
   (b) Let \( H \) be a Hilbert space, and \( A : H \to H \) a linear operator. If for all \( x, y \in H \)
   \[
   (x, Ay)_H = (Ax, y)_H,
   \]
   prove that \( A \) is bounded.

5. Find the infimum of
\[
\frac{\int_{-1}^{1} (u'(x))^2 \, dx}{\int_{-1}^{1} (u(x))^2 \, dx}
\]
among all non-zero functions in \( H_0^1(-1, 1) \).

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6. Show that $H^1(-\pi, \pi)$ is compactly contained in $L^2(-\pi, \pi)$. Hint: Use Fourier series and reduce the problem to the finite dimensional case (finite number of coefficients) where there is compactness.

7. Use the Contraction Mapping Theorem to prove local existence and uniqueness for the initial-value problem

$$\begin{cases} 
q(0) = 1, \\
\frac{dq}{dt} = q^2 + t, \quad t \in (0, T).
\end{cases}$$

Give a lower bound for $T$, the length of the time interval for which the solution is guaranteed to exist.
1. By the Gauss' Theorem:

$$\int_{\partial V} \mathbf{n} \cdot dS = \int_{V} \text{div} \mathbf{u} \, dV$$

In cylindrical coordinates $\gamma$

$$\text{div} \mathbf{u} = \sum_{i=1}^{3} \frac{\partial \mathbf{u}^i}{\partial y^i} \cdot \hat{y}^i$$

Cylindrical coordinates $(r, \theta, z)$

$$\hat{z} = (\cos \theta, \sin \theta, 1)$$

$$\hat{r} = \frac{\partial z}{\partial r} = (\cos \theta, \sin \theta, 0) = \mathbf{e}_r = \hat{z}$$

$$\hat{\theta} = \frac{\partial r}{\partial \theta} = (-\sin \theta, \cos \theta, 0)$$

$$\hat{\theta} = \frac{\partial \theta}{\partial \theta} = (-\sin \theta, \cos \theta, 0), \quad \hat{\theta} = \frac{1}{r} (-\sin \theta, \cos \theta, 0)$$

$$\hat{z} = \mathbf{e}_z = \mathbf{e}_z = (0, 0, 1)$$

So

$$\text{div} \mathbf{u} = \frac{\partial}{\partial r} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) \cdot \mathbf{e}_r$$

$$+ \frac{2}{\partial r} (\mathbf{e}_\theta) \cdot \frac{1}{r} \mathbf{e}_\theta$$

$$+ \frac{2}{\partial r} (\mathbf{e}_z) \cdot \mathbf{e}_z$$

$$= \frac{\partial}{\partial \theta} (\cos \theta, \sin \theta, 0) \cdot \frac{1}{r} \mathbf{e}_\theta = \frac{1}{r}$$
Parametrization of the cone:

\[ 0 < z < a \]
\[ 0 < \theta < 2\pi \]
\[ 0 < r < z \]

\[ 2\pi \int_0^a \left( \int_0^{\frac{a}{z}} r \, dr \right) \, dz = 2\pi \int_0^a \frac{z}{2} \, dz = \pi a^2 \]
2. Facts:

1. \( N(\nabla) = \text{constants, trivial} \)

2. \( \nabla = \nabla q \iff \nabla \times \overrightarrow{E} = 0 \)
   is true in any simply-connected domain, including \( \mathbb{R} \).
   Differentiation lowers the order of polynomials, so the operators are well-defined.

3. We need to show that \( \nabla x \) is a surjection. One way to show it is to compare the dimensions of the spaces:

\[
\begin{align*}
\dim \mathbb{R}(\nabla x) &= \dim \mathbb{P}^b - \dim N(\nabla x) \\
&= \dim \mathbb{P}^b - \dim \mathbb{R}(\nabla) \\
&= \dim \mathbb{P}^b - (\dim \mathbb{P}^{b-1} - \dim N(\nabla)) \\
&= \left( \tfrac{p+1}{2} \right) \left( \tfrac{p+2}{2} \right) - \frac{b^2 + 2 + b + 2}{2} + 1 \\
&= \frac{p^2 + 2p + 2}{2} - \frac{p^2 + 2p + 2}{2} + 1 \\
&= \frac{p^2 + 2p - 3}{2} + 1 \\
&= \frac{(p^2)(p^2 - 1) + 2}{2} = \frac{p^2 + p}{2} = \frac{p(p + 1)}{2}
\end{align*}
\]

\[\dim \mathbb{P}^{b-1} = \frac{p(p + 1)}{2} \quad \text{OK!} \]
3. For \( x \in X \), let \( A x_n \xrightarrow{n \to \infty} x \). Now
\[
\| A x_n - A x_m \|_Y = \| A(x_n - x_m) \|_Y \leq \| A \| \| x_n - x_m \|_Y
\]
so we conclude \( A x_n \xrightarrow{n \to \infty} \) is Cauchy in \( Y \). Thus \( \exists y \in Y \) s.t.
\[
\lim_{n \to \infty} A x_n = y
\]
Let \( \hat{A} x = y \). We need to check that \( \hat{A} \) is well-defined, i.e., independent of the sequence chosen. Let \( E \subseteq \mathbb{R} \), \( E \xrightarrow{m \to \infty} x \). Then, since norms are continuous,
\[
\| A E_m - y \|_Y = \lim_{n \to \infty} \| A E_m - A x_n \| \leq \lim_{n \to \infty} \| A \| \| E_m - x_n \| = \| A \| \| E_m - x \| \to 0 \text{ as } m \to \infty.
\]
Thus \( \hat{A} \) is well-defined, and \( \hat{A} \big|_D = A \).
Trivially \( \hat{A} : X \to Y \) is linear: For \( x, \xi \in X \) with \( x_n \to x \), \( \xi_n \to \xi \), \( x_n \xi_n \in D \) and \( \alpha \in \mathbb{R} \)
\[
\hat{A}(\alpha x + \beta \xi) = \lim_{n \to \infty} A(\alpha x_n + \beta \xi_n)
\]
\[
= \alpha \lim_{n \to \infty} A x_n + \beta \lim_{n \to \infty} A \xi_n
\]
\[
= \alpha \hat{A} x + \beta \hat{A} \xi.
\]
Now
\[
\| \hat{A} \| = \sup_{\| x \| = 1} \| \hat{A} x \| \geq \sup_{\| x \| = 1} \| A x \| = \| A \|.
\]
and for \( x \in X, \| x \| = 1 \) and \( c > 0 \), there exists \( \bar{x} \in D \) such that

\[
\| \tilde{A} x - A \bar{x} \| < \varepsilon \quad \text{and} \quad \| x - \bar{x} \| < \varepsilon
\]

Thus

\[
\| \tilde{A} x \| \leq \| A \bar{x} \| + \varepsilon \leq \| A \| \| x \| + \varepsilon \leq \| A \| (1 + \varepsilon) + \varepsilon
\]

and

\[
\| \tilde{A} \| = \sup_{\| x \| = 1} \| \tilde{A} x \| \leq \| A \| (1 + \varepsilon) + \varepsilon
\]

Thus \( \| \tilde{A} \| = \| A \| \) and \( \tilde{A} \) is continuous. Finally \( \tilde{A} \) is unique since if \( B : X \to Y \) satisfies the properties,

\[
\| \tilde{A} x - B x \| = \lim_{n \to \infty} \| A x_n - B x_n \|
\]

\[
= \lim_{n \to \infty} \| A x_n - A \bar{x} \| = 0,
\]

so \( B = \tilde{A} \).
4. (a) Let $X$ and $Y$ be Banach spaces and $A : X \to Y$ a linear operator. Then

$A$ is continuous $\iff A$ is closed.

(b) By (a), we need only show $A$ is closed. That is, if $\exists x_n^2_{n=1} \in H$ and $x_n \to x$ and $Ax_n \to y \in H$, then we must show $y = Ax$.

But for $z \in H$,

$$
(y, z)_H = \lim_{n \to \infty} (Ax_n, z)_H = \lim_{n \to \infty} (x_n, Az)_H
$$

$$
= (x, Az)_H = (Ax, z)_H,
$$

so $Ax = y$. 

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5. Let
\[ q = \inf_{u \in H_0^1} \frac{\int u'^2}{\int u^2} = \inf_{u \in H_0^1} \frac{\int u'^2}{\|u\|^2} = 1 \]

Thus we solve the constrained minimization problem. Let
\[ H(y, \lambda) = \int (u'^2 - \lambda (u^2 - 1)) \, dx \equiv \int dH \]
for the Lagrange multiplier \( \lambda \).

The Euler-Lagrange equations lead to
\[ \frac{\partial H}{\partial u'} = \left( \frac{\partial H}{\partial u} \right)' \]
\[ -2u'' = (2u')' = 2u'' \]
\[ \Rightarrow u'' = \lambda u \]

The eigenvalues of \(-A\) are \( \lambda \geq 0 \), so
\[ \lambda \geq 0 \]
and, with \( \mu = \frac{\lambda}{2} \),
\[ u(x) = a \sin(\mu x) + b \cos(\mu x) \]
Now \( u \in H_0^1(-1,1) \), so
\[ 0 = u(\pm 1) = \pm a \sin \mu + b \cos \mu \]
\[ \Rightarrow b \cos \mu = 0 \text{ and } a \sin \mu = 0. \]
So \( b = 0 \) or \( \mu = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \)
and \( a = 0 \) or \( \mu = m\pi, m > 1, 2, \ldots \)

Now note that \(-u'' = \lambda u \) \( \Leftrightarrow \)
\[ \int u'u' = \lambda \int u^2 \quad \forall u \in H_0^1 \]
\[ \Rightarrow \int u'^2 = \lambda \int u^2 \quad \Rightarrow \quad q = \inf_{\lambda} \lambda = \inf_{\mu} \mu^2 = \frac{\pi^2}{4} \]
Problem 6:

1. Since \( H'(-\pi, \pi) = \{ f \in L^2(-\pi, \pi), Df \in L^2(-\pi, \pi) \} \), then \( H'(-\pi, \pi) \subset L^2(-\pi, \pi) \) by def.

2. To see that \( H^1(-\pi, \pi) \) is compactly contained in \( L^2(-\pi, \pi) \) it is enough to show that any bounded set \( K \) on \( H^1(-\pi, \pi) \) is precompact in \( L^2(-\pi, \pi) \). That is, \( K \) is compact in \( L^2 \).

Since \( L^2 \) and \( H^1 \) on \( (-\pi, \pi) \) are metric space then, \( K \) is compact iff it is sequentially compact: every sequence in \( K \) has a convergent subsequence in \( K \) in \( L^2(-\pi, \pi) \).

Let \( \{ f_n \} \) be a sequence in \( K \), a bounded set in \( H^1(-\pi, \pi) \). Using the identification of the Thom and Fourier coeff. 

there exists a constant \( C \), depending only on \( K \) s.t.

\[
\| f_n \|_{H^1(-\pi, \pi)} \leq C \left( \sum_{n \geq 0} n^2 (a_n^2 + b_n^2) \right) \leq C(K).
\]

then, given an \( \varepsilon > 0 \), there exists an \( N_0 \) s.t.

\[
n^2 (a_n^2 + b_n^2) \leq \frac{\varepsilon}{2k}
\]

with \( k = \sum_{n \geq N_0} \frac{1}{n^2} \), a finite positive number.
so \[ a_n^2 + b_n^2 \leq \frac{\xi}{2k} \frac{1}{n^2} \text{ for } n \geq N_k. \]

Now, we take \( S = \text{Span} \{ \sin nx, \cos nx, 0 \leq n < m \} \) as a finite dimensional subspace of \( L^2(-\pi, \pi) \) (so that any bounded sequence of a finite dimensional space is bounded).

Then take \( \{ g_n \} = \{ \sum_{n=0}^{N_0} a_n \sin nx + b_n \cos nx \} \)

\( g_n \in S \) and, because of the above estimate, we obtain that \( \| g_n \|_{L^2(-\pi, \pi)} \)

\[ \| g_n \|_{L^2(-\pi, \pi)} = \sum_{n=0}^{N_0} a_n^2 + b_n^2 \leq N_0^2 \mathcal{C}(k), \]

is bounded in \( S \). Hence, there exists a convergent subsequence \( g_{n_k} \to g \) in \( L^2(-\pi, \pi) \).

Set \( \| g - g_{n_k} \|_{L^2} \leq \frac{\varepsilon}{2} \) for \( n_k > N_0 \).

Then taking \( \lim_{n_k \to \infty} \), we estimate

\[ \| f - g \|_{L^2} \leq \| f - g_{n_k} \|_{L^2(-\pi, \pi)} + \| g_{n_k} - g \|_{L^2(-\pi, \pi)} \]
\[ \frac{2}{n^2} \frac{e}{2k} \frac{1}{n^2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \varepsilon \frac{1}{2} = \varepsilon. \]

So \( f_{n_k} \to g \) in \( L_2 \ (-\pi, \pi) \).

Then \( K \) is compact in \( L_2(-\pi, \pi) \).
Define a map

\[ q \rightarrow Aq, \quad Aq = 1 + \frac{t^2}{2} + \int_0^t q(\eta) \, d\eta \]

We shall identify a domain \( D(A) \subset C[0, T] \) in which the map is contractive.

- Natural candidate for \( D(A) \) — a ball in \( C[0, T] \) centered at \( q \equiv 1 \) with some radius \( \varepsilon \).

Estimate:

\[ \| Aq - 1 \| = \sup_{t \in [0, T]} \left| \int_0^t q(\eta) \, d\eta + \frac{t^2}{2} \right| \]

\[ \left| q_1(s) - 1 \right| < \varepsilon \Rightarrow \]

\[ \left| q_1(s) \right| \leq \left| q_1(s) - 1 \right| + 1 \leq 1 + \varepsilon \]

\[ \left| q_1(s) \right| \leq (1 + \varepsilon)^2 \]

\[ \leq T (1 + \varepsilon)^2 + \frac{T^2}{2} \]

For operator \( A \) to map the ball into itself, it is necessary that

\[ T (1 + \varepsilon)^2 + \frac{T^2}{2} < \varepsilon \]
\[ \| a_1 - a_2 \| = \sup_{[0,T]} \int_0^T (q_1 - q_2) (q_1 + q_2) \, ds \]
\[ \leq 2(1+\varepsilon) \| a_1 - a_2 \| \]

Now, given any \( \varepsilon > 0 \), say e.g. \( \varepsilon = 1 \), we can determine \( T \) that satisfies both conditions:

\[ T (1+\varepsilon)^2 + \frac{T^2}{2} = 4T + \frac{T^2}{2} < 1 \]
\[ 2T(1+\varepsilon) = 4T < 1 \]

\[ T = \frac{1}{16} - 4 \quad \text{provides a lower bound for interval} \]
\[ (0, T) \text{ in which the solution exists and is unique} \]
CAM AREA A PRELIMINARY EXAM (CAM 385C–D)
May 29, 2007, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Prove the Mazur Separation Lemma, which says that if $X$ is a normed linear space, $Y$ a linear subspace of $X$, $w \in X$ but $w \not\in Y$, and
   
   \[ d = \text{dist}(w, Y) = \inf_{z \in Y} \|w - z\|_X > 0, \]
   
   then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: work in $Z = Y + \mathbb{F}w$, and extend to $X$ using the Hahn-Banach Theorem.]

2. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose $K \in C(\overline{\Omega} \times \overline{\Omega})$ and $T : L_2(\Omega) \to L_2(\Omega)$ is defined by
   
   \[ T f(x) = \int_{\Omega} K(x, y) f(y) \, dy. \]
   
   (a) Show that $T$ is well defined.
   (b) State the Ascoli-Arzelà Theorem about continuous functions defined on a compact metric space.
   (c) Prove that $T$ is a compact operator. [Hint: use the density of $C(\overline{\Omega})$ in $L_2(\Omega)$ and apply Ascoli-Arzelà.]

3. Consider the heat equation
   
   \[ \frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = f(x), \]
   
   on the domain $(x, t) \in \mathbb{R} \times [0, T]$ where $f$ is a continuous function that vanishes (i.e., is zero) outside the closed interval $[-1, 1]$.
   (a) Find a solution to this equation. [Hint: use the Fourier transform.]
   (b) Show that the initial data controls the solution in the sense that
   
   \[ \int_{-\infty}^{\infty} |u(x, t)|^2 \, dx \leq \int_{-\infty}^{\infty} |f(x)|^2 \, dx. \]
   (c) By looking at the form of your solution, explain why the solution is infinitely smooth in the space variable and that heat dissipates with time.

4. Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. For $y \in \mathbb{R}^d$, the translation operator $\tau_y$ is defined by $\tau_y \phi(x) = \phi(x - y)$.
   (a) Show that
   
   \[ u(\tau_y \phi) - u(\phi) = \int_0^1 \sum_{j=1}^d y_j \frac{\partial u}{\partial x_j}(\tau_y \phi) \, dt. \]
   (b) Apply this to
   
   \[ f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) = \left\{ f \in L_{1,\text{loc}}(\mathbb{R}^d) : \frac{\partial f}{\partial x_j} \in L_{1,\text{loc}}(\mathbb{R}^d) \text{ for all } j \right\} \]
   to show that
   
   \[ f(x + y) - f(x) = \int_0^1 y \cdot \nabla f(x + ty) \, dt. \]
5. The symmetric gradient of a vector function \( \mathbf{v} \in (H^1(\Omega))^d \) is a \( d \times d \) matrix of the form
\[
\epsilon(\mathbf{v})_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]
It arises often in structural mechanics problems. Assuming that \( \Omega \in \mathbb{R}^2 \) is smooth and bounded, prove Korn's inequality, which says that there is some \( C > 0 \) such that
\[
\|\mathbf{v}\|_{H^1(\Omega)} \leq C \|\epsilon(\mathbf{v})\|_{L^2(\Omega)} \quad \text{for all } \mathbf{v} \in (H^1_0(\Omega))^2.
\]
[Hint: You will need to show that all rigid motions, i.e., vector fields of the form \( \mathbf{v} = \alpha(x_2, -x_1) + (\beta, \gamma) \) vanish in \((H^1_0(\Omega))^2\).]

6. Consider a stream between the lines \( x = 0 \) and \( x = 1 \), with speed \( v(x) \) in the \( y \)-direction. A boat leaves the shore at \((0, 0)\) and travels with constant speed \( c > 0 \), relative to the water, to the given terminal point \((1, \beta)\). The problem is to find the path \( y(x) \) of minimal crossing time.
   (a) Let \( \frac{dx}{dt} = u \) and \( \frac{dy}{dt} = w \). Then \( y'(x) = w/u \) and \( c^2 = u^2 + (w - v)^2 \). Show that the crossing time is
\[
t = \int_0^1 \frac{dx}{u} = \int_0^1 \frac{\sqrt{c^2[1 + (y')^2] - v^2 - vy'}}{c^2 - v^2} \, dx.
\]
   [Hint: you will need to use the quadratic formula.]
   (b) Find the Euler-Lagrange equations, and reduce them to a single equation for \( y' \).
   (c) If \( v \) is constant, find \( y \).
CAM AREA A PRELIMINARY EXAM (CAM 385C–D)
May 30, 2008, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Hahn-Banach Theorem.
   (a) State the Hahn-Banach Theorem.
   (b) Let $X$ be a NLS and $x, y \in X$. Prove that $f(x) = f(y)$ for all $f \in X^*$ if and only if $x = y$.
   (c) Prove the Mazur Separation Lemma: Let $X$ be a NLS and $Y$ a linear subspace. If $w \in X \setminus Y$ and
      \[ d = \text{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0, \]
      then there is $f \in X^*$ such that $\|f\|_X \leq 1$, $f(w) = d$, and $f(y) = 0$ for all $y \in Y$. [Hint: work in $Z = Y + \mathbb{F}w$, and extend to $X$ using the Hahn-Banach Theorem.]

2. Let $H$ be a Hilbert space and $Z \subset H$ a closed linear subspace. Recall that we define the Hilbert space of cosets of $Z$ as $H/Z = \{x + Z \subset H : x \in H\}$ with the norm $\|\cdot\|_{H/Z} : H/Z \to \mathbb{R}$ given by
   \[ \|\hat{x}\|_{H/Z} = \inf_{y \in Z} \|y\|_H. \]
   (a) If $T \in H^*$, let $Z = \ker(T)$ and define the function $\hat{T} : H/Z \to \mathbb{R}$ by $\hat{T}(\hat{x}) = T(x)$, where $\hat{x} = x + Z$. Show that this function is well defined and injective.
   (b) Find a continuous linear map that takes $H/Z$ one-to-one and onto $Z^\perp = \{x \in H : \langle x, z \rangle = 0 \forall z \in Z\}$ such that norms are preserved.

3. Let the underlying field be real and let $V \in L_2((0, 1) \times \Omega)$, where $\Omega$ is some domain. Suppose that we define $T : L_2(0, 1) \to L_2(0, 1)$ by
   \[ T f(x) = \int_0^1 \int_{\Omega} V(x, \omega) V(y, \omega) f(y) \, d\omega \, dy. \]
   (a) Justify that $T$ is well defined, compact, symmetric, and positive (semi-definite).
   (b) State the Spectral Theorem for compact, positive, self-adjoint linear operators.
   (c) Apply the theorem to $T$ to show that we can express
   \[ V(x, \omega) = \sum_{\alpha \in \mathcal{K}} a_\alpha(\omega) v_\alpha(x), \]
   for some orthonormal basis $\{v_\alpha(x)\}_{\alpha \in \mathcal{K}}$ of $L_2(0, 1)$. Also give an expression for the coefficients $a_\alpha(\omega)$. 
4. Suppose that $f$ is a real, periodic, and continuous function on $[-\pi, \pi]$. Define the Fourier coefficients for $n$ an integer by

$$
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.
$$

(a) Show that

$$
|\hat{f}(n)| \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(t + \pi/n) - f(t)| dt.
$$

[Hint: $\cos(x + \pi) = -\cos(x)$ and $\sin(x + \pi) = -\sin(x)$]

(b) Explain how this formula implies that the Fourier coefficients of $f$ must decay to zero for large $|n|$.

5. Suppose that $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. Suppose also that $\epsilon \in (0, 1]$, $a_\epsilon \in L_\infty(\Omega)$ satisfies $0 < a_* \leq a_\epsilon \leq a^* < \infty$ on $\bar{\Omega}$, and $f \in L^2(\Omega)$. For each $\epsilon$, consider the boundary value problem

$$
-\nabla : (a_\epsilon \nabla u_\epsilon) = f \quad \text{in} \quad \Omega,
$$

$$
u_\epsilon = 0 \quad \text{on} \quad \partial \Omega.
$$

(a) State the Lax-Milgram Theorem (for Hilbert spaces).

(b) Prove that for each $\epsilon$, there is a unique solution to the boundary value problem.

(c) Show that there exists some $u \in H^1_0(\Omega)$ and a single subsequence $u_{\epsilon_n}$ for which both $u_{\epsilon_n} \rightharpoonup u$ weakly in $H^1_0(\Omega)$ and $u_{\epsilon_n} \to u$ strongly in $L^2(\Omega)$. Be sure to state the theorems that you use to show these results.

6. Suppose that $X$ is a Banach space and we wish to solve $F(x) = 0$, where $F : X \to X$. Suppose there is at least one root $x$. If we use Newton’s Method, we are given $x_0$ and define

$$
x_{n+1} = G(x_n), \quad n = 0, 1, 2, \ldots, \quad \text{where} \quad G(y) = y - D F(y)^{-1} F(y).
$$

Assume that $F$ is $C^1$ on all of $X$, $DF(y)$ is invertible for all $y \in X$, and $\|DF(y)^{-1}\|$ is uniformly bounded in $y \in X$. Let the error be denoted by $e_n = x_n - x$.

(a) Show that

$$
\|e_{n+1}\| = o(\|e_n\|).
$$

[Hint: note that $(x_{n+1} - x) = (x_n - x) - D F(x_n)^{-1} [F(x_n) - F(x)]$, since $x$ is a root, and apply the definition of the Fréchet derivative.]

(b) State the Mean Value Theorem for NLS’s.

(c) Now assume that $F$ is $C^3$ on all of $X$. You are given that

$$
DG(y)(k) = DF(y)^{-1} D^2 F(y)(DF(y)^{-1} F(y), k),
$$

and we assume that $\|D^2 G(y)\|$ is uniformly bounded in $y \in X$. Show that $DG(x) = 0$, and use this fact to show that there is some $C > 0$ such that

$$
\|e_{n+1}\| \leq C \|e_n\|^2.
$$

That is, if Newton’s method converges, it does so quadratically. [Hint: note that $(x_{n+1} - x) = G(x_n) - G(x)$ and apply the Mean Value Theorem twice.]

(d) Under the extra hypothesis given in (c), give a reasonable condition on $x_0$ to insure that $x_n \to x$. 

CAM AREA A PRELIMINARY EXAM (CAM 385C–D)
May 29, 2009, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Suppose \( f \in L_p(\mathbb{R}^d) \) and \( g \in L_q(\mathbb{R}^d) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( 1 < p < \infty \). Show that
   \[
   f \ast g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \, dy
   \]
is a continuous function with
   \[
   \lim_{|x| \to \infty} f \ast g(x) = 0.
   \]
   [Hint: Recall that if \( h \in L_1(\mathbb{R}^d) \), then \( \int_{|x| > R} h(x) \, dx \to 0 \) as \( R \to \infty \).]

2. Suppose that \( H \) is a Hilbert space and \( V \subset H \) is a nonempty, closed convex set. Let \( x \in H \) such that \( x \not\in V \).
   (a) Define what it means for \( V \) to be convex.
   (b) Prove that there is a unique \( y \in V \) such that \( \|x - y\| \) is minimal.
   (c) If \( V \) is a linear subspace, prove that \( x - y \) is orthogonal to \( V \).

3. Consider an operator \( A : L^2[0, 1] \to L^2[0, 1] \) defined by
   \[
   Af(x) = \int_0^x f(t) \, dt.
   \]
   (a) Show that \( A^* f(x) = \int_x^1 f(t) \, dt \) and \( A^* Af(x) = \int_x^1 [1 - \max(x, t)] f(t) \, dt \).
   (b) Show that \( A^* A \) is self-adjoint, positive, and compact on \( L^2[0, 1] \).
   (c) Show that if \( \lambda \neq 0 \) is an eigenvalue of \( A^* A \) with eigenfunction \( f \), then \( \lambda f'' = -f \) almost everywhere on \([0, 1]\), and also \( f(0) = 0 \) and \( f'(1) = 0 \).
   (d) Show that \( \|A^* A\| = 4/\pi^2 \) and \( \|A\| = 2/\pi \).

4. Suppose \( X \) is a Banach space and \( T \in B(X, X) \) has norm \( \|T\| < 1 \). For \( n \) a positive integer, define \( T^n \) to be the composition of \( T \) with itself \( n \) times.
   (a) If \( R \in B(X, X) \), show that \( \|RT\| \leq \|R\| \|T\| \).
   (b) Show that
   \[
   S = I + T + T^2 + \ldots = \sum_{n=0}^{\infty} T^n = \lim_{N \to \infty} \sum_{n=0}^{N} T^n
   \]
is a well defined element of \( B(X, X) \). We call \( S \) the Neumann series of \( T \).
   (c) Show carefully that \( I - T \) is invertible and that \( (I - T)^{-1} = S \).
5. Consider the partial differential equation

\[(\nabla \cdot a\nabla)(\nabla \cdot a\nabla)u = f \quad \text{in } \Omega \subset \mathbb{R}^d,\]

where $\Omega$ is bounded with a smooth boundary, scalar function $a \in C^\infty_0(\bar{\Omega})$, $a(x) \geq a_* > 0$, and $f \in L^2(\Omega)$.

(a) What are the homogeneous essential boundary conditions for this problem?
(b) Prove that there is $C > 0$ such that, if $u \in H^1_0(\Omega) \cap H^2(\Omega)$, then

\[\|u\|_{H^2} \leq C\|\nabla \cdot a\nabla u\|_{L^2}.\]

[Hint: Use the Elliptic Regularity Theorem.]

(c) If we impose the boundary conditions $u = 0$ and $\nabla \cdot a\nabla u = g$ on $\partial \Omega$, prove that there is a unique solution to the weak or variational form of the problem.

6. Let $X$ be a Banach space and $F : X \to X$ be a smooth map. Suppose that $x_*$ is a simple root of $F$ in the sense that $F(x_*) = 0$ and the derivative $DF(x_*)$ is invertible. Given any starting point $x_0$, consider the full Newton iteration scheme

\[x_{k+1} = G(x_k) \quad \text{where} \quad G(x) = x - DF(x)^{-1}F(x).\]

Here we prove that if $x_0$ is sufficiently close to $x_*$, then $x_k \to x_*$ as $k \to \infty$.

(a) Show that $G(x_*) = x_*$, $DG(x_*) = 0$ and that there is a closed ball $B$ about $x_*$ such that $\|DG(x)\| \leq \frac{1}{2}$ for all $x \in B$. [Hint: You do not need to compute $DG(x)$, only $DG(x_*)$.]

(b) Show that $G(x) \in B$ for all $x \in B$.

(c) Show that $G : B \to B$ is a contraction.

(d) Prove that $x_k \to x_*$ as $k \to \infty$ for any $x_0 \in B$. 
CSEM Area A-CAM Preliminary Exam (CAM 385C-D)
June 1, 2010, 9:00 a.m.-12:00 noon

Work any 5 of the following 6 problems.

1. Consider the operator $T : C([0, 1]) \to C([0, 1])$ defined by

$$Tf(x) := \int_0^x f(t) \, dt$$

(a) Determine the spectral radius of $T$.
(b) Determine the norm of $T$.
(c) Let $M := \{ f \in C([0, 1]) : f(0) = 0 \}$. Prove that the Banach quotient space $C([0, 1])/M$ is isomorphic to $\mathbb{C}$.

2. Let $X$ be an infinite dimensional Banach space over $\mathbb{C}$, and assume that $f \in X^*$ and $f_1, \ldots, f_n \in X^*$. Assume that there exists a constant $C > 0$ such that whenever $|f_j(x)| < C$ holds for all $j = 1, \ldots, n$, with $x \in X$, then $|f(x)| < 1$.

(a) Prove that $\bigcap_{j=1}^n \ker(f_j) \subseteq \ker(f)$.
(b) Prove that $f$ is a linear combination of the $f_j$. [Hint: Using (a), study the range $R(F) \subseteq \mathbb{C}^{n+1}$ of the map $F : X \to \mathbb{C}^{n+1}$ defined by $F(x) := (f_1(x), \ldots, f_n(x), f(x))$.]

3. Let $D(\mathbb{R})$ denote the set of test functions on $\mathbb{R}$, and $D'(\mathbb{R})$ the space of distributions.

(a) Assume that $\psi \in D(\mathbb{R})$ is a test function such that $\psi(0) \neq 0$. For $n \in \mathbb{N}$, consider the functions

$$n \psi(nx), \quad \psi(x/n), \quad \text{and} \quad \sum_{k=0}^n \psi(x-k).$$

If they exist, find their limits as $n \to \infty$ pointwise and in $L^\infty(\mathbb{R})$ and $D'(\mathbb{R})$.

(b) Assume that $u \in D'(\mathbb{R})$ satisfies $u(x^2 \phi) = 0$ for all $\phi \in D(\mathbb{R})$. Prove that

$$u(\phi) = c_0 \phi(0) + c_1 \phi'(0) \quad \forall \phi \in D(\mathbb{R}).$$

4. Let the field be complex but $m \in L^\infty(\mathbb{R}^d)$ be real; in fact, let $0 < m_0 \leq m(x)$ for a.e. $x \in \mathbb{R}^d$. Define the multiplier operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $Tu := (m\hat{u})^\vee$. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary, we can define a similar operator $\tilde{T} : L^2(\Omega) \to L^2(\Omega)$ by $\tilde{T}u := (mE_0u)^\vee|_{\Omega}$, where $E_0$ is extension by zero and we restricted the result back to $\Omega$.

(a) Consider the differential equation

$$-\Delta u + Tu = f \quad \text{in } \mathbb{R}^d.$$  

Prove that if $f \in H^{-1}(\mathbb{R}^d)$, then there is a unique solution $u \in H^1(\mathbb{R}^d)$.

(b) Show that $\tilde{T}$ is a bounded, symmetric, and strictly positive definite operator.

(c) Consider the boundary value problem

$$-\Delta u + \tilde{T}u = f \quad \text{in } \Omega \quad \text{and} \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega.$$  

Prove that if $f \in H^{-1}(\Omega)$, then there is a unique solution $u \in H^1(\Omega)$.  

Area A-CAM
June 2010
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, and define

$$H(\text{div}; \Omega) := \{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \}.$$

(a) Show that $H(\text{div}; \Omega)$ is a Hilbert space with the inner-product

$$(u, v)_{H(\text{div})} := (u, v)_{L^2(\Omega)} + (\nabla \cdot u, \nabla \cdot v)_{L^2(\Omega)}.$$

(b) The trace theorem does not imply that $\partial_x v = v \cdot \nu$ exists on $\partial \Omega$. Nevertheless, show that $\partial_u : H(\text{div}; \Omega) \to H^{-1/2}(\partial \Omega) = (H^{1/2}(\partial \Omega))^*$ is a well defined bounded linear operator in the sense of integration by parts:

$$\int_{\partial \Omega} v \cdot \nu \phi d\sigma(x) = \int_{\Omega} \nabla \cdot v \phi dx + \int_{\Omega} v \cdot \nabla \phi dx.$$

[Hint: In what space must $\phi$ lie?]

(c) Prove the following inf-sup condition: there exists $\gamma > 0$ such that

$$\inf_{w \in L^2(\Omega)} \sup_{v \in H(\text{div}; \Omega)} \frac{(w, \nabla \cdot v)_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)} \|v\|_{H(\text{div}; \Omega)}} \geq \gamma > 0.$$

6. Let the field be real and $G : C^0_B(\mathbb{R}) \to C^0_B(\mathbb{R})$ be defined by

$$G(u)(x) := \int_{\mathbb{R}} e^{-|x-y|} \frac{u^2(y)}{1 + u^2(y)} dy + \cos x.$$

(a) Show that $G$ has at least one fixed point.

(b) Find $DG(u)$ [You do not need to justify your result.]

(c) Let $G_0(u) := G(u)(0)$, so $G_0 : C^0_B(\mathbb{R}) \to \mathbb{R}$. Find all critical points of $G_0$. 
CSEM Area A-CAM Preliminary Exam (CAM 385C–D)
May 31, 2011, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let $f_n$ converge weakly to $f$ in $L^2([0,1])$ and define

$$F_n(x) := \int_0^x f_n(\xi) \, d\xi \quad \text{and} \quad F(x) := \int_0^x f(\xi) \, d\xi.$$ 

(a) Show that $F_n(x)$ converges to $F(x)$ pointwise.
(b) Show that $F_n(x)$ converges to $F(x)$ weakly in $H^1([0,1])$.
(c) Show that $F_n(x)$ converges uniformly on $[0,1]$ to $F(x)$. [Hint: Given $\varepsilon > 0$ and fixed $x \in [0,1]$, consider the points $x_j = \varepsilon j$ and $x^* = \varepsilon j^*$ nearest $x$.]

2. Let $X$ be a Banach space and $T : X \to X$ a bounded linear operator. Suppose further that $\|T\| < 1$.
(a) State the meaning of $\|T\|$ and show that $\|T^2\| \leq \|T\|^2$.
(b) Prove carefully that $I - T$ has an inverse, given by the Neumann series

$$I + T + T^2 + T^3 + \cdots.$$ 

(c) For $S : X \to X$ a bounded linear operator, show that the spectrum of $S$ is contained in the closed circle of radius $\|S\|$ in the complex plane.

3. For $f \in L^2(\mathbb{R}^d)$, consider a solution $u \in H^2(\mathbb{R}^d)$ to the problem

$$u - \Delta u = f.$$ 

(a) Show that the solution is unique using the Fourier Transform.
(b) If $f \in H^s(\mathbb{R}^d)$, $s \geq 0$, find $r$ such that $u \in H^r(\mathbb{R}^d)$. Justify your answer.
(c) For what values of $s$ can you be sure that a fundamental solution is continuous?

4. Suppose that $u \in H^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain. Define the $H^1(\Omega)$-seminorm by

$$|u|_{H^1(\Omega)} = \left\{ \sum_{|\alpha| = 1} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2}.$$ 

(a) Show that there is some constant $C_\Omega$, depending on $\Omega$ but not on $u$, such that

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\Omega)} \leq C_\Omega |u|_{H^1(\Omega)}.$$ 

[Hint: You may take $c = \bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) \, dx$ and argue by contradiction.]
(b) Let $\Omega = (0,h)^d$. Show that there is a constant $C$, independent of $\Omega$ and $u$, such that

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\Omega)} \leq C h |u|_{H^1(\Omega)}.$$ 

[Hint: Change variables to integrate over $(0,1)^d$, and use (a).]
5. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a smooth boundary. The equations of linear elasticity can be formulated for the displacement vector \( \mathbf{u} = (u_1, u_2)^T \) using the symmetric gradient tensor \( \mathbf{\varepsilon}(\mathbf{u}) \) defined by

\[
\mathbf{\varepsilon}_{ij}(\mathbf{u}) = \frac{1}{2}\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]
as

\[
\sum_i \frac{\partial}{\partial x_i} \left[ 2\mu \mathbf{\varepsilon}_{ij}(\mathbf{u}) + \lambda \mathbf{\nabla} \cdot \mathbf{u} \delta_{ij} \right] = f_j \quad \text{in } \Omega,
\]

where \( \mu \) and \( \lambda \) are positive constants. We assume the boundary condition \( \mathbf{u} = 0 \) on \( \partial \Omega \).

(a) Develop a weak form for the equations that involve the integrals \( (\mathbf{\nabla} \cdot \mathbf{u}, \mathbf{\nabla} \cdot \mathbf{v}) \) and \( \sum_{ij} (\mathbf{\varepsilon}_{ij}(\mathbf{u}), \mathbf{\varepsilon}_{ij}(\mathbf{v})) \). For the latter, you will need to use the fact that \( \mathbf{\varepsilon}(\mathbf{u}) \) is indeed symmetric.

(b) In which spaces should \( \mathbf{u} \) and \( \mathbf{f} = (f_1, f_2)^T \) lie?

(c) Show that there is a unique solution to the problem using the Lax-Milgram Theorem.

You will need to use Korn's Inequality, which says that there is \( \gamma > 0 \) such that

\[
\| \mathbf{\varepsilon}(\mathbf{u}) \|_{(L^2(\Omega))^{2 \times 2}} \geq \gamma \| \mathbf{u} \|_{(H^1(\Omega))^2} \quad \forall \mathbf{u} \in (H^1_0(\Omega))^2.
\]

6. Set up and apply the contraction mapping principle to show that the boundary value problem

\[
\begin{align*}
\mathbf{u}'' - \epsilon \mathbf{u}^2 &= f(x), \quad x \in (0, 1), \\
\mathbf{u}(0) &= \alpha \text{ and } \mathbf{u}'(1) = \beta,
\end{align*}
\]

has a unique, continuous solution if \( \epsilon > 0 \) is small enough, where \( f(x) \) is a smooth function on \([0, 1] \).
CSEM Area A-CAM Preliminary Exam (CSE 386C/D)
May 31, 2012, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.
Remark: \( \mathbb{N} \) is the set of natural numbers \( \{1, 2, 3, \ldots \} \).

   (a) Show that \( Y_1 = \{x = (x_1, x_2, x_3, \ldots) \in \ell^2 \mid x_{2n} = 0, n \in \mathbb{N}\} \) is a closed subspace of \( \ell^2 \), and find \( Y_1^\perp \).
   (b) What is \( Y_2^\perp \) if \( Y_2 = \text{span}\{e_1, \ldots, e_n\} \subset \ell^2 \), where \( e_{j,k} = \delta_{j,k} \)?
   (c) Take \( x = (1, 2, 3, 4, 0, 0, 0, \ldots) \). What are the orthogonal projections \( P_{Y_1}(x) \) and \( P_{Y_2}(x) \), and the distances \( d(x, Y_1) \) and \( d(x, Y_2) \)?

2. Let \( T_t \) be the operator \( T_t(\varphi)(x) = \varphi(x + t) \) on \( L^2(\mathbb{R}) \).
   (a) What is the norm of \( T_t \)?
   (b) Show that \( T_t \) does not converge as \( t \to \infty \) in \( B \left( L^2(\mathbb{R}), L^2(\mathbb{R}) \right) \).
   (c) To what operator does \( T_t \) converge as \( t \to \infty \) if the Hilbert space is \( L^2(\mathbb{R}, e^{-x^2}dx) \)?

3. Let \( H \) be a Hilbert space and \( A \) a bounded linear operator on \( H \). Recall that \( |A| = (A^*A)^{1/2} \) is a self-adjoint, bounded linear operator. A bounded linear operator \( U \) on \( H \) is a partial isometry if \( \|Ux\| = \|x\| \) for all \( x \in N(U)^\perp \) (i.e., \( U \) is an isometry except on its nullspace, where it is zero).
   (a) Show that \( \|A|x\| = \|Ax\| \) for all \( x \in H \).
   (b) Show that \( H = \overline{R(|A|)} \oplus N(|A|) \) and that \( N(|A|) = N(A) \).
   (c) Show that there exists a partial isometry \( U \) such that \( A = U|A| \). [Hint: define \( U : R(|A|) \to R(A) \) by \( U(|A|x) = Ax \) (is this well defined?) and extend \( U \) first to \( R(|A|) \) and then to all of \( H \).]

4. Consider \( x \in \mathbb{R}^{n \times n} \) with associated \( \ell^2 \)-norm \( \|x\|_2 = \left( \sum_{j=1}^{n} \sum_{k=1}^{n} |x_{j,k}|^2 \right)^{1/2} \) and total-variation semi-norm \( |x|_{TV} = \sum_{j=1}^{n} \sum_{k=1}^{n-1} |x_{j,k+1} - x_{j,k}| + \sum_{j=1}^{n-1} \sum_{k=1}^{n} |x_{j+1,k} - x_{j,k}| \).
   (a) Why is \( \cdot |_{TV} \) not a norm?
   (b) Prove directly the following Sobolev inequality: if \( x_{1,j} = x_{j,1} = 0 \) for all \( 1 \leq j \leq n \), then \( \|x\|_2 \leq \frac{1}{\sqrt{n}} |x|_{TV} \). [Hint: Find two ways to write \( x_{j,k} \) as a sum of discrete differences.]
   (c) Does the result in (b) extend to \( \ell^2(\mathbb{N} \times \mathbb{N}) \)?
5. Consider the following problem in \( \Omega \subset \mathbb{R}^d \), a \( C^1 \) domain. Given functions \( f, g, \) and \( c \geq 0 \), find a solution pair \( (u, w) \) to the partial differential boundary value problem (BVP)

\[
\begin{align*}
-\Delta u &= c(w - u) + f \quad \text{in } \Omega, \\
-\Delta w &= c(u - w) + g \quad \text{in } \Omega, \\
w &= 0 \quad \text{and} \quad u = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(a) Determine appropriate Hilbert spaces within which the solution pair \( (u, w) \) and functions \( f, g, \) and \( c \) should lie, and formulate an appropriate variational problem for the BVP.
(b) Show that the BVP and your variational problem are equivalent.
(c) Show that there is a unique solution to the variational problem.

   (a) State the Implicit Function Theorem.
   (b) Let \( X \) and \( Y \) be Banach spaces. Let both \( F : X \to Y \) and \( G : X \to Y \) be \( C^1 \) on \( X \), and \( H(x, \epsilon) = F(x) + \epsilon G(x) \), for \( \epsilon \in \mathbb{R} \). If \( H(x_0, 0) = 0 \) and \( DF(x_0) \) is invertible, show that there exists \( x \in X \) such that \( H(x, \epsilon) = 0 \) for \( \epsilon \) sufficiently close to 0.
   (c) For small \( \epsilon \), prove that there is a solution \( w \in H^2(0, \pi) \) to

\[
w'' = w + \epsilon w^2, \quad w(0) = w(\pi) = 0.
\]
Work any 5 of the following 6 problems.

1. Let $H$ be a real Hilbert space and suppose that $P$ is a bounded linear projection on $H$. Let $Q = I - P$ and define $M = P(H)$ and $N = Q(H)$. Suppose that $M$ and $N$ are closed.
   (a) Show that there exists $C > 0$ such that
   \[ \|x - Px\| \leq C \inf_{y \in M} \|x - y\| \quad \text{for all } x \in H. \]
   [Hint: Relate this to the orthogonal projection $P_M$.]
   (b) Prove that $P$ is an orthogonal projection if and only if
   \[ \inf_{\substack{y \in N, \|y\| = 1 \atop x \in M}} \|y - x\| = 1. \]
   [Hint: For the converse, it is enough to show that for any $z \in H$, $z - Pz = Qz \perp M$. Consider $y = Qz/\|Qz\|$.]

2. Let $X$ be a Banach space with dual $X^*$. Let $\{L_n\}_{n=1}^\infty \subset X^*$ and $\{x_n\}_{n=1}^\infty \subset X$. Assume that $L_n \to L \in X^*$ in the weak-* sense, and $x_n \to x$ in the norm of $X$.
   (a) State the Uniform Boundedness Principle.
   (b) Show that if $X$ is a reflexive Banach space, then $L_n(x_n) \to L(x)$.

3. Let $\Omega = (-1, 1)^2$ and define
   \[ H = \{u \in H^2(\Omega) : u(x, 0) = u_y(0, y) = 0 \text{ for a.e. } x, y \in (-1, 1)\}. \]
   (a) Why is $H$ a well defined, complete linear subspace of $H^2(\Omega)$?
   (b) Prove that there is some $C > 0$ such that for $u \in H$,
   \[ \|u\|_{H^1(\Omega)} \leq C \left\{ \|u_{xx}\|_{L^2(\Omega)} + \|u_{xy}\|_{L^2(\Omega)} + \|u_{yy}\|_{L^2(\Omega)} \right\}. \]

4. Consider the Telegrapher’s equation
   \[ u_{tt} + 2u_t + u = \varepsilon^2 u_{xx} \quad \text{for } x \in \mathbb{R} \text{ and } t > 0; \]
   with $u(x; 0) = f(x)$ and $u_t(x; 0) = g(x)$ given in $L^2(\mathbb{R})$.
   (a) Use the Fourier transform (in $x$ only) and its inverse to find a representation of the solution. [Hint: The solution to $y'' + 2y' + (1 + \alpha^2)y = 0$ is $y(t) = e^{-\alpha t}(A \cos(\alpha t) + B \sin(\alpha t))$.]
   (b) Justify that your representation is indeed a solution. You may assume that $f$ and $g$ are in the Schwartz space.
5. Contraction mappings.
   (a) State the contraction-mapping theorem.
   (b) Consider $K \in C^0([0,1]^2)$. Show that for $\lambda$ small enough, for any $g \in C^0([0,1])$, there
       exists a unique solution $f \in C^0([0,1])$ to
       \[
       f(x) = g(x) + \lambda \int_0^1 K(x, y) f(y) \, dy.
       \]

6. Let $\Omega$ be a bounded smooth domain with $\nu$ being the normal vector on its boundary.
   Consider the solution $(u, v)$ of the differential problem
   \[
   \begin{align*}
   u - \Delta u &= f + aw, & \text{in} \, \Omega, \\
   -\Delta w &= g - au, & \text{in} \, \Omega, \\
   \nabla u \cdot \nu &= \gamma \text{ and } w = 0, & \text{on} \, \partial \Omega,
   \end{align*}
   \]
   where $a \in L^\infty(\Omega)$.
   (a) Provide an appropriate weak form for the problem. In what Sobolev spaces should $f$, $g$, and $\gamma$ lie?
   (b) Prove that there exists a unique solution to the problem.
1. \( \mathbf{H}, \mathbf{P} : \mathbf{H} \rightarrow \mathbf{M}, \quad Q = I - \mathbf{P} : \mathbf{H} \rightarrow \mathbf{N} \)

(a) \( \inf_{y \in \mathbf{M}} \|x - y\| = \|x - \mathbf{P} x\| = \|\mathbf{P}^\perp x\| \)

\( \mathbf{x} = \mathbf{P} x + \mathbf{P}^\perp x \)

\( \mathbf{P} x = \mathbf{P}^\perp x + \mathbf{P} \mathbf{P}^\perp x \)

\( \mathbf{x} - \mathbf{P} x = \mathbf{x} - \mathbf{P}^\perp x + \mathbf{P} \mathbf{P}^\perp x = (\mathbf{I} + \mathbf{P}) \mathbf{P}^\perp x \)

\( \Rightarrow \|\mathbf{x} - \mathbf{P} x\| \leq (1 + \|\mathbf{P}\|) \|\mathbf{P}^\perp x\| \)

(b) \( \mathbf{P} = \mathbf{P}^\perp \iff \inf_{y \in \mathbf{N}, \|y\|=1} \|y - x\| = 1 \)

(\(\Rightarrow\)) \( \mathbf{P} = \mathbf{P}^\perp \Rightarrow \mathbf{N} = \mathbf{M}^\perp \)

\( \Rightarrow \|y - x\|^2 = \|y\|^2 + \|x\|^2 = 1 + \|x\|^2 \)

\( \Rightarrow \inf_{\mathbf{x} \in \mathbf{M}} \|\mathbf{x}\| = 1 \)

(\(\Leftarrow\)) For any \( \mathbf{z} \in \mathbf{H} \), \( \mathbf{Qz} \neq \mathbf{0} \), let \( \mathbf{y} = \frac{\mathbf{Qz}}{\|\mathbf{Qz}\|} \)

\( \Rightarrow 1 \leq \inf_{\mathbf{x} \in \mathbf{M}} \|\frac{\mathbf{Qz}}{\|\mathbf{Qz}\|} - \mathbf{x}\| = \|\mathbf{P}^\perp \left(\frac{\mathbf{Qz}}{\|\mathbf{Qz}\|}\right)\| \)

\( \leq \frac{\|\mathbf{Qz}\|}{\|\mathbf{Qz}\|} = 1 \)

\( \Rightarrow \|\mathbf{Qz}\| = \|\mathbf{Qz}\| \Rightarrow \mathbf{Qz} \in \mathbf{M}^\perp \)

Thus \( \mathbf{z} - \mathbf{Pz} = \mathbf{Qz} \perp \mathbf{M} \)

If \( \mathbf{Qz} = \mathbf{0} \), then \( \mathbf{z} = \mathbf{Pz} \in \mathbf{M} \)

In general, \( \mathbf{z} - \mathbf{Pz} \perp \mathbf{M} \), so \( \mathbf{P} = \mathbf{P}^\perp \).
2. \( \exists L_n \subseteq X^*, \forall x_n \in X \)

\[ L_n \xrightarrow{w^*} L, \quad \alpha_n \xrightarrow{} X \]

(b) \( X = X^{**} \)

\[ L_n \xrightarrow{w^*} L \Leftrightarrow L_n(x) \rightarrow L(x) \quad \forall x \in X \]

\[ L_n(\alpha_n) = L_n(x) + L_n(x_n - x) \]

\[ \frac{\overline{L(x)}}{L(x)} \]

UBP \implies \|L_n\| \leq M \text{ or } \exists x \text{ s.t. } \sup_x |L_n(x)| = \infty.

But \( L_n(x) \rightarrow L(x) \implies L_n(x) \text{ bounded.} \)

Thus \( \|L_n(x_n - x)\| \leq M \|x_n - x\| \rightarrow 0. \)

\[ \Rightarrow L_n(x_n) \rightarrow L(x). \]

(a) UB: \( X \) Banach, \( Y \) NLS

\( \exists T \in B(X,Y) \Rightarrow \)

(ii) \( \exists M \text{ s.t. } \|T\| \leq M \forall 2 \)

or (ii) \( \exists x \in X \text{ s.t. } \sup_x \|T(x)\| = +\infty. \)
3. $\Omega = (-1,1)^2$, $H = \{u \in H^2 : u(x,0) = y(0,y) = 0 \text{ a.e.} \}$

(a) The trace operator is well defined down $\frac{1}{2}$ dimension, so $u(x,0)$ and $u_y(0,y)$ exist and are cont. That is, $\gamma_0 = u(x,0)$, $\gamma_1 = u_y(0,y)$ are well-defined and cont.

$\Rightarrow$

$H = \{u \in H^2 : \gamma_0 u = \gamma_1 u = 0 \}$

$= \gamma_0 H^1(\partial \Omega) \bigcap \gamma_1 H^1(\partial \Omega)$

is closed, so a complete lin. subspace.

(b) Suppose not. Then $\exists$ seq. $u_n \in H$

$|u_{nx}||^2 + |u_{nxy}|^2 + |u_{nyy}|^2 \leq \frac{1}{n}$

But $\|u_n\|_{H^1(\Omega)} = 1$.

But then $u_{nx}, u_{nxy}, u_{nyy} \to 0$

Now $\|u_n\|^2 \leq \frac{1}{n} \Rightarrow \exists$ subseq. $u_{n_k} \to u$

and $u_{nx} \to u$. Thus $u$ is linear: $u = a + \beta x + \gamma y$

$\Rightarrow a + \beta x = 0$ a.e., $\Rightarrow a = 0$

Contradicting that $u_n \to u$

$\|u_n\|^2 = 1$ $\|u\|^2 = 0$. $\times$
4. \[ \begin{aligned} u_{tt} + 2u_t + u &= c^2 u_{xx}, \quad t > 0 \\ u(x,0) &= f, \quad u_t(x,0) = g \end{aligned} \]

(a) \[ \begin{aligned} \hat{u}_{tt} + 2\hat{u}_t + \hat{u} &= -c^2 |\xi|^2 \hat{u} \\ \Rightarrow \quad \hat{u}_{tt} + 2\hat{u}_t + (1 + c^2 |\xi|^2) \hat{u} &= 0 \\ \Rightarrow \quad \hat{u} &= e^{-t} \left( A(\xi) \cos(c\xi t) + B(\xi) \sin(c\xi t) \right) \\ \hat{u}(x,0) &= \hat{f} = A(\xi), \\ \hat{u}_t &= -\hat{u} + e^{-t} \left( -c^2 A(\xi) \sin(c\xi t) + c\xi B(\xi) \cos(c\xi t) \right) \\ \hat{u}_t(x,0) &= -\hat{f} + c\xi B(\xi) = \hat{g} \\ \Rightarrow \quad \hat{u}(x,t) &= e^{-t} \left( \hat{f}(\xi) \cos(c\xi t) + \hat{g} + \hat{f} \frac{c\xi}{c^2} \sin(c\xi t) \right) \\ \Rightarrow \quad u(x,t) &= f^{-1} \left( \hat{f}(\xi) \cos(c\xi t) + \hat{g} + \hat{f} \frac{c\xi}{c^2} \sin(c\xi t) \right) \\ &= \frac{1}{\sqrt{2\pi}} \left[ f * f^{-1}(\cos(c\xi t)) + (g+f) * f^{-1} \left( \frac{\sin(c\xi t)}{c^2} \right) \right] \\ (b) \text{ Since } \cos(c\xi t), \frac{\sin(c\xi t)}{c^2} \in C^\infty \cap L^\infty, \\
\text{ the Fourier Inverses exist.} \]
5. (a) \((X, d)\) complete metric, \(g : X \to X\) contraction (i.e.,
\[ d(g(x), g(y)) \leq \theta \, d(x, y), \quad \forall x, y \in X \]
for some \(\theta < 1\).
\[ \implies \exists! \text{ fixed pt} \quad g(x) = x. \]

(b) \(f(x) = g(x) + 2 \int_0^1 k(x, y) f(y) \, dy\)
\(X = C^0([0, 1])\) is complete metric.
\[ \Phi (f) = g(x) + 2 \int_0^1 k(x, y) f(y) \, dy \]
\[ \Phi : X \to X \]
\[ \| \Phi (f) - \Phi (f_2) \|_{L^\infty} \]
\[ = \| 2 \int_0^1 k(x, y) (f(y) - f_2(y)) \, dy \| \]
\[ \leq \| f - f_2 \|_{L^\infty} \int_0^1 \| k(x, y) \| \, dy \]
\[ \leq \| f - f_2 \|_{L^\infty} \int_0^1 \| k(x, y) \| \, dx \, dy \]
\[ \leq \text{small}, \; \text{small for } |s| \text{ small}. \]
6. \[
\begin{aligned}
\mathcal{L}u - \Delta u &= f + a w, \\
\Delta w &= g - a u, \\
\nabla u \cdot \mathbf{v} &= \mathbf{x}, \\
w &= 0,
\end{aligned}
\]

(a) \[
\begin{aligned}
(u, \phi) + (\nabla u, \nabla \phi) - \langle \nabla u, \nabla \phi \rangle - (aw, \phi) &= \langle f, \phi \rangle \\
(b, \psi) + (au, \psi) - \langle b, \psi \rangle &= \langle g, \psi \rangle.
\end{aligned}
\]

Let \( u \in H^1(\Omega) \), \( w \in H^1_0(\Omega) \), \( \phi \in H^1(\Omega) \), \( \psi \in H^1_0(\Omega) \), \( b \in (H^1(\Omega))^2 \), \( g \in H^{-1}(\Omega) \), \( r \in H^{-1/2}(\Omega) \).

(b) \text{ Lax–Milgram}

\[
\begin{aligned}
a((u, w), (\phi, \psi)) &= (u, \phi) + (\nabla u, \nabla \phi) + (\nabla w, \nabla \psi) \\
&\quad + (au, \psi) - (aw, \phi) \\
|a| &\leq (\|u\|_H^2 + \|w\|_H^2) (\|\phi\|_H^2 + \|\psi\|_H^2) \\
\Rightarrow &\text{ bounded.}
\end{aligned}
\]

\[
a((u, w), (u, w)) = \|u\|_H^2 + \|\nabla w\|_H^2 \geq c(\|u\|_H^2 + \|w\|_H^2)
\]

since \( w \in H^1_0(\Omega) \).

\Rightarrow \text{ coercive.}

\[
b(\phi, \psi) = (f, \phi) + \langle g, \phi \rangle + (g, \psi)
\]

\[
\leq \|f\|_{H^{-1}} \|\phi\|_H + a\|\|w\|_H + \|g\|_{H^{-1}} \|\psi\|_H
\]

\Rightarrow \text{ bounded}

\Rightarrow L \text{ solv.}
CSEM Area A-CAM Preliminary Exam (CSE 386C / 386D)
May 29, 2014, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let \( H \) be a separable Hilbert space, \( \{v_i\}_{i=1}^{\infty} \) a countable orthonormal base, and \( L : H \to H \) the operator such that \( Lv_i = \sum_{j=1}^{\infty} 2^{-(i+j)}v_j \). Show that \( L \) is compact.

2. Let \( w \) be a distribution on \( \mathbb{R} \) such that

   (i) \( |\langle w, \phi \rangle| \leq \|\phi\|_{L^2} \) for any test function \( \phi \);  

   (ii) \( |\langle w, \phi' \rangle| \leq \|\phi\|_{L^1} \) for any test function \( \phi \).

   (a) Assuming only (i), show that \( w \) can be represented by a function in \( L^2 \) with norm less than or equal to 1. [Hint: Riesz Representation Theorem.]

   (b) Assuming both (i) and (ii), show that \( w \) can be represented by a Lipschitz function \( u \) with Lip seminorm (\( L^\infty \) norm of the derivative) less than or equal to 1.

3. Let \( \Omega = [-1,1]^2 \) and define \( \bar{u} \) as the local average in each quadrant, i.e.,

\[
\bar{u}(x) = \int_b^{b+1} \int_a^{a+1} u(y) \, dy_1 \, dy_2 \quad \text{when } x \in (a,a+1) \times (b,b+1) \text{ and } a,b = -1,0.
\]

   (a) Prove that there is a constant \( C > 0 \) such that

\[
\|u\|_{H^2} \leq C \left\{ \|\bar{u}\|_{L^\infty} + \sum_{|\alpha| = 2} \|D_\alpha u\|_{L^2} \right\} \quad \text{for all } u \in H^2(\Omega).
\]

   (b) Will the result in (a) hold in spatial dimension \( d > 2 \)? Why or why not?

4. Let \( \Omega \subset \mathbb{R}^2 \) be a domain with a smooth boundary and consider the variational problem: Find \( u \in V \) such that

\[
(au,v) + (\nabla \cdot u, \nabla \cdot v) = (f, \nabla \cdot v) \quad \text{for all } v \in V,
\]

where \( u \) and \( v \) are vectors in \( \mathbb{R}^2 \), \( a \in L^\infty(\Omega) \), \( a(x) \geq a_* > 0 \) for some constant \( a_* \), \( (au,v) = \int_{\Omega} a(x) \left( u_1(x)v_1(x) + u_2(x)v_2(x) \right) \, dx \), and \( \nabla \cdot u = \text{div} \, u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \).

   (a) For the problem to make sense, define \( V \) and a space for \( f \).

   (b) State the Lax-Milgram theorem for Hilbert spaces.

   (c) Show that the hypotheses of the Lax-Milgram theorem hold for this problem. What norm do we use for \( V \)?

   (d) Define \( p = f - \nabla \cdot u \) and determine the strong form of the equation represented by the variational problem. What boundary condition should you impose on \( p \)?
5. Given \( I = [0, b] \), consider the problem of finding \( u : I \to \mathbb{R} \) such that

\[
\begin{cases}
  u'(t) = g(t)f(u(t)), & \text{for a.e. } t \in I, \\
  u(0) = \alpha,
\end{cases}
\]

where \( \alpha \in \mathbb{R} \) is a given constant, \( g \in L^p(I), p \geq 1 \), and \( f : \mathbb{R} \to \mathbb{R} \) are given functions. We suppose that \( f \) is Lipschitz continuous and satisfies \( f(0) = 0 \). Consider the functional

\[ F(u) = \alpha + \int_0^t g(s) f(u(s)) \, ds. \]

(a) Show that \( F \) maps \( C^0(I) \) into \( C^0(I) \cap W^{1,p}(I) \). Moreover, show that \( u \in C^0(I) \cap W^{1,p}(I) \) is solution to (1) if and only if it is a fixed point of \( F \).

(b) Show that there exists \( b \) small enough, not depending on \( \alpha \), such that \( F \) has a unique fixed point in \( C^0(I) \).

(c) Show that (1) has a unique solution \( u \in C^0(I) \cap W^{1,p}(I) \) for any \( g \in L^p(I) \) and \( b > 0 \).

6. Let

\[ F(u) = \int_{-1}^{5} [(u'(x))^2 - 1]^2 \, dx. \]

(a) Find all extremals in \( C^1([-1, 5]) \) such that \( u(-1) = 1 \) and \( u(5) = 5 \).

(b) Decide if any extremal from (a) is a minimum of \( F \). [You may consider \( u(x) = |x| \).]
Solutions

1. \( H, \sum \int v_i \frac{\partial^2}{\partial x_i^2} \int \sum \frac{1}{j} 2^{-(i+j)} v_j \)

Note: \( Li: H \rightarrow H \) is well defined.

\[ u \in H \Rightarrow u = \sum u_i v_i \]

\[ Lu = \sum (2^{-i} u_i 2^{-j} v_j \text{ converged.} \]

Now if \( u_n \in H \), \( \| u_n \| \leq 1 \) (say), then

\[ Lu_n = \sum (2^{-i} u_{n_i} v_j) 2^{-j} v_j , \quad u_{ni} = \langle u_n, v_i \rangle \]

Let \( u_n \rightarrow u \in H \).

Then

\[ \| Lu_n - u \| \leq \sum_{i \leq M} 2^{-2(i+j)} |K u_n - u_j v_i| \]

Note \( \langle u_n - u_j, v_i \rangle \rightarrow 0 \) for \( i \) (as \( n \rightarrow \infty \)).

Let \( \varepsilon > 0 \) be given. Then choose

\[ M \text{ s.t. } \sum_{i \leq M} 2^{-2(i+j)} \leq \varepsilon \]

and \( N \text{ s.t. } \| u_n - u_j, v_i \| \leq \varepsilon \)

for \( i < M, n \geq N \).

Then

\[ \| Lu_n - u \| \leq \left( \sum_{i \leq M} + \sum_{i > M} \right) \sum_{j} |K u_n - u_j v_i| \]

\[ \leq \sum_{i \leq M} 2^{-2(i+j)} + \sum_{i > M} 2^{-2(i+j)} \varepsilon \]

\[ \leq C \varepsilon \rightarrow 0. \]

Thus \( Lu_n \rightarrow Lu \) and so \( L \) is compact.
2. \( w \in \mathcal{D} \)

(i) \( |\langle w, \varphi \rangle| \leq \|\varphi\|_2 \quad \forall \varphi \in \mathcal{D} \)

(ii) \( |\langle w, \varphi' \rangle| \leq \|\varphi\|_2 \quad \forall \varphi \in \mathcal{D} \)

(a) Define \( T : L^2 \rightarrow L^2 \) as

\[
T(f) = \lim_{n \to \infty} T(f_n) = \lim_{n \to \infty} \langle w, f_n \rangle
\]

where \( f_n \xrightarrow{L^2} f, f_n \in \mathcal{D} \).

Then \( T \) is a bounded linear functional and so \( T(f) = (f, g) \)

for some \( g \in L^2 \). Hence

\[
T(\varphi) = \langle w, \varphi \rangle = (g, \varphi) \quad \forall \varphi \in \mathcal{D}
\]

\( \iff w = g \in L^2 \).

Moreover, \( \|T\| \leq 1 \Rightarrow \|g\|_2 \leq 1 \).

(b) We know from (a) that \( \langle w, \varphi \rangle = (g, \varphi) \)

Thus \( \langle w, \varphi' \rangle = \langle g', \varphi \rangle \)

\[
\Rightarrow |\langle g', \varphi \rangle| \leq \|g'\|_2 \|\varphi\|_2 \quad \forall \varphi \in \mathcal{D}
\]

\[
\Rightarrow \sup_{\varphi} \frac{|\langle g', \varphi \rangle|}{\|\varphi\|_2} = \|g'\|_2 \leq 1 \quad \text{[C(f) = 1]}
\]

Thus \( g \in W^{1,\infty} \) and \( \|g'\|_\infty \leq 1 \).
3. \( \bar{u} = \sum_{b} \int_{a}^{a+1} u(x) \, dx \), \( a \leq x \leq b, a, b = 1, 2, 3 \)

(a) \[ \| u \|_{H^2} \leq C \left\{ \| u \|_{L^0} + \sum_{|k|=2} \| D_k u \|_{L^2} \right\} \]

Suppose not. Then \( \exists u_n \in H^2 \)

\[ \| u_n \|_{H^2} = 1 \] and

\[ \| u_n \|_{L^0} + \sum_{|k|=2} \| D_k u_n \|_{L^2} \leq \frac{1}{n}, n = 1, 2, 3, \ldots \]

Now \( u_n \rightarrow u \) in \( H^2 \)

and \( D_k u_n \rightarrow 0 \), \( \| u_n \|_{L^0} \rightarrow 0 \)

Thus \( u \in P_1 \) is a linear polynomial.

Now \( \bar{u}_n \rightarrow \bar{u} = 0 \)

So, e.g., \( \bar{u} = 0 \Rightarrow u = ax_1 + bx_2 \)

But each quadrant avg. = 0 \( \Rightarrow u = 0 \).

However, \( u_n \rightarrow u \) in \( H^1 \) and \( D_k u_n \rightarrow D_k u \)

\( \Rightarrow \bar{u}_n \rightarrow \bar{u} \)

Thus \( \| u_n \|_{H^1} = 1 \Rightarrow \| u \|_{H^1} = 1 \), a contradiction.

Thus the result holds.

(b) Yes, if we define \( \bar{u} \) as the average over each unit cube in \( \mathbb{R}^d \) with one vertex at \( v \) and \( v + 1 \), since the argument above will continue to hold.
4. \( \Omega \subseteq \mathbb{R}^2, u \in L^\infty, a > 0 \)

Find \( u \in V : (au, v) + (\nabla u, \nabla v) = (f, v) \forall v \in V \)

(a) \( V = \{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega) \} \)

(b) Thm: Let \( H \) be a real Hilbert space, let \( B : H \times H \to \mathbb{R} \) be bilinear, and \( X \subseteq H \) be a closed subspace. Assume

(i) \( |B(u, v)| \leq M \| u \| \| v \| \quad \forall u, v \in H \)

(ii) \( |B(u, v)| \geq \alpha \| u \|^2 \quad \forall u \in X \).

If \( u_0 \in H \) and \( f \in H^* \), then \( f! u \) solving

Find \( u \in X + u_0 : \quad B(u, v) = \langle f, v \rangle \forall v \in H \)

and \( \| u \| \leq \frac{1}{\alpha} \| f \| + (\frac{M}{\alpha} + 1) \| u_0 \| \)

(c) \( 0 \leq (au, v) + (\nabla u, \nabla v) \)

\( \leq M \| u \| \| v \| \leq + \| \nabla u \| \| \nabla v \| \)

\( \leq C \| u \| \| v \| \) \(
\text{where } \| u \|^2 = \| u \|^2 + \| \nabla u \|^2 \)

\( \geq \alpha \| u \|^2 \quad \geq \alpha \| u \|^2 \quad \geq \min (a, 1) \| u \|^2 \)

(d) \( f \in L^2 = (L^2)^* \)

\( p = -\nabla u + f \)

\( (au, v) + (\nabla p, v) = 0 \)

\( = (au, v) + (\nabla p, v) - (p, v) \quad \Rightarrow \quad (au + \nabla p = 0, \Omega) \quad p = 0, \partial \Omega \)

\( \{ \begin{align*}
    
    & p + \nabla u = f, \Omega, \\
    & p = 0, \partial \Omega
\end{align*} \)
5. \( I = [0, b], \ u: I \to \mathbb{R}, \ g \in L^p (p \geq 1), \ f: \mathbb{R} \to \mathbb{R} \)

\[
\begin{align*}
1) & \ \int u' = gf(u) \quad \text{a.e. } t \in I \\
2) & \ u(0) = \alpha \\
3) & \ f(\alpha) = 0, \ f \ \text{Lipschitz} \\
4) & \ F(u) = \alpha + \int_0^t g(s) f(u(s)) \, ds
\end{align*}
\]

(a) \( u \in C^0(I) \Rightarrow f(u) \in C^0(I) \)

\[
\lim_{s \to t} \left| F(u(t)) - F(u(s)) \right| = \lim_{s \to t} \left| \int_s^t g(s) f(u(s)) \, ds \right|
\]

\[
\leq \lim_{s \to t} \|g\|_p \left( \int_s^t |f(u(s))|^p \, ds \right)^{1/p}
\]

\[
\leq \lim_{s \to t} \|g\|_p \|f(u)\|_\infty (t-s) = 0
\]

since \( |f(u)| \leq \max_{-b \leq s \leq b} |f(s)| < \infty \).

Thus \( F(u) \) is continuous on \( I \).

\( F(u) \in L^p(I) \) since \( u \in C^0(I) \)

\[
\frac{d}{dt} F(u) = g(t) f(u(t)) \in L^p(I).
\]

Moreover,

\( u \in C^0 \cap W^{1,p} \) solves (1) \( \Rightarrow \)

\[
\int_0^t u' = \int_0^t g f(u) = u(t) - u(0) \Rightarrow F(u) = u.
\]

\( F(u) = u \Rightarrow u' = \frac{1}{g} f(u) = g f(u). \sqrt{\text{and } u(0) = F(u(0)) = \alpha.}
\]

(Continued)
(b) \( F : \mathcal{C}^0(\mathbb{I}) \to \mathcal{C}^0(\mathbb{I}) \). Need \( F \) is a contraction.
\[
\| F(u) - F(v) \|_{L^\infty} = \left\| \int_0^t g(s)(f(u(s)) - f(v(s))) \, ds \right\|_{L^\infty}
\]

\[
\leq L \int_0^t |g(s)||u(s) - v(s)| \, ds
\]

\[
L = \text{Lipschitz constant for } R
\]

\[
\leq L \|g\|_{L^\infty(\mathbb{I})} \|u - v\|_{L^\infty}
\]

Now \( \|g\|_{L^\infty([0,b])} \to 0 \) as \( b \to 0 \), so

\[
\int_0^t g(s)B(s) \, ds \to 0 \quad \text{as} \quad b \to 0^+
\]

Thus \( F \) is a contraction

and \( \exists ! \) fixed point in \( \mathcal{C}^0(\mathbb{I}) \).

(c) Suppose \( u, v \in \mathcal{C}^0(\mathbb{I}) \cap W^{1,p}(\mathbb{I}) \) solve (1).

Then by (a) \( u = F(u) \) and \( v = F(v) \).

Thus

\[
(u - v)(t) = F(u) - F(v) = \int_0^t g(s)(f(u(s)) - f(v(s))) \, ds
\]

\[
\leq L \|g\|_{L^\infty([0,b])} \|u - v\|_{L^p([0,b])} \quad \forall t \leq b^*
\]

\[
\frac{1}{L} \Rightarrow u(b) = v(b^*)
\]

Start from \( t = b^* \) and repeat the argument.

to get \( u = v \) on \([b^*, 2b^*]\).

Etc. \( \Rightarrow u = v \) on \([b^*, 12]\).
6. \( F(u) = \sum_{x=1}^{5} (u'(x))^2 - 1)^2 \ dx \)

(a) Euler-Lagrange equation are:
\[ D_2 f = (D_3 f) \]
\[ f(x, u, u') = (u')^2 - 1 \]
\[ \Rightarrow 0 = (2((u')^2 - 1) 2u')' \]
\[ \Rightarrow \]
\[ u'((u')^2 - 1) = c \]
\[ c = 0 \Rightarrow u' = 0 \text{ or } u' = \pm 1. \]
\[ u(-1) = 1, u(5) = 5 \Rightarrow u \neq \text{const}. \]
\[ u \in C^1 \Rightarrow u' \neq \pm 1. \]

\[ c \neq 0 \Rightarrow 3 \text{ values solve } \frac{3}{2} - \frac{3}{3} - c = 0 \]
\[ \Rightarrow u' = \text{const.} = \frac{4}{6} = \frac{2}{3} \]
\[ \Rightarrow u(t) = \frac{2}{3}(t + \frac{5}{2}) \]

(b) \( u(t) = \frac{2}{3}(t + \frac{5}{2}) \) is not a min. For example:
\[ v(t) = \begin{cases} -t & -1 \leq t < 0 \\ t & 0 \leq t \leq 5 \end{cases} \]
\[ \Rightarrow F(v) = 0. \text{ For } u \in C^1, v \in C^1, F(w) \geq 0. \]
\[ \text{But } \]
\[ F(u) = \sum_{x=1}^{5} (u'(x))^2 - 1)^2 = \frac{25}{49} \gg 0. \]
CSEM Area A-CAM Preliminary Exam (CSE 386C-D)
May 28, 2015, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let $1 \leq p < \infty$ and define, for each $s > 0$, $T_s : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by $T_s(f)(x) = f(sx)$.
   (a) Verify that $T_s(f) \in L^p(\mathbb{R}^d)$ and that $T_s$ is bounded and linear. What is the norm of $T_s$?
   (b) Show that as $r \to s$, $\|T_r f - T_s f\|_{L^p} \to 0$. [Hint: Use that the set of continuous functions with compact support is dense in $L^p$ for $p < \infty$.]

2. Let $X$ and $Y$ be Banach spaces, and let $A : X \to Y$ be a linear operator.
   (a) Define the topological dual of the Banach space $X$.
   (b) Define the weak topology on the Banach space $X$.
   (c) Prove that the operator $A$ is (strongly) continuous if and only if it is weakly continuous (i.e., it is continuous when $X$ and $Y$ are equipped with their weak topologies).

3. Let $\tilde{H}^1 = \{ u \in H^1(0, 1) : u(0) = 0 \}$ and define the first order linear operator $A : \tilde{H}^1 \to L^2(0, 1)$ by $Au = u' - 2u$, where the derivative is understood in the sense of distributions.
   (a) Show that $A$ is bounded.
   (b) Show that the null space of the adjoint operator $A^*$ is trivial.
   (c) Prove that the operator $A$ is bounded below in $L^2(0, 1)$.
   (d) For an appropriate right-hand side $f$, discuss the well-posedness of the problem

\[
u \in \tilde{H}^1(0, 1), \quad Au = f.
\]

4. Modify the previous problem to $Au = u' - 2u = f$, where now $u \in H^1_0(0, 1)$. Consider the “ultra-weak” variational formulation of this problem: Find $u \in U = L^2(0, 1)$ such that

\[
\int_0^1 u A^* v \, dx = \int_0^1 f v \, dx \quad \forall v \in V = H^1(0, 1),
\]

where $A^*$ denotes the formal adjoint of $A$. $A^* v = -v' - 2v$.

   (a) Define operator $B : U \to V^*$ and its conjugate corresponding to the bilinear form $b(u, v)$.
   (b) State the Babuška-Nečas Theorem for Hilbert spaces.
   (c) Use this theorem to investigate the well-posedness of the variational formulation.

5. We want to prove that for any $f \in L^2(\mathbb{R}^d)$, there exists a unique solution $u \in H^1(\mathbb{R}^d)$ to

\[-\Delta u + u = f, \quad \text{in } \mathbb{R}^d.
\]

   (a) State the Lax-Milgram Theorem for Hilbert spaces.
   (b) Find the Variational problem associated to the PDE, and show carefully the equivalence for a function to be both the solution of the variational problem and a weak solution of the equation.
   (c) Show the existence and uniqueness of the solution of the variational problem.
   (d) Using the Fourier transform, shows that the solution is actually bounded in $H^2(\mathbb{R}^d)$.

6. Let $X$ and $Y$ be Banach spaces, and let $F$ and $G$ take $X$ to $Y$ be $C^1$.

   (a) Let $H(x, \epsilon) = F(x) + \epsilon G(x)$ for $\epsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \epsilon) = 0$ for $\epsilon$ sufficiently close to 0.
   (b) For small $\epsilon$, prove that there is a solution $w \in H^2(0, \pi)$ to

\[w'' = w + \epsilon w^2, \quad w(0) = w(\pi) = 0.
\]
1. Let $X$ and $Y$ be normed linear spaces and $T : X \to Y$ a linear operator. We say that $T$ is bounded if it takes bounded sets to bounded sets.

(a) Prove that $T$ is bounded if and only if there is a constant $C > 0$ such that
$$\|Tx\|_Y \leq C\|x\|_X \quad \forall x \in X.$$  

(b) Prove that $T$ is continuous if and only if $T$ is bounded.

2. Let $f_n$ be a sequence bounded both in $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. Assume that $f_n$ converges pointwise almost everywhere to $f \in L^2(\mathbb{R}^d)$.

(a) Prove that the entire sequence $f_n$ converges weakly to $f$ in $L^2(\mathbb{R}^d)$. [Hint: consider compactly-supported test functions.]

(b) If additionally $\|f_n\|_{L^2} \to \|f\|_{L^2}$, prove that the entire sequence $f_n$ converges strongly to $f$ in $L^2(\mathbb{R}^d)$.

3. Define the linear operator $T : L^2([0, 1]) \to L^2([0, 1])$ by
$$Tf(x) = \int_0^x \int_y^1 f(z) \, dz \, dy.$$  

(a) Show that $T$ is self-adjoint.

(b) Show that $T$ is compact.

(c) Find an orthogonal basis for $L^2([0, 1])$ based on the eigenvalues of this operator. [Hint: differentiate twice and consider carefully the boundary conditions that must be satisfied.]

4. Let $\Omega \subset \mathbb{R}^d$ be a domain and let $w \in L^\infty(\Omega)$. Define
$$H_w(\Omega) = \{ f \in L^2(\Omega) : \nabla(wf) \in (L^2(\Omega))^d \}.$$  

(a) Give reasonable conditions on $w$ so that $H_w(\Omega) = H^1(\Omega)$.

(b) Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner product?

(c) Suppose that $\Omega$ is bounded. Prove that there is a constant $C > 0$ such that for all $f \in H_w(\Omega)$ satisfying $\int_\Omega w(x) f(x) \, dx = 0$,
$$\|f\|_{L^2(\Omega)} \leq C \{ \|\nabla(wf)\|_{L^2(\Omega)} + \|(1-w)f\|_{L^2(\Omega)} \}.$$
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$ and unit outward normal $\nu$. Given smooth functions $f(x)$, $g(x)$ and $a(x)$, consider the following boundary-value problem (BVP) in non-divergence form:

$$-a\Delta u + u = f \quad \text{in } \Omega$$

$$a\nabla u \cdot \nu = g \quad \text{on } \partial \Omega.$$ 

(a) Reformulate the BVP as a variational problem for $u \in H^1(\Omega)$. Indicate precisely the spaces for $f$ and $g$. Is the variational problem equivalent to the BVP?

(b) State reasonable (but not necessarily optimal) conditions on $a$ for which the Lax-Milgram Theorem would be applicable to the variational problem.

(c) Prove the existence of a solution to the variational problem.

6. Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $K(x) \in L^1(\mathbb{R})$. Use the contraction mapping principle to prove that the initial-value problem

$$\partial_t u = K \ast (u + u^3), \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x, 0) = \phi(x)$$

has a continuous and bounded solution $u = u(x, t)$, at least up to some time $T < \infty$. 
CSEM Area A-CAM Preliminary Exam (CSE 386C–D)
May 30, 2017, 9:00 a.m. – 12:00 noon

Work any 5 of the following 6 problems.

1. Let $X$ be a NLS. Suppose $x \in X$, $\{x_n\}_{n=0}^{\infty}$ is a sequence in $X$, and $M \subset X'$ is such that its span is dense in $X'$. Prove that $x_n \to x$ in $X$ if and only if
   (i) the sequence $\{\|x_n\|\}_{n=0}^{\infty}$ is bounded, and
   (ii) for every $f \in M \subset X'$, $f(x_n) \to f(x)$.

2. Up to a constant multiple, the Legendre polynomial of degree $n$ is
   \[ P_n(x) = \frac{d^n}{dx^n} (x^2 - 1)^n. \]

   The Weierstrass approximation theorem says that for any function $g \in C^0([-1,1])$ and $\epsilon > 0$, there is a polynomial $p$ such that $|g(x) - p(x)| \leq \epsilon$ for any $x \in [-1,1]$.

   (a) Show that $P_n$ has exact degree $n$.

   (b) Show that the Legendre polynomials form an orthogonal base for $L^2((-1,1))$. [Hint: For orthogonality, show that $P_n$ is orthogonal to $x^m$ for $m < n$ using integration by parts.]

3. Let $X$ be a Banach space and consider $GL(X, X)$, the set of all isomorphisms from $X$ to $X$. Show that $GL(X, X)$ is an open set of $B(X, X)$. [Hint: Recall that $(1 + x)^{-1} = \sum_{n=0}^{\infty} (-x)^n$.]

4. Consider the boundary value problem:
   \[-u_{xx} + (1 + y)u = f, \quad \text{for } (x, y) \in (0, 1)^2,\]
   \[u(0, y) = 0, \quad u(1, y) = \cos(y), \quad \text{for } y \in (0, 1).\]

   (a) Find the associated variational problem. In which space should $f$ lie?
   (b) Show that there exists a unique solution to this problem.
5. Let \((X, d)\) be a complete metric space and \(T : X \to X\) be a contraction with contraction constant \(\theta \in [0, 1)\) and fixed point \(x \in X\). Suppose that \(S : X \to X\) is an approximation to \(T\) in the sense that for some \(\epsilon > 0\),

\[ d(T(z), S(z)) \leq \epsilon \quad \text{for all } z \in X. \]

For fixed \(x_0 = y_0 \in X\) and integer \(m \geq 1\), let \(x_m = T(x_{m-1})\) and \(y_m = S(y_{m-1})\).

(a) Use induction to show that

\[ d(x_m, y_m) \leq \frac{1 - \theta^m}{1 - \theta}. \]

(b) We know that \(d(x_m, x) \leq \frac{\theta^m}{1 - \theta} d(x_0, x_1)\). Use this fact to prove that

\[ d(y_m, x) \leq \frac{1}{1 - \theta} (\epsilon + \theta^m d(y_0, y_1)). \]

6. Fix \(g \in L^2(\mathbb{R}^d)\). For any \(u \in H^1(\mathbb{R}^d)\), we define

\[ J(u) = \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2 - gu) \, dx. \]

(a) Find the Euler-Lagrange equation associated to \(J\).

(b) Find all the critical points of \(J\) [Hint: You may use the Fourier transform.]

(c) Are those critical points maxima or minima of \(J\)?
1. X NLS, \( x \in X, \sum_{n=1}^{\infty} x_n \leq X, M \subseteq X, \overline{\text{span}}(M) = X \)

\[ x_n \to x \iff \begin{cases} (i) \quad \|x\| \text{ bounded} \\ (ii) \quad f(x_n) \to f(x) \quad \forall f \in M. \end{cases} \]

(\Rightarrow) If \( x_n \to x \), we know that \( f(x_n) \to f(x) \quad \forall f \in X' \), so (ii) holds.

Now for a fixed \( f \in X' \),

\[ |f(x_n)| \text{ is bounded, (since } f(x_n) \text{ converges)} \]

so \( |f(x_n)| \leq |E_{x_n}(f)| \leq C_f \quad \forall f \in X' \)

By UBP,

\[ |E_{x_n}(f)| \leq C \]

That is, \( \|E_{x_n}\| = \|x_n\| \text{ bounded.} \)

(\Leftarrow) Let \( g \in X', \varepsilon > 0 \) and choose \( n, x_i \in M \),

\[ f_i \in M \quad \text{for } i = 1, 2, \ldots, n \text{ s.t.} \]

\[ \|g - \sum_{i=1}^{n} a_i f_i\| \leq \varepsilon \]

Then

\[ g(x_n) - g(x) = g(x_n - x) \]

\[ = (g - h)(x_n - x) + h(x_n - x) \]

\[ \Rightarrow \]

\[ |g(x_n - x)| \leq \|g - h\| (\|x_n\| + \|x\|) + |h(x_n - x)| \]

\[ \leq \varepsilon \left( M + \|x\| \right) + |h(x_n - x)| \]

\[ \to 0 \quad \text{as } \varepsilon \to 0, n \to \infty. \]
2. \( P_n = \frac{d^n}{dx^n} (x^2 - 1)^n \)  
\( g \in C^0, \epsilon > 0 \Rightarrow \exists \eta \text{ st. } |f(x) - p(x)| \leq \epsilon \quad \forall x \in [-1, 1] \)

(a) \((x^2 - 1)^n \in \mathcal{P}_{2n} \Rightarrow P_n \in \mathcal{P}_n \)
Leading term of \((x^2 - 1)^n\) is \(x^{2n}\)
\Rightarrow leading term of \(P_n\) is \(\frac{(2n)!}{n!} x^n\)

(b) The set is clearly lin. indep.
For \(1, ETS \perp \) of \(P_n\) to \(x^m, m < n\).
\[ S_{-1}^1 P_n x^m = S_{-1}^1 D^n (x^2 - 1)^n x^m \]
\[ = D^{n-1} (x^2 - 1)^n x^m \frac{1}{n} - m S_{-1}^1 D^{n-1} (x^2 - 1)^n x^{m-1} \]
\[ \underline{all terms have} \quad (x^2 - 1) \Rightarrow \text{term vanishes} \]
\[ = -\left( \frac{1}{n-m+1} \right) \]
\[ = \sum_{-1}^{n-m+1} D(x^2 - 1)^n, 0 = 0. \]

For density, note for \(f \in L^2, \exists g \in L^2\) st. \(\|f - g\|\leq \epsilon, x \in [-1, 1]\), Wannier's theorem gives \(p \perp g\).
Now \(p \in \text{span } P_1, \ldots, P_n\) for some \(n < \infty\), so
\[ \|f - p\| \leq \|f - g\| + \|g - p\| \leq \epsilon + 2\epsilon = 3\epsilon \rightarrow 0. \]
Thus we have \(p \perp g\) hence.
3. Let $A \in GL(X, X)$.

For $\varepsilon > 0$ to be determined, consider

$$B_\varepsilon(A) = \{T \in B(X, X) : \|T - A\| < \varepsilon\}.$$

Now

$$T = T - A + A = A(I + A^{-1}(T - A))$$

This is the composition of 2 invertible maps if (claim) $\|A^{-1}(T - A)\| < 1$ which is true if $\|T - A\| < \frac{1}{\|A^{-1}\|} \equiv \varepsilon$.

To prove the claim (i.e., $I + R$ inv. if $\|R\| < 1$)

$$S_N = \sum_{n=0}^{N} (-R)^n = I - R + R^2 - \cdots + (-1)^N R^N$$

Thus $S_N(I + R) = I + (-1)^N R^{N+1} = (I + R)S_N$.

Now $\|R^{N+1}\| \leq \|R\|^{N+1} \rightarrow 0$ as $N \rightarrow \infty$.

Thus $S_N$ is Cauchy $\Rightarrow S_N \rightarrow S \in B(X, X)$.

So $S_N \rightarrow (I + R)^{-1}$. 
4. \[ \begin{cases} -u_{xx} + (1+y)u = f & (x, y) \in (0, 1)^2 \\ u(0, y) = 0, u(1, y) = \alpha y \end{cases} \]

(a) Let

\[ \mathcal{H} = \{ v \in L^2((0, 1)): v_x \in L^2((0, 1)^2) \} \]

\[ \mathcal{H}_0 = \{ v \in \mathcal{H}: v(0, y) = v(1, y) = 0 \ \forall y \} \]

The wave at \( x = 0, 1 \) exists because for a.e. \( y \), \( v(\cdot, y) \in H^1(0, 1) \).

Find \( u \in \mathcal{H}_0 + x \alpha y \) s.t.

\[ \left( u_x, v_x \right) + ((1+y)u, v) = (f, v) \ \forall v \in \mathcal{H}_0 \]

We want \( f \in (\mathcal{H}_0)' \).

(b) Let \( a(u, v) = (u_x, v_x) + ((1+y)u, v) \)

Note: \( \mathcal{H} \) is Hilbert with IP

\[ \langle u, v \rangle = (u_x, v_x) + (uf, v) \]

Completeness follows from the completeness of \( L^2 \) : \( \{ u_n | u_n \to u \}_n \to u \)

But

\[ \langle u_n x, \phi \rangle = \langle u_n, \phi_x \rangle \to \langle u x, \phi_x \rangle = \langle u_x, \phi \rangle \]

Thus \( u \equiv u_x \). Thus \( u_n \xrightarrow{H} u \).

Now \( |a(u, v)| \leq \|u_x\| \|v_x\| + 2\|u\| \|v\| \leq 3\|u\| \|v\| \)

and

\[ a(u, v) = \|u_x\|^2 + (1+y)\|uv\| \geq \|u_x\|^2 + \|uv\|^2 \]

\[ \text{Poincaré} \Rightarrow \|u_x(\cdot, y)\| \geq C \|u(\cdot, y)\| \forall y \Rightarrow \|u_x\| \geq C \|u\| \]

\[ \text{Thus} \quad a(u, v) \geq \frac{1}{2} \text{min}(\frac{C}{2}, y) \|uv\|^2 \]

\[ \text{Lax-Milgram} \Rightarrow \exists ! \text{ soln.} \]
5. \((x, d)\) \(T: x \Rightarrow x\) contraction, \(\theta, T_x = x.\)

\(S: x \Rightarrow x,\ d(T(2), S(2)) \leq \varepsilon \forall x \in X.\)

\(x_0 = x_0, \quad x_m = T(x_{m-1}), \quad y_m = S(y_{m-1}).\)

(a) We have that

\[d(T(x), T(y)) \leq \theta d(x, y).\]

Now

1. \(d(x_0, y_0) = 0 \leq \varepsilon \frac{1 - \theta^0}{1 - \theta} = 0.\)

2. Suppose

\[d(x_m, y_m) \leq \varepsilon \frac{1 - \theta^m}{1 - \theta}.\]

Consider

\[d(x_{m+1}, y_{m+1}) = d(Tx_m, Sy_m) \leq d(Tx_m, T_{x_m}) + d(T_{x_m}, Sy_m) \leq \theta d(x_m, y_m) + \varepsilon \leq \varepsilon (\theta \frac{1 - \theta^m}{1 - \theta} + 1) = \varepsilon \frac{1 - \theta^{m+1}}{1 - \theta}.\]

(b) \(d(x, x_j) \leq \frac{\theta^m}{1 - \theta} d(x_0, x_i).\)

\[d(x, y) \leq d(x, x_j) + d(x_j, y) \leq \varepsilon \frac{1 - \theta^m}{1 - \theta} + \frac{\theta^m}{1 - \theta} d(x_0, x_i) = \frac{1}{1 - \theta} \left[ \varepsilon (1 - \theta^m) + \theta^m (d(y_0, y_1) + d(y_1, x_i)) \right] \leq \frac{1}{1 - \theta} (\varepsilon + \theta^m d(y_0, y_1))\]
6. \( g \in L^2, u \in H^1, J(u) = \int_{\mathbb{R}^d} \left( |\nabla u|^2 + |u|^2 - g(u) \right) dx \)

(a) \( F(u) = |\nabla u|^2 + |u|^2 - g(u) \)
\[
\frac{\partial}{\partial u} F = - \frac{\partial}{\partial x_j} \left( \frac{\partial F}{\partial u_x} \right) = 0
\]
\[
\Rightarrow \quad 2u - g - 2 \sum_j u_{x_j} x_j = 0
\]
\[
\Rightarrow \quad -\Delta u + u = \frac{1}{2} g
\]

(b) \( (1 + |x|^2)^{\frac{1}{2}} \hat{u} = \frac{1}{2} \hat{g} \)
\[
\Rightarrow \quad \hat{u} = \frac{1}{2} \frac{\hat{g}}{1 + |x|^2} \quad \Rightarrow \quad u = \frac{1}{2} \left( \frac{\hat{g}}{1 + |x|^2} \right)^{\frac{1}{2}}
\]
\[
\Rightarrow \quad u = \frac{1}{2} \left( 2\pi \right)^{-\frac{1}{2}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{1}{2}} + g
\]

(c) \( \text{Since} \)
\[
J(u + \epsilon v) - J(u) = \int_{\mathbb{R}^d} \left( |\nabla (u + \epsilon v)|^2 + |u + \epsilon v|^2 - g(u + \epsilon v) \right) dx
\]
\[
= \epsilon \int_{\mathbb{R}^d} \left( |\nabla u|^2 + |u|^2 - g(u) \right) dx
\]
\[
+ \epsilon^2 \int_{\mathbb{R}^d} \left( |\nabla v|^2 + |v|^2 \right) dx
\]
\[
\Rightarrow \quad 0 \quad \Rightarrow \quad \text{minima.}
\]