CAM Area B Exam
Spring 2006

Numerical Linear Algebra:

1. (12 points) Suppose that we are given a real, rectangular matrix $A$, i.e., $A \in \mathbb{R}^{m \times n}$, with $m < n$.
   
   a. Describe the Gram-Schmidt method for producing an orthonormal basis of the rows of $A$ (you should write down the formulae involved).
   
   b. Show that the above method yields a decomposition $A = LQ$, where $L \in \mathbb{R}^{m \times m}$ is a lower triangular matrix, and $Q \in \mathbb{R}^{m \times n}$ is a row orthogonal matrix, i.e., $QQ^T = I$.
   
   c. When $A$ has rank $< m$, will Gram-Schmidt succeed in producing an $LQ$ decomposition? If not, can you modify it to produce $A = LQ$?

2. (8 points)
   
   a. Is the matrix 
      \[
      A = \begin{pmatrix}
      1 + \epsilon & 1 \\
      1 & 1 \\
      \end{pmatrix}
      \]
      positive definite? Explain your answer.
   
   b. Given an $n \times n$ real, symmetric matrix $A$, briefly outline any two different methods of testing if $A$ is positive definite. Are your methods guaranteed to return the correct result (hint: what role do roundoff errors play?)?

3. (10 points)
   
   Determine a spacing $h$ so that equispaced cubic Hermite interpolation of the sine function on the interval $[0, \pi/2]$ has absolute error less than or equal to $10^{-8}$.

4. (10 points)
   
   Given an inner product space $X$ (where the inner product is denoted by $(,)$ and its associated norm by $\|\|$), let $G$ be a subspace of $X$ and $f$ be an element of $X$. Prove that an element $g^*$ of $G$ minimizes $\|g - f\|$ over all $g \in G$ if and only if $(g^* - f, g) = 0$ for every element $g \in G$. 
1. Consider the solution of a system of linear equations using the original Gaussian elimination algorithm with no strategy for pivot selection (i.e., "diagonal pivoting"). Further consider the effects of employing this algorithm in a floating point arithmetic environment.
   a. With a very simple example, show how the algorithm can produce solutions that are very inaccurate.
   b. Using the terminology of backward error analysis, discuss the source of the inaccuracy in the example in part a.
   c. Need such inaccuracies in the diagonally pivoting algorithm be associated with large systems?, with the effects of a large number of accumulated floating point errors?, with nearly singular systems? (Discuss each.)

2. Prove the following:

**Theorem:** Given an $n \times n$ matrix $A$, for $i = 1, \ldots, n$ let $D_i = \left\{ z : |z - a_{ii}| \leq \sum_{i \neq j, j \neq i}^n |a_{ij}| \right\}$. If

$\lambda$ is an eigenvalue of $A$ then $\lambda \in \bigcup_{i=1}^n D_i$.

3. Given a set $\{g_1, g_2, \ldots, g_n\}$ of $n$ real-valued functions and a set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ pairs of real values (such that the set $\{x_1, x_2, \ldots, x_n\}$ has $n$ distinct values), prove that there exists a unique linear combination

$$g = \sum_{i=1}^n a_i g_i$$

satisfying $g(x_i) = y_i$, for $i = 1, \ldots, n$, if and only if there is no non-null such linear combination $\vec{g}$, satisfying $\vec{g}(x_i) = 0$, for $i = 1, \ldots, n$.

4.a. Give the general form for a $p$-stage Runge-Kutta methods for taking one step of approximating the solution of the initial value problem:

$$y'(x) = f(x, y(x))$$

$$y(x_k) = y_k$$

from some $x_k$ to $x_{k+1} > x_k$.

b. Derive the constraints on the parameters for a two stage second-order Runge-Kutta method.

c. Use b to display some particular two stage second-order Runge-Kutta method.
5. Prove the following:

**Theorem:**

Given a real interval \([a, b]\) (where \(a < b\)) and a function \(v\) that is defined and positive on \([a, b]\), if \(\{x_1, \ldots, x_n\}\) are the \(n\) distinct real roots of an \(n\)th degree polynomial \(\tilde{p}\) satisfying

\[
\int_a^b v(x)p(x)\tilde{p}(x)dx = 0
\]

for all polynomials \(p\) of degree \(n - 1\), then there exist values \(\{w_1, \ldots, w_n\}\) so that

\[
\int_a^b v(x)p(x)dx = \sum_{i=0}^n w_i v(x_i)p(x_i)
\]

for all polynomials \(p\) of degree \(2n - 1\).
Problem 1. (a) (6 pts) Let $A, Q_1 \in \mathbb{R}^{m_1 \times n}$, $B \in \mathbb{R}^{m_2 \times n}$, $Q_2 \in \mathbb{R}^{(m_1 + m_2) \times n}$, $Q_3 \in \mathbb{R}^{(n + m_2) \times n}$, and $R_1, R_2, R_3 \in \mathbb{R}^{n \times n}$. Assume that $Q_1$, $Q_2$, and $Q_3$ have orthonormal columns, $R_1$, $R_2$, and $R_3$ are upper triangular, and that

$$A = Q_1 R_1, \quad \begin{pmatrix} B \\ A \end{pmatrix} = Q_2 R_2, \quad \text{and} \quad \begin{pmatrix} B \\ R_1 \end{pmatrix} = Q_3 R_3.$$

Show that under mild conditions $R_2 = R_3$. What are those mild conditions?

(b) (4 pts) Sketch a QR factorization algorithm using Householder transformations that computes the QR factorization of a matrix of the form

$$\begin{pmatrix} B \\ R \end{pmatrix}$$

that takes advantage of the fact that $R$ is upper triangular. Focus on how the computation can take advantage of the zeroes below the diagonal of $R$ rather than the minute details of how individual Householder transformations are computed and applied.

Problem 2. (a) (4 pts) Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Show that $\|A^T A\|_2 = \|A\|_2^2$ and $\kappa_2(A^T A) = \kappa_2(A)^2$. (For the last part involving $\kappa_2$ you may assume $m = n$ and $A$ is nonsingular, although it holds in general.)

(b) (3 pts) In practical computation, the two norm of a matrix is rarely used. Why?

(c) (3 pts) The one-norm of a matrix is defined as $\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1$. How is it computed in practice? Prove why this practical computation computes the one-norm.
Problem 3. Given this:

**Lemma:** Given $a < b$, a function $f$ with $N$ continuous derivatives on $[a, b]$ and zeros on $[a, b]$ of total multiplicity $N+1$, then there exists a point $\xi \in [a, b]$ so that $h^{(N)}(\xi) = 0$.

Prove this:

**Theorem:** Given $n \geq 1$, $a < b$, a function $f$ with $2n$ continuous derivatives on $[a, b]$, and a polynomial $p$ of degree $2n-1$ so that

$$p(x_i) = f(x_i) \quad \text{and} \quad p'(x_i) = f'(x_i) \quad \text{for} \quad i = 1, \ldots, n,$$

where the elements $x_i \in [a, b]$ and are distinct, then for $x \in [a, b]$ there exists a point $\xi \in [a, b]$ so that

$$f(x) - p(x) = \frac{(x-x_1)^2 \cdots (x-x_n)^2}{(2n)!} f^{(2n)}(\xi)$$

Problem 4. a. Given $n \geq 1$, determine equations for $x_1, x_2, \ldots, x_n$ and $w_1, w_2, \ldots, w_n$ so that

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

for all functions $f$ of the form $f(x) = \left(1 - x^2\right)p(x)$ for polynomials $p$ of highest possible degree.

b. Solve the equations for $n = 1, 2$. 
Problem 1. (a) (6 pts) Let \( A, Q_1 \in \mathbb{R}^{m_1 \times n} \), \( B \in \mathbb{R}^{m_2 \times n} \), \( Q_2 \in \mathbb{R}^{(m_1 + m_2) \times n} \), \( Q_3 \in \mathbb{R}^{(n + m_2) \times n} \), and \( R_1, R_2, R_3 \in \mathbb{R}^{n \times n} \). Assume that \( Q_1 \), \( Q_2 \), and \( Q_3 \) have orthonormal columns, \( R_1 \), \( R_2 \), and \( R_3 \) are upper triangular, and that
\[
A = Q_1 R_1, \quad \left( \frac{B}{A} \right) = Q_2 R_2, \quad \text{and} \quad \left( \frac{B}{R_1} \right) = Q_3 R_3.
\]
Show that under mild conditions \( R_2 = R_3 \). What are those mild conditions?

(b) (4 pts) Sketch a QR factorization algorithm using Householder transformations that computes the QR factorization of a matrix of the form
\[
\left( \frac{B}{R} \right)
\]
that takes advantage of the fact that \( R \) is upper triangular. Focus on how the computation can take advantage of the zeroes below the diagonal of \( R \) rather than the minute details of how individual Householder transformations are computed and applied.

Problem 2. (a) (4 pts) Let \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \). Show that \( \| A^T A \|_2 = \| A \|_2^2 \) and \( \kappa_2(A^T A) = \kappa_2(A)^2 \). (For the last part involving \( \kappa_2 \) you may assume \( m = n \) and \( A \) is nonsingular, although it holds in general.)

(b) (3 pts) In practical computation, the two norm of a matrix is rarely used. Why?

(c) (3 pts) The one-norm of a matrix is defined as \( \| A \|_1 = \max_{\|x\|_1 = 1} \| Ax \|_1 \). How is it computed in practice? Prove why this practical computation computes the one-norm.
Problem 3. Consider solving the model problem using a fourth-order differential equation with one variable:

$$\begin{align*}
\frac{d^4 u}{dx^4} &= f, \quad x \in I = (0,1), \\
u(0) &= u(1) = 0, \\
u''(0) &= u''(1) = 0,
\end{align*}$$

(1)

where $f \in L^2(I)$. This is the problem of bending a simply supported beam.

a) Show that the problem can be formulated by two variational statements.

**Formulation 1:**
Find $u \in V$ such that

$$(u'', v'') = (f, v), \quad v \in V,$$

where

$$V = \left\{ v; \quad v \text{ and } v' \text{ are continuous on } [0,1], \quad v'' \text{ is piecewise continuous, and } v(0) = v(1) = 0 \right\}.$$

**Formulation 2:**
Let $w = u''$ and $w'' = f$ with $u(0) = u(1) = 0$ and $w(0) = w(1) = 0$.
Find $u$ and $w$ satisfying

$$(u', v') + (w, v) = 0, \quad v \in H,$$

$$(w', \phi') + (f, \phi) = 0, \quad \phi \in H,$$

where $H = H_0^1(I)$. Discuss appropriate trial spaces.

b) Formulate finite element approximations for each of the two variational formulations. Discuss convergence and possible advantages and disadvantages of each of the two schemes.
Problem 4. Consider the one-dimensional heat conduction problem on the unit internal $I = (0,1)$:

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \sigma \frac{\partial^2 u}{\partial x^2}, \quad x \in I, \quad t > 0, \\
\left\{ \begin{array}{l}
\left. u(0,t) = u(1,t) = 0, \\
\left. u(x,0) = u_0(x), 
\end{array} \right. \\
\end{aligned}
\]  

(2)

where $u(x,t)$ is the temperature at time $t$ at a point $x$ and $\sigma$ is the heat diffusion coefficient. The heat diffusion coefficient $\sigma$ is assumed to be constant, and $u_0(x)$ is the initial distribution of the temperature.

a) Formulate a discrete-in-time Galerkin finite element approximation of (2) using a continuous piecewise linear basis.

b) Discuss convergence of the scheme.

c) Formulate a flux approximation at a point $\bar{x} = x$, for some interior node in the mesh. (Hint: Use Green's theorem.)
CAM AREA B PRELIMINARY EXAM
May 29, 2009. 9:00 a.m. 12:00 noon

Work Part I and the appropriate Part II. Each Part is counted equally in the final score.

Part I. CAM 385C Numerical Linear Algebra

There are 3 problems.

1. (10 points) Let \( A \in \mathbb{R}^{m \times n}, m > n, \) and let \( A \) have rank \( r < m. \)
   (a) [3 points] What is the SVD (Singular Value Decomposition) of \( A? \) Give the orthogonal
   basis, in terms of the singular vectors of \( A, \) for each of the four fundamental subspaces
   associated with \( A: (i) \) column space of \( A, \) (ii) row space of \( A, \) (iii) null space of \( A, \) and
   (iv) null space of \( A^T. \)
   (b) [2 points] What is the condition number of \( A \) (use the operator/induced 2-norm)? What
   is the condition number of \( A, \) when \( A \) is restricted to operate on the subspace spanned
   by the leading \( r \) right singular vectors of \( A? \)
   (c) [5 points] Suppose we need to solve (for \( x \)) the least squares problem: \( \min_x \|Ax - b\|_2. \)
   Show how to use the SVD to solve for \( x. \) Is the solution unique?

2. (10 points) Suppose we need to solve a linear system \( Ax = b, \) where \( A \) is an \( n \times n \) matrix.
   (a) [3 points] Explain backward error analysis in the context of solving the above linear
   system of equations.
   (b) [5 points] Suppose \( A \) is a general \( n \times n \) dense matrix. Give the pros and cons of
   the following three pivoting strategies if \( Ax = b \) is to be solved by \( LU \) decomposition: (i)
   no pivoting, (ii) partial pivoting, (iii) complete pivoting. Consider both accuracy of the
   solution and time complexity. Which methods guarantee a small backward error?
   (c) [2 points] How would your answer to part (b) change if \( A \) is known to be positive definite?

3. (10 points) Assume that \( \{q_1, q_2, \ldots, q_k\} \) are mutually orthonormal vectors of length \( n \)
   and that \( x \) is also a vector of length \( n. \)
   (a) [3 points] Show that \( y = x - (q_1^T x q_1 + q_2^T x q_2 + \cdots + q_k^T x q_k) \) is the component of \( x \)
   orthogonal to \( q_1, \ldots, q_k. \)
   (b) [3 points] Let \( y \) be defined by the following iteration:
   \[
   y = x \\
   \mathrm{do} \ i = 1, \cdots, k \\
   \quad y = y - q_i^T y q_i \\
   \mathrm{enddo}
   \]
   Show that the resulting \( y \) is also the component of \( x \) orthogonal to \( q_1, \ldots, q_k. \)
   (c) [2 points] Discuss how the computations in the previous parts are used as part of the
   Gram-Schmidt (GS) and Modified Gram-Schmidt (MGS) processes.
   (d) [2 points] Briefly (informally) discuss which approach is better in practice, and why.
Part II. CAM 394F Finite Element Methods (Two problems.)

1. Consider the triangular domain shown in Fig. 1 and the following boundary-value problem.

![Diagram of a triangular domain with a four element FE mesh](image)

Figure 1: A triangular domain with a four element FE mesh

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \Gamma_D \\
\frac{\partial u}{\partial n} = 1 & \text{on } \Gamma_N \\
\end{cases}
\]

- Explain why this is an elliptic problem. Is it uniformly elliptic?
- Derive the corresponding variational formulation and discuss its equivalence with the classical formulation. Identify bilinear and linear forms corresponding to the variational formulation.
- Demonstrate that the bilinear form is symmetric and positive-definite on the space of test functions.
- Explain why the problem admits a minimum energy principle and identify the corresponding equivalent minimization problem.

2. Consider the problem above and approximate it with a simple mesh of four linear ($p = 1$) triangular elements shown in the picture.

- Compute the element stiffness matrix and load vector for elements in the mesh (be clever enough to avoid computing terms that will get eliminated because of Dirichlet boundary conditions).
- Assemble the global matrices and solve the final system of equations.
- Plot by hand the FE solution.
Part II. CAM 385D Numerical Analysis

There are 2 problems.

1. a. Suppose you are given six data pairs \( \{(x_1, y_1), (x_1, \overline{y}_1), (x_2, y_2), (x_2, \overline{y}_2), (x_2, \overline{y}_2), (x_2, \overline{y}_2)\} \) and assume \( x_1 < x_2 \). Describe how to construct a quintic (i.e. fifth degree) function \( s \) so that

\[
\begin{align*}
  s(x_i) &= y_i, \\
  s'(x_i) &= \overline{y}_i, \\
  s''(x_i) &= \overline{y}_i,
\end{align*}
\]

for \( i = 1, 2 \).

b. With \( f(x) = \cos(x) \) and \( x_i = (i - 1)b \) for \( i = 1, \ldots, n \), determine \( n \) and a spacing \( b \) so that a piecewise quintic function \( s \) satisfying

\[
\begin{align*}
  s(x_i) &= f(x_i), s'(x_i) = f'(x_i), \text{ and } s''(x_i) = f''(x_i), \text{ for } i = 1, 2, \ldots, n,
\end{align*}
\]

also satisfies

\[
|s(x) - f(x)| \leq 10^{-10} \text{ for all } x \in [0, \pi/2].
\]

The value of \( n \) you determine may not be the minimal such value, but at least it should be close. Also you may specify \( b \) as a solution to a nonlinear equation without actually solving it and your value of \( n \) may depend upon that value of \( b \).

2. a. Show how to derive the set of Legendre polynomials

\[
L_0(x), L_1(x), \ldots
\]

where for \( n = 0, 1, \ldots \), \( L_n \) is a monic (i.e. leading coefficient 1) polynomial of degree \( n \) so that

\[
\int_{-1}^{1} L_n(x)p(x)dx = 0
\]

for all polynomials \( p \) of degree \( n - 1 \) or less.

b. How could the Legendre polynomials be used to determine a Gaussian quadrature formula of the form

\[
\int_{-1}^{1} f(x) = \sum_{i=1}^{n} w_i f(x_i)
\]

that is exact if \( f \) is a polynomial of degree \( 2n - 1 \) or less. Specifically, how should the nodes \( \{x_i\}_{i=1}^{n} \) and the weights \( \{w_i\}_{i=1}^{n} \) be determined?

c. Prove your claim from part b: if the nodes \( \{x_i\}_{i=1}^{n} \) and the weights \( \{w_i\}_{i=1}^{n} \) are determined according to your specifications then

\[
\int_{-1}^{1} p(x) = \sum_{i=1}^{n} w_i p(x_i)
\]

for all polynomials \( p \) of degree \( 2n - 1 \) or less.
Part II. CS 395T Computational Statistics

There are 3 problems. You may use a calculator.

1. There are two boxes. Each contains some number of balls. In one box (the “Good” box), 2/3 of the balls are Red, 1/3 are Blue. In the other box (the “Bad” box), 1/3 of the balls are Red, 2/3 are Blue. You choose one box at random and draw (with replacement) 2 balls from it.
   (a) What is the probability of getting two different colors?
   (b) Now suppose that you drew Red balls both times. What is the probability that you have selected the Good box?
   (c) Now suppose that you draw a third ball from the same box. What is the probability that it is Red?

2. At \( N \) values of the independent variable \( x \), that is, \( x_i, i = 1, \ldots, N \), you measure the value of a dependent variable \( y \), obtaining values \( y_i, i = 1, \ldots, N \). Each measurement \( y_i \) has a corresponding accuracy \( \sigma_i \), independently normally distributed. In other words, the data model is
   \[
   y_i = y(x_i) + e_i, \quad \text{with} \quad e_i \sim N(0, \sigma_i)
   \]  
   where \( e_i \) is the unknown error of measurement \( i \).
   You don’t know the function \( y(x) \), but you think it might be in the three-parameter family of functions \( \tilde{y}(x|\lambda_1, \lambda_2, \lambda_3) \) where \( \lambda_1, \lambda_2, \lambda_3 \) are the parameters. You are going to fit this model to the data.
   (a) As a Frequentist, in terms of the above quantities, write down a statistic \( \chi^2(\lambda_1, \lambda_2, \lambda_3) \) that measures the mean-square deviation of the fitted model from the data, and that is ChiSquare distributed.
   (b) Next suppose that you have found values \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \) that minimize your answer to part (a), and that the minimum \( \chi^2 \) value is \( \hat{\chi}^2 \). If the model is indeed correct, and if \( N \) is large, how do you expect \( \chi^2 \) to be distributed? (Either write down a distribution explicitly, or at least give a mean and standard deviation or variance.)
   (c) Now, as a Bayesian, start over and write down the probability (or, if you will, probability density) of getting exactly the data seen, as a function of \( \lambda_1, \lambda_2, \lambda_3 \).
   (d) What is the relation between your answers to (c) and (a)?
   (e) Write a Taylor series for \( \chi^2(\lambda_1, \lambda_2, \lambda_3) \), through second derivatives, around the minimum values \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \). Write down (or just circle in your Taylor series) the Hessian matrix.
   (f) Characterize the accuracy with which the parameters \( \lambda_1, \lambda_2, \lambda_3 \) can be found. For example, what can you say about their variances or covariances? If possible, don’t just assert the answer. Use your answers to parts (c), (d), and (e) to prove it. Hint: multivariate Gaussian!

3. Consider the 2-state Hidden Markov Model shown below. with states A and B, each of which can emit symbols X and Y. The figure shows the transition probabilities. State A emits the symbol X with probability 0.9, and the symbol Y with probability 0.1. State B emits the symbol X with probability 0.3, and the symbol Y with probability 0.7.
(a) Write down the transition matrix for the Markov model. What makes it a stochastic matrix? You observe the model at times 0 and 1 (that is, just one transition) and see the symbol X at both times. The figure below shows the model “unrolled” for this case. The remaining parts of this question refer to this figure.

(b) What are the values of $\alpha$ associated with states $A_0$ and $B_0$. Hint: $\alpha$ is (proportional to) the probability of a state at time $i$ given the symbols emitted at time $i$ and all earlier times.

(c) What are the values of $\beta$ associated with states $A_1$ and $B_1$. Hint: $\beta$ is (proportional to) the probability of a state at time $i$ given the symbols emitted at all times later than $i$.

(d) What are the remaining values of $\alpha$ and $\beta$, namely $\alpha$ for states $A_1$ and $B_1$, and $\beta$ for states $A_0$ and $B_0$. Hint: You can simplify the arithmetic by realizing that you only need to calculate, for each given time, a quantity proportional to each $\alpha$ or $\beta$.

(e) What is the probability, given all the data, that the model is in state A at time 0? state B at time 0?
Part I. CAM 383C – Numerical Linear Algebra

1. For an \( m \)-vector \( u \neq 0 \) and scalar \( \alpha = \frac{2}{u^T u} \), consider the \( m \times m \) matrix:
   \[
   A = I - \alpha uu^T.
   \]

   a. Show that
      i. \( A \) is symmetric.
      ii. \( A \) is orthogonal.
      iii. \( A \) is its own inverse.

   b. Given a positive integer \( k \leq m \) and an \( m \)-vector \( x \), show how to determine \( u \) so that
      \( y = Ax \) has the properties:
      i. \( y_i = x_i \) for \( i = 1, \ldots, k - 1 \).
      ii. \( y_i = 0 \) for \( i = k + 1, \ldots, m \).

   c. Given a positive integer \( n < m \), show how such matrices could be used to factor a given \( m \times n \) matrix \( C \) (with linearly independent columns) into the product of an \( m \times m \) orthogonal matrix \( Q \) and an \( m \times n \) upper triangular matrix \( R \).

2. For an \( n \times n \) nonsingular matrix \( A \) and \( n \)-vector \( b \), let matrices \( M \) and \( N \) satisfy:
   \[
   M \text{ is nonsingular}
   \]
   and
   \[
   A = M - N.
   \]
   Consider the iteration:
   \[
   Mx^k = Nx^{k-1} + b.
   \]
   for \( k = 1, 2, \ldots \) and a given \( x^0 \).

   a. Prove that if \( \|M^{-1}N\| < 1 \) for some matrix norm \( \| \| \) that is consistent with a vector norm
   then for any given \( x^0 \) it follows that
   \[
   \lim_{k \to \infty} x^k = A^{-1}b.
   \]

   b. Let \( M \) be the diagonal portion of matrix \( A \), and \( N = M - A \). If \( A \) is strictly row diagonally dominant (i.e.
   \[
   |a_{ii}| > \sum_{j \neq i} |a_{ij}| \text{ for } i = 1, \ldots, n
   \]
   then with the iteration above for any given \( x^0 \) it follows that
   \[
   \lim_{k \to \infty} x^k = A^{-1}b.
   \]
Part II. CAM 383D – Numerical Analysis

1. a. You are given real values $\alpha$ and $\beta$ so that $\alpha < \beta$ and $f$, a real valued function with four continuous derivatives on the interval $[\alpha, \beta]$. For $m = \frac{\alpha + \beta}{2}$, let $p$ be a cubic polynomial satisfying:

\[
\begin{align*}
p(\alpha) &= f(\alpha), \\
p(m) &= f(m), \\
p'(m) &= f'(m), \\
p(\beta) &= f(\beta).
\end{align*}
\]

Use the Hermite Interpolation Remainder Theorem to prove:

\[
|p(x) - f(x)| \leq \frac{(\beta - \alpha)^4 M}{4 \cdot 4!},
\]

where $M = \max_{x \in [\alpha, \beta]} |f^{(4)}(x)|$. (A tighter bound can be proved but you need not do so for this problem.)

b. What are the coefficients $w_1$, $w_2$, and $w_3$ for Simpson’s Rule:

\[S(f; \alpha, \beta) = w_1 \cdot f(\alpha) + w_2 \cdot f(m) + w_3 \cdot f(\beta)\]

for approximating $\int_{\alpha}^{\beta} f(x) \, dx$?

c. Assume the simple Simpson’s Rule in part b is exact for all cubic polynomials (i.e. $S(p; \alpha, \beta) = \int_{\alpha}^{\beta} p(x) \, dx$ for all cubic polynomials $p$). Use part a to prove,

\[
|S(f; \alpha, \beta) - \int_{\alpha}^{\beta} f(x) \, dx| \leq \frac{(\beta - \alpha)^4 M}{4 \cdot 4!}.
\]

d. Now consider applying Simpson’s Rule in a $2n+1$ point composite fashion to $f$ on an interval $[c, d]$. Expressed as

\[S^*(f; c, d, n) = \sum_{i=0}^{2n} w_i \cdot f(c + ih),\]

where $b = \frac{d - c}{2n}$, what are the coefficients $w_i$?

e. Use part c to prove

\[
|S^*(f; c, d, n) - \int_{c}^{d} f(x) \, dx| \leq \frac{(2b)^4 \bar{M}(d - c)}{4 \cdot 4!},
\]

where $\bar{M} = \max_{x \in [c, d]} |f^{(4)}(x)|$. (Hint: You may want to express $S^*$ in terms of a summation of $S$.)
2. Given an array $g_1, g_2, ..., g_n$ of linearly independent functions and an inner product $(.,.)$ defined for all linear combinations of those functions, specify an algorithm to determine an array $h_1, h_2, ..., h_n$ of functions orthonormal with respect to the inner product and spanning the same space as the span of $g_1, g_2, ..., g_n$. 
Part II. CAM 394F – Finite Element Methods

1. Consider the stationary advection-diffusion equation:

\[ u + \nabla \cdot (qu) - \Delta u = f, \quad \Omega \]  

(1)

with boundary conditions:

\[ (qu - \nabla u) \cdot n = qu_I \cdot n, \quad \Gamma_I, \]  

\[ \nabla u \cdot n = 0, \quad \Gamma_O, \]  

(2)

(3)

where \( \Gamma_I \) denotes the inflow part of the boundary \( \partial \Omega \), and \( \Gamma_O \) denotes the outflow/noflow portion. Here \( q \) is a specified vector-valued function, \( f \) is a source function, \( u_I \) is a specified inflow value of \( u \), and \( n \) is the unit outward normal to the boundary. We note that \( \Gamma_I \) is the portion of the boundary where \( q \cdot n < 0 \) and \( \Gamma_O \) is the remainder of the boundary.

- Derive a weak formulation of the problem by multiplying by a test function \( v \in H^1(\Omega) \) and integrating by parts. Determine appropriate test and trial spaces for the weak formulation, including the imposition of boundary conditions.
- Based on your weak form, develop a finite element approximation of \( u \) based on continuous, piecewise polynomials.
- Discuss the existence and uniqueness of the finite element solution \( u_h \). If necessary, determine conditions on \( q \) which guarantee a solution exists.

2. Consider the Galerkin FEM approximation to the solution of the diffusion equation

\[ u - \nabla \cdot (D\nabla u) = f, \quad \Omega \]

with \( u = 0 \) on \( \partial \Omega \). Assume \( D \) is symmetric and positive definite. Let \( u_h \) be the FEM solution based on continuous, piecewise polynomials defined on a finite element mesh. Derive the error bound

\[ ||u - u_h||^2 + ||D^{1/2}\nabla (u - u_h)||^2 \leq ||u - \psi||^2 + ||D^{1/2}(u - \psi)||^2, \]

where \( \psi \) is any function in the finite element space.
Part II. CS 395T – Computational Statistics

This is intentionally a long exam, so you might want to do the easier parts of each question before circling back and doing the harder parts of each.

1. A univariate probability distribution is given by

\[ p(x) = \text{const} \times \begin{cases} 1 - \cos(x) & \text{for} \ 0 < x \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \]

a) Sketch the distribution.
b) What is the normalizing constant? (That is, what is the value of “\text{const}”?)
c) Write the CDF (cumulative distribution function) as an explicit function \( P(x) \). (Don’t just write its definition).
d) Write clear computer code, or pseudo-code, that returns a random deviate drawn from \( p(x) \). You can assume a function \text{uniform()} \ that returns a uniform deviate in \((0, 1)\). Comment any line that is not completely obvious. Hint: my answer to this has five lines, two of which are a while loop and its corresponding endwhile (or close bracket).

2. You are given a list of \((x, y \pm \sigma)\) data points, and you want to fit a straight line \( y = a + bx \) through them, obtaining parameters \( a \) and \( b \). Here \( x \) is the independent variable, \( y \) is the measured, dependent variable, and \( \sigma \) is the measurement error, assumed to be the standard deviation of a Normal \( N(0, \sigma) \) distribution of errors. The data are:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

a) Graph the points, draw a line by eye, and write down your best guess for the parameters \( a \) and \( b \).
b) Write down the probability (or probability density) of the observed data as a function of \( a \) and \( b \)? (You can ignore any normalizing constant.)
c) Write down the function \( \chi^2(a, b) \) that a frequentist would minimize in finding the best parameter values. (Your answer should be an explicit function containing \( a \), \( b \), and numerical values only – no other symbols allowed.)
d) What numerical values of \( a \) and \( b \), call them \( a_{\text{min}} \) and \( b_{\text{min}} \), minimize \( \chi^2 \)? (The arithmetic here has been designed to come out simple.)

e) What is covariance matrix of the parameters \( a \) and \( b \)? Hint: Hessian. Another hint: The inverse of the matrix

\[
\begin{pmatrix}
x & w \\
w & y \\
\end{pmatrix}
\]

is

\[
\frac{1}{xy - w^2}
\begin{pmatrix}
y & -w \\
-w & x \\
\end{pmatrix}
\]

f) Write down the joint probability distribution of the parameter uncertainties \( a - a_{\text{min}} \) and \( b - b_{\text{min}} \). Your answer should be an explicit function of \( (a - a_{\text{min}}), (b - b_{\text{min}}) \), and numerical values only. (However, you don’t need to multiply out any matrices that might occur in your answer. Again, OK to leave out the normalizing constant.) Why does your answer have the functional form that it does?

3. A Supreme Court has of 10 Justices, consisting of 7 boys and 3 girls. A committee of 4 Justices is selected randomly to organize the Holiday Party, with the outcome (showing column and row marginals)

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>on the Party Committee</td>
<td>2</td>
</tr>
<tr>
<td>not on the Party Committee</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>7</td>
</tr>
</tbody>
</table>

a) What is the total number of distinguishable ways that the committee of 4 could have been chosen?

b) What is the number of distinguishable ways that it could have been chosen with 2 boys and 2 girls?

c) What is the probability of getting exactly 2 boys and 2 girls if the selection was random?

It is later alleged that the committee was NOT drawn randomly, but that girls were favored, perhaps in the scurrilous belief that they are better at organizing parties.

d) What are the number of distinguishable ways to have 0, 1, 2, 3, or 4 boys on the committee? (Your answer should be a list of five integers.)

e) What is the p-value with which the null hypothesis (of random selection) can be rejected? Explain why you choose either a 1-tailed or 2-tailed test. Do you think that the null hypothesis is rejected?
Part II. CS 395T – Computational Statistics

f) In class, we made a big deal about how experimental protocols never fix all the marginals, so that Fisher’s Exact Test (which you have just done) is not really exact. Does that hold here, or have we found an exception? Explain why or why not.

4. For each part below, write at most two sentences, or one sentence and an equation, that show that you know something about the indicated topic.
   a) characteristic function of a probability distribution
   b) Wiener filter
   c) ROC curve
   d) marginalization of a nuisance parameter
   e) EM method
1. Let \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \) and assume that \( A \) has \( n \) linearly independent columns.
(a) Give the (Reduced) Singular Value Decomposition (SVD) of \( A \).
(b) Determine the (Reduced) SVD of the matrices \((A^T A)^{-1}\), \((A^T A)^{-1} A^T\), \(A(A^T A)^{-1}\), and \(A(A^T A)^{-1} A^T\).
(Pay attention to the dimensions of the different matrices, as a sanity check!)
(c) Compute \(\|(A^T A)^{-1}\|_2\), \(\|(A^T A)^{-1} A^T\|_2\), \(\|A(A^T A)^{-1}\|_2\), and \(\|A(A^T A)^{-1} A^T\|_2\).
(d) Discuss how to use the SVD to compute the linear least-squares solution to \(Ax = b\).

2. 
(a) Define the machine epsilon, \( \epsilon_{\text{mach}} \) (sometimes called the unit roundoff).
(b) Consider the matrix
\[
\begin{pmatrix}
1 & 1 & 1 \\
\epsilon & 0 & 0 \\
0 & \epsilon & 0 \\
0 & 0 & \epsilon \\
\end{pmatrix}
\]
where \( \epsilon^2 < \epsilon_{\text{mach}} \). Compute the QR factorization of this matrix using two different methods: (1) Classical Gram-Schmidt and (2) Modified Gram-Schmidt. Assume that when you compute the only error that is made is that every time you compute \(1 + \epsilon^2\) this is rounded to 1. No other errors are incurred.
(c) Explain in simple terms (understandable by an undergraduate who has seen Gram-Schmidt but does not know yet about roundoff error) why there is a different in the resulting QR factorization.
3. Consider the initial value problem \( y'(t) = f(y(t)) \) for \( t > 0 \) and \( y(0) = y_0 \). As usual, let \( \Delta t > 0 \) and \( t_n = n\Delta t \). We approximate \( y_n \approx y(t_n) \) by the two-step method (for \( n = 0, 1, 2, \ldots \))

\[
\begin{align*}
\tilde{y}_{n+1} &= y_n + \Delta t \, f(y_n), \\
y_{n+1} &= y_n + \frac{1}{2} \Delta t \left[ f(y_n) + f(\tilde{y}_{n+1}) \right].
\end{align*}
\]

(a) Show that the order of the local truncation error for the method is \( O(\Delta t^2) \).
(b) Find a bound on \( \Delta t \) so that the method is linearly stable, i.e., so that it is stable when \( f(y) = \lambda y \), where \( \lambda < 0 \) is real. [This is the real axis part of the region of linear stability.]

4. When using rectangular grids, one sometimes changes mesh spacing by using what are known as hanging nodes. These are nodal points that do not actually have a linear functional associated with them. To fix ideas, consider the three elements as in the figure, where the hanging node is at (1,1), and we use continuous piecewise bilinear functions over the three elements.

(a) Describe the nodal basis function at (0,0) for all \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 2 \). [You can ignore the point (1,1) in this case.] What is the value of the basis function at (1,1)?
(b) Describe the nodal basis function at (1,0) for all \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 2 \). What is the value of the basis function at (1,1)? [Note that the value at (1,1) must be defined so that the function is continuous.]

5. For \( \mu > 0 \) a constant, consider the modified Stokes problem on a bounded domain \( \Omega \subset \mathbb{R}^2 \)

\[
\begin{align*}
-\mu \Delta u + \nabla p &= f & \text{in } \Omega, \\
p + \nabla \cdot u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( u(x) \) and \( f(x) \) are vector valued in \( \mathbb{R}^2 \). A variational form for this problem can be written as: Find \( u \in (H^1_0(\Omega))^2 \) and \( p \in L^2(\Omega) \) such that

\[
\begin{align*}
\mu (\nabla u, \nabla v) - (p, \nabla \cdot v) &= (f, v) \quad \forall v \in (H^1_0(\Omega))^2, \\
(p, q) + (\nabla \cdot u, q) &= 0 \quad \forall q \in L^2(\Omega).
\end{align*}
\]

(a) In terms of the coordinates \( \partial/\partial x_i, u_i, \) and \( v_i, \) what is the meaning of \( (\nabla u, \nabla v) \)?
(b) Suppose we use a conforming triangular mesh over \( \Omega \). Describe simple finite element subspaces of \( (H^1_0(\Omega))^2 \) and \( L^2(\Omega) \) that you might use to discretize the equations. Does \( p_h \) need to be continuous?
(c) Show that the finite element method is stable, provided that \( f \in (L^2(\Omega))^2 \).
(d) Show that there exists a unique solution \( (u_h, p_h) \) to the finite element method.
Work Part I and the appropriate Part II.

CSEM Prelim 2012: Part 1 - Numerical Linear Algebra

1. Consider solving the equation \( A\tilde{x} = \tilde{b} \) iteratively, where \( A \in \mathbb{R}^{n \times n} \) is nonsingular and \( \tilde{b} \in \mathbb{R}^n \) is given. We will use the following notations:
   - \( \tilde{x} = A^{-1}\tilde{b} \) is the true solution,
   - \( A = M - K \) is the splitting of \( A \) (with \( M \) nonsingular),
   - \( R = M^{-1}K \),
   - \( || \cdot || \) is any induced matrix norm.

We can then run the iterative scheme

\[
(*) \quad \tilde{x}_{k+1} = M^{-1}(K\tilde{x}_k + \tilde{b}).
\]

(a) We know that if \( ||R|| < 1 \), then \((*)\) converges for any initial guess \( \tilde{x}_0 \) (i.e. \( ||\tilde{x}_k - \tilde{x}|| \to 0 \) as \( k \to \infty \)). Discuss why this statement alone is inconvenient in practice to use as a convergence criterion.

(b) Let \( \rho(R) \) denote the spectral radius of \( R \). Show that \( \rho(R) \leq ||R|| \).

(c) Show that \((*)\) converges for any initial guess \( \tilde{x}_0 \), if and only if, \( \rho(R) < 1 \).

Hint: For one of the directions, you will need to use the following fact:

For a given \( \epsilon > 0 \), there exists an induced matrix norm \( || \cdot ||_* \) such that \( ||R||_* \leq \rho(R) + \epsilon \).

2. This question concerns the QR factorization of a matrix via Householder transformations.

(a) Let \( x \in \mathbb{R}^k \). How should we choose \( u \in \mathbb{R}^k \) so that \((I - 2uu^T)x = \pm\|x\|e_0\), where \( e_0 \) is the first unit basis vector?

(Note: the rest of this question can be answered without answering this first part. So don’t panic if you forgot.)

(b) Let \( u \in \mathbb{R}^k \) and \( X \in \mathbb{R}^{k \times k} \) be given.
   - How should one in practice compute \((I - 2uu^T)X\)?
   - What is the approximate cost of this computation, in floating point operations?
      (A \( O(k^3) \) expression is not exact enough. Count a multiply as one flop and count an add/subtract as one flop.)

Note: if your cost is \( O(k^3) \), you did it wrong... The leading term in the cost function must have the correct constant to get full credit.

(c) Compute

\[
\begin{pmatrix}
  I & 0 \\
  0 & H_k
\end{pmatrix}
\begin{pmatrix}
  I & 0 \\
  0 & X_k
\end{pmatrix}
\]

where the two matrices are partitioned conformally \((H_k, X_k \in \mathbb{R}^{k \times k})\)

(d) Recall how Householder transformations are used to compute a QR factorization of an \( n \times n \) matrix:
   - A sequence of \( n \) "Householder vectors" \( \{v_0, \ldots, v_{n-1}\} \) are computed.
   - \( \tilde{H}_{n-1}\tilde{H}_{n-2}\cdots\tilde{H}_0A = R \).
   - Each \( \tilde{H}_k \) has the form of the left matrix in (1), with \( k = n - j \).
   - Each \( H_k \) has the form \( H_k = I - 2v_nv_n^T \).

Consider forming \( Q = \tilde{H}_0\tilde{H}_1\cdots\tilde{H}_{n-1} \). Sketch out an algorithm that computes this and discuss its approximate cost. I am most interested in how you make sure that you do NOT to compute with (most of the) zeroes. You do not need to prove your result, but I do want to you show intermediate steps so that I can give partial credit.

Hint: \( Q = (\tilde{H}_0(\tilde{H}_1\cdots(\tilde{H}_{n-1}I)\cdots)) \).
Part II – CSE 383L, Num Anly: Diff Eq

1. Approximate a smooth function \( f(x) \) by a polynomial \( p_n(x) = \sum_{j=0}^{n} a_j x^j \).
   
   (a) Formulate the equations for the least square approximation over the interval \(-1 < x < 1\).
   
   (b) Prove that there exists a unique \( p_n(x) \) interpolating \( f(x) \) at points \( \{x_j\}_{j=0}^{n} , -1 < x_0 < x_1 < \cdots < x_n < 1 \).

2. Consider the hyperbolic partial differential equation

\[
\begin{align*}
  u_t + au_x + b(x)u & = 0, \quad 0 < x < 1, t > 0, \quad a \leq 0, b(x) \geq 0, \\
  u(0, t) = u(1, t), & \quad t > 0, \\
  u(x, 0) = u_0(x), & \quad 0 \leq x \leq 1.
\end{align*}
\]

   (a) Derive an upwind finite difference scheme on the grid \( x_j = j\Delta x, j = 1, \cdots, N, N\Delta x = 1, \) and \( t^k = k\Delta t, k = 0, 1, \cdots \).
   
   (b) Prove a maximum principle for this scheme, under suitable conditions on \( \Delta x \) and \( \Delta t \).

3. Consider the energy

\[
E(u) := \frac{1}{2} \int_0^1 |u_x(x)|^2 dx + \frac{\lambda}{2} \int_0^1 (u(x) - f(x))^2 dx, \quad \lambda > 0,
\]

declared for \( u \in C^2([0, 1]; \mathbb{R}), f \in C([0, 1]; \mathbb{R}), \) and \( u(0) = u(1) = f(0) = f(1) = 0 \). Consider a discrete approximation of \( E \) as follows: \( U = (u_1, u_2, \cdots, u_{N-1})^T \),

\[
E_h(U) := \frac{1}{2} \sum_{j=1}^{N-1} |D^+ u_j|^2 h + \frac{\lambda}{2} \sum_{j=1}^{N-1} |u_j - f_j|^2 h,
\]

where \( h = 1/N, u_0 = u_N = 0, D^+ u_j = (u_{j+1} - u_j)/h, \) and \( f_j = f(jh) \).

   (a) Do \( E \) and \( E_h \) have unique minimizers?
   
   (b) Derive the linear system
   \[
   AU = b
   \]

   whose solution minimizes \( E_h \).

   (c) How would you solve the above system with a theoretical optimal computational complexity?

   (d) Does the solution of (1) approximate the minimizer of \( E \)?

Justify your answers.

Please do all three problems. Provide complete enough answers to allow partial credit to be reasonably given. Please write your answers on blank sheets of paper, not on this page.

1. Consider a random variable $X$ with the triangular probability density function $p(x)$ on $0 \leq x \leq 1$ shown below. Its maximum is at $x = a$.

![Triangular PDF](image)

- a) What is the value of $p(a)$, shown in the figure as $p_{\text{max}}$?
- b) In terms of $a$, what is the expectation (or mean) of $X$, that is, the value of $\langle x \rangle$? If you get a result that is not very simple, you can give its numerical values for the cases $a = 0$ and $a = 1$ instead of trying to simplify it.
- c) Explain how you would draw random deviates $x_i$, $i = 1, \ldots, N$, from $p(x)$ with a known value $a$. ($N$ is some large integer.)
- d) Conversely, explain how you might estimate $a$, and also an uncertainty for your estimate of $a$, given a bunch of draws $x_i$, $i = 1, \ldots, N$ from the distribution.

2. You want to fit a straight line $y = a + bx$ to a set of measurements $(x_i, y_i, \sigma_i)$, obtaining values for the parameters $a$ and $b$. Here $x_i$ is precisely known, but $y_i$ is measured with an uncertainty $\sigma_i$. That is, its measurement error is drawn from the Normal distribution $N(0, \sigma_i)$. Here is the data:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$\sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>4.0</td>
<td>4.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Here is a graph showing the data and (as a hint) the line that I got when I did this problem.
a) As a function of the parameters $a$ and $b$, what is the probability (or probability density) of the data? (You don’t have to simplify this, just write it in an understandable form.)

b) As a function of $a$ and $b$, what is the $\chi^2$ statistic that you should minimize to find the most probable (or likely) values of $a$ and $b$. This answer should be closely related to your answer for part (a). It should contain $a$’s and $b$’s, but no other symbolic quantities (i.e., you should substitute numerical values for all $x$’s, $y$’s, or $\sigma$’s).

c) Show that the minimization with respect to $a$ and $b$ implies two linear equations in the two unknowns $a$ and $b$. Write those equations.

d) Solve the two equations for $a$ and $b$. Did you get the same line that I did?

3. “Tell me about...” the following concepts. That is, write two or three sentences or equations on each that roughly tells what it is about and how you might use it.

a) Bayes denominator.

b) EM (or expectation-maximation) methods.

c) one-sided versus two-sided $p$-value test.

d) Metropolis-Hastings algorithm.

e) characteristic function of a distribution.
2013 CSEM Area B Preliminary Exam

Part I: Numerical Linear Algebra

1. (10 points) Consider the linear least squares problem

\[ \min_x \|Ax - b\|_2, \tag{1} \]

where \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \).

(a) (2 points) Derive the normal equations for solving (1).

(b) (3 points) Show how to use QR decomposition and SVD (singular value decomposition) to solve (1).

(c) (3 points) Suppose \( A \) does not have full column rank. Is the least squares solution unique? Characterize all solutions in terms of the SVD of \( A \).

(d) (2 points) Suppose \( A \) does have full column rank, but many of its singular values are small (for example, \( m = 100, n = 50, \sigma_1 = 2, \sigma_{25} > 1 \) and \( \sigma_{26}, \ldots, \sigma_{30} < 10^{-13} \)). How will you solve the least squares problem (1) in this case? Discuss.

2. (10 points) Let \( A \) be a square, non-singular matrix.

(a) (4 points) Let \( Ax = b \) and \( (A + \delta A)\hat{x} = b + \delta b \). Show that

\[ \frac{\|x - \hat{x}\|}{\|\hat{x}\|} \leq \kappa(A) \left( \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|A\|\|\hat{x}\|} \right). \tag{2} \]

(b) (2 points) Explain backward error analysis in the context of solving \( Ax = b \), and discuss how the above bound is useful in analyzing the accuracy of solving a system of linear equations in floating point arithmetic.

(c) (4 points) Give the pros and cons of the following three pivoting methods if \( Ax = b \) is to be solved by \( LU \) decomposition: (i) no pivoting, (ii) partial pivoting, (iii) complete pivoting. Consider both accuracy of the solution and time complexity. What can you say about the backward error for these three methods (explain your answer in terms of the bound (2))?
Part II: Numerical Analysis: Diff Eq

1. Consider \( g(x) = \frac{1}{2} x + 2x^2 - \frac{3}{2} x^3 \) and the iteration defined by \( x_{n+1} = g(x_n) \).

   (a) Show that for any \( x_0 \in [0,1] \), the sequence \( x_n \) converges, and find the limiting function \( g_\infty(x_0) \) of the initial value.

   (b) For each \( x_0 \in [0,1] \), determine the order of convergence of \( \{x_n\} \).

2. Given the system of differential equations
   \[
   \begin{pmatrix}
   u \\
   v
   \end{pmatrix}
   _t + \begin{pmatrix}
   0 & 1 \\
   1 & 0
   \end{pmatrix}
   \begin{pmatrix}
   u \\
   v
   \end{pmatrix}
   _x = 0
   \]
   to be solved for \( t \geq 0 \) and \( 0 \leq x \leq 1 \) with periodic boundary conditions in \( x \) and smooth initial data
   \[
   u(x,0) = f(x),
   v(x,0) = g(x).
   \]
   Construct a convergent explicit finite difference scheme on a uniform grid which uses a two point stencil for \( u + v \) and possibly another two point stencil for \( u - v \) in space. Prove the convergence of the scheme in a suitable norm.

3. Consider the following heat equation in two space dimensions,
   \[
   u_t = \Delta u, \quad |x| < 1, |y| < 1, t > 0,
   u = 0, \quad |x| \leq 1, |y| = 1, t > 0,
   u_x = u, \quad x = -1, |y| < 1, t > 0,
   u_x = -u, \quad x = 1, |y| < 1, t > 0,
   u(x,y,0) = u_0(x,y), \quad |x| \leq 1, |y| \leq 1.
   \]
   (a) Derive a numerical approximation \( u_h^n(x,y) \) by rewriting the heat equation in a variational form and formulating a method based on finite elements in space and Crank-Nicolson (trapezoidal rule) in time, \( t_n = n\Delta t, n = 0,1,\ldots \).
   (b) Prove \( L^2 \)-stability: \( ||u_h^n||_{L^2(|x|<1,|y|<1)} \leq ||u_0||_{L^2(|x|<1,|y|<1)} \).
Part II: Computational Statistics

Do all three problems.

1. Suppose that \( \lambda \) and \( \nu \) are independent normal random variables,

\[
\lambda \sim N(0,1) \quad \text{and} \quad \nu \sim N(0,1)
\]

and that \( x, y, \) and \( z \) are random variables constructed as follows:

\[
x = 4 + 5\nu \\
y = 1 - 2\lambda - 2\nu \\
z = 3 + 3\lambda
\]

(a) What are the means of each of \( x, y, \) and \( z \)? (Give numerical answers.)
(b) What are the variances of each of \( x, y, \) and \( z \)? (Give numerical answers.)
(c) What is the covariance matrix of the joint distribution of \( (x, y, z) \)? (Give numerical answer.)
(d) What is the correlation (or “R”) matrix of \( (x, y, z) \)? (Give numerical answer.)
(e) Make a scatter plot of what a bunch of draws from the joint distribution would look like in the \( (x, y) \) plane. Be sure to label all axes with actual numerical values. If your points are in some sense elliptically distributed, be sure to indicate numerical values for the center of the ellipse and its various dimensions. (A complete answer will include enough positions and dimensions to specify the ellipse completely.)

2. A value \( x \) is drawn with probability \( c \) from the distribution

\[
p_1(x) = 2x, \quad (0 \leq x \leq 1)
\]

and otherwise (with probability \( 1 - c \)) from the distribution

\[
p_2(x) = 2(1 - x), \quad (0 \leq x \leq 1)
\]

(a) Draw the two distributions, labeling all axes.
(b) What is the pdf for \( x \), given a value of \( c \). That is, what is \( P(x|c) \)?
(c) What is your choice of a non-informative prior for \( c \), that is, \( P(c) \)?

A single value of \( x \) is now drawn. It is 0.75. Call this the “data”.
(d) What is \( P(c|\text{data}) \)?
(e) What is the probability that the data point was drawn from $p_1$ and not $p_2$? (Give numerical answer.)

(f) What is the probability distribution for $x$ on the next draw. (Hint: the answer should not contain any $c$'s.)

3. For each part below, write at most two sentences, or one sentence and an equation or picture, that show that you know something about the indicated topic.

   (a) chi-square fitting a model $y(x|\mathbf{b})$ to obtain MLE estimates of the parameters $\mathbf{b}$.

   (b) rejection method for generating random deviates

   (c) false discovery rate (FDR)

   (d) Markov chain Monte Carlo (MCMC)

   (e) hypergeometric distribution
1. [35 points.] For the following questions give the name of the method, a brief explanation why (one or two sentences) and its work complexity.

(a) [7 points.] Given \( n \) vectors \( \{v_i\}_{i=1}^n \), which algorithm would you use to orthogonalize them? What is its work complexity as a function of \( n \)?

(b) [7 points.] Let \( A \in \mathbb{R}^{m \times n} \), \( m > n \) and \( \text{rank}(A) = m \). Given \( b \in \mathbb{R}^n \), which algorithm would you use to solve \( \min_{x \in \mathbb{R}^m} \|Ax - b\|_2^2 \)? What is its complexity as a function of \( m \) and \( n \)?

(c) [7 points.] Let \( A \in \mathbb{R}^{m \times m} \), \( \text{rank}(A) = m \). Given \( b \in \mathbb{R}^m \), which algorithm would you use to solve \( Ax = b \)? What is its complexity as a function of \( n \)?

(d) [7 points.] Let \( A \in \mathbb{R}^{m \times m} \), \( \text{rank}(A) = m \). In addition, \( A \) is symmetric and positive definite and the total number of the nonzero entries in \( A \) is \( 10m \). Given \( b \in \mathbb{R}^m \), which algorithm would you use to solve \( Ax = b \)? What is its complexity as a function of \( n \)?

(e) [7 points.] Let \( A \in \mathbb{R}^{m \times m} \). Assuming you have no other information about \( A \), what algorithm would you use to compute all the eigenvalues of \( A \)? What is its complexity as a function of \( n \)?

2. [30 points.] Suppose that you’re designing a new numerical algorithm and you want to test it on matrices \( A \) that have a specific form of SVD factorization. To test your algorithm, you need to generate random matrices \( A \) that have some specific properties.

(a) [15 points.] Give an algorithm to generate a random matrix \( A \in \mathbb{R}^{m \times n} \) in terms of its reduced SVD factorization with the following properties: \( m < n \), \( \text{rank}(A) = r \leq m \), the condition number of \( A \) is equal to 10, and the two-norm of \( A \) is equal to one.

(b) [10 points.] Given a vector \( x \), what is the expected relative error in computing \( y = Ax \) due to floating point errors?

(c) [5 points.] Now in addition to the above properties, assume you also want \( m \) to be a power of two and \( \text{rank}(A) = \log_2(m) + 1 \). You also want the absolute value of each component of all singular vectors that span the range space of \( A \) is equal to a constant. That is, let \( z \) be any one of those singular vectors. Then abs\((z(i))\) = constant, for all vectors and all \( i \).

3. [35 points.] Let \( v_1, v_2 \in \mathbb{R}^n \) be two linearly independent vectors. Give an algorithm for constructing an orthonormal basis \( \{w_i\}_{i=1}^n \) for \( \mathbb{R}^n \) so that \( \text{span}\{w_1, w_2\} = \text{span}\{v_1, v_2\} \). The complexity of your algorithm for computing any one of these basis vectors should be \( O(n) \).

Also, if \( W \in \mathbb{R}^{n \times n} \) is defined as \( W(:,i) = w_i \), give algorithms to compute \( Wz \) and \( W^Tz \) in \( O(n) \) cost for any vector \( z \in \mathbb{R}^n \). If you can not find algorithms with the above complexity constants, you will receive half credit if you state any algorithm (and its complexity) for the construction of \( \{w_i\}_{i=1}^n \).
1. Consider the following ODE initial value problem and the related class of numerical algorithms,

\[ y'(t) = f(y(t)), \quad t > 0, \quad y(0) = y_0, \]
\[ y_{n+1} = y_n + h(s f(y_n) + (1-s)f(y_{n+1})) \]
\[ 0 \leq s \leq 1, \quad t_n = nh, \quad n = 0, 1, ... \]

(a) Determine the order of the local truncation error as a function of s and check if the Dahlquist stability condition is satisfied.
(b) For which s-values is the algorithm A-stable.
(c) Show that for \( s = \frac{1}{2} \) and \( f = i\omega y \) with real \( \omega \), the algorithm is an exact approximation of an ODE with slightly different \( \omega \).
2. Given the parabolic equation below,

\[ u_t = (a(x)u_x)_x - u, \quad 0 < x < 1, \quad 0 < a(x) \leq A \]
\[ u(x,0) = u_0(x), \quad u(0,t) = 0, \quad u_x(1,t) = \alpha. \]

(a) Develop the general form of a finite element approximation and a discontinuous Galerkin approximation of this parabolic problem. Use trapezoidal rule (Crank-Nicolson) for time discretization.

(b) Show that the bilinear form in the finite element approximation of the related stationary problem \(-(a(x)u_x)_x + u = f(x), \quad u(0) = 0, \quad u_x(1) = \alpha\) is coercive and continuous.

(c) Show that the discontinuous Galerkin satisfies a maximum principle if \(P_0\) elements are used and if \(u_x(x)\) in the flux terms are approximated by \((u(x + \epsilon) - u(x - \epsilon)) / h, \epsilon \to 0\) for an element boundary point \(x\). Use explicit Euler in time and assume appropriate relation of the space and time steps.
3. (a) Develop a Lax-Friedrichs and an upwind finite difference approximation of the hyperbolic system below. (Simplifying the problem by replacing the matrix $A$ by the scalar $a > 0$ will give partial credit.)

$$u_t = A u_x + f(x)$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

(b) Determine the order of accuracy of the approximations based on their local truncation errors.
(b) Use von Neumann analysis to show that with periodic boundary conditions the Lax-Friedrichs scheme is $L^2$ stable.
Problem 1. There are two boxes. Each contains some number of balls. In one box (the “Good” box) 4/5 of the balls are white and 1/5 of the balls are black. In the other box (the “Bad” box), 1/10 of the balls are white and 9/10 are black. You choose one box at random and draw (with replacement) 2 balls from it.

a) What is the probability that both drawn balls are the same color?

b) Now suppose the two drawn balls are both black. What is the probability that you have selected the Good box?

c) Now suppose you draw a third ball from the same box. What is the probability that it is white?

Problem 2. You are given the values of $N$ i.i.d. draws $x_i$, $i = 1, 2, \ldots, N$ from a Normal distribution of mean zero and standard deviation $\sigma$ of unknown magnitude.

a) What is the probability (or probability density) of the data as a function of the $x_i$’s given some value of $\sigma$?

b) What is a reasonable Bayes prior $P(\sigma)$ for $\sigma$?

c) What is the maximum a posteriori estimate (MAP) for $\sigma$ in terms of the $x_i$’s? Hint: I can save you some algebra by telling you that the function $f(x) = x^k e^{-ax}$ has a single maximum at $x = k/\alpha$.

d) Would the maximum likelihood estimate (MLE) be the same as your answer for part c)? Why or why not?

e) If you chose a uniform prior in parts b) and c), how would your answer be different if you had chosen a log-uniform prior. Conversely, if you chose a log-uniform prior, how would your answer be different for a uniform prior? (If you chose some other prior, my advice is to redo parts b) and c)!

Problem 3.

a) If $y = y(x)$, write down the law of transformation of probabilities that relates the p.d.f.s $p_Y(y)$ and $p_X(x)$. If you can’t do this, you are hosed on this problem.
b) What is the p.d.f. of a random variable $Y$ that is the square of another random variable $X \sim \text{Exponential}(\lambda)$ (which has, I'm sure you know, the p.d.f. 
\[ p_X(x) = \lambda e^{-\lambda x}, \quad 0 \leq x \leq \infty \])? 

c) Sketch the p.d.f.s of $x$ and $y$ on the same graph. Explain how they are qualitatively different near the origin $x = y = 0$. 

**Problem 4.** For each part below, write at most three sentences, or two sentences and an equation or picture, that show that you know something about the indicated topic. 

a) Normal approximation to the binomial distribution 
b) covariance matrix 
c) multiple hypothesis correction 
d) chi-square statistic 
e) longitudinal study
1. Consider the Singular Value Decomposition (SVD) \( A = U\Sigma V^T \) where
   
   - \( A \in \mathbb{R}^{m \times n} \);
   - \( \Sigma \in \mathbb{R}^{m \times n} \);
   - \( U \in \mathbb{R}^{m \times m} \) with \( U^TU = I \); and
   - \( V \in \mathbb{R}^{n \times n} \) with \( V^TV = I \).

   Partition
   
   \[
   U = \begin{pmatrix} U_L & U_R \end{pmatrix}, \quad V = \begin{pmatrix} V_L & V_R \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{TL} & \text{0} \\ 0 & 0 \end{pmatrix}
   \]

   so that \( \Sigma_{TL} \in \mathbb{R}^{r \times r} \) has no zero elements on its diagonal and \( A = U_L\Sigma_{TL}V_L^T \).

   (a) Prove that the row space of \( A \) equals the column space of \( V_L \).

   (b) Derive the formula for the vector minimizes the linear least-squares problem

   \[
   \min_x \|Ax - b\|_2
   \]

   and has minimal length.

   (Or state the solution and then prove that it meets the requirements. If you can’t do this for the general case, assume \( r = n \).) Be very careful in your explanation: if you state something that is not (always) true, you lose points!

2. Consider \( A, B, Q \in \mathbb{R}^{n \times n} \) where

   \[ B = QAQ^T \]

   with \( Q^TQ = I \) and let \( B \) be upper Hessenberg.

   Prove that given matrix \( A \) and the first column of \( Q \) the remaining columns of \( Q \) and \( B \) are uniquely determined (modulo the sign of the subdiagonal elements of \( B \).) This is known as the implicit \( Q \) theorem.

3. Consider a matrix \( A \in \mathbb{R}^{m \times n} \) with \( n \leq m \).

   (a) Describe a method for determining whether \( A \) has linearly independent columns that uses Gauss transforms (or, equivalently, some variant of LU factorization). For this part, assume you compute in exact arithmetic.

   (b) Describe a method for determining whether \( A \) has linearly independent columns that uses orthogonalization of the columns of \( A \). For this part, assume you compute in exact arithmetic.

   (c) Describe a method for determining whether \( A \) has linearly independent columns that is appropriate when you are using floating point arithmetic. Discuss pros and cons of the method that you choose. (Notice that different people may choose different methods. What is important is that you discuss the pros and cons for the method you choose.)
1. Analyze the two numerical methods below for the ordinary differential equation,

\[
\begin{align*}
\frac{dy}{dt} &= f(y) , \quad 0 < t < T, \\
y(0) &= y_0, \\
y_{n+2} - y_n &= 2hf(y_{n+1}), \quad n = 0,1,..., \quad t_n = nh, \\
(y_0 &= y_0 + hf(y_0)), \\
y_{n+1} - y_n &= h\left(f(y_n) + f(y_{n+1})\right)/2 \quad n = 0,1,..., \quad t_n = nh.
\end{align*}
\]

(a) Describe how the methods can be combined into a predictor corrector scheme.

(b) How would adaptive step-size control work in this predictor corrector setting?

(c) Investigate order of accuracy, stability (in the sense of Dahlquist) and A-stability of the two methods.
2. Given the Stokes equation for incompressible flow with Dirichlet boundary conditions,

\[
\begin{align*}
-\mu \Delta u + \nabla p &= f(x), \quad x \in \Omega \subset \mathbb{R}^2, \\
\nabla \cdot u &= 0, \\
\n&\quad x \in \partial \Omega.
\end{align*}
\]

(a) Derive a weak formulation of this differential equation with divergence free function spaces for \( u \) and the test function, which can be used in a FEM approximation.

(b) Prove that the relevant bilinear and linear forms in the weak formulation are continuous and that the bilinear form is also coercive.

(b) Derive a weak mixed formulation without the divergence free condition on the function spaces and discuss potential choices of finite element basis functions.
3. Consider the following difference approximation,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{b\Delta t}{\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \Delta t f(x_j),$$

$b > 0$, periodic boundary conditions,

$x_j = j\Delta x, j = 0, 1, \ldots, J, \ t_n = n\Delta t, n = 0, 1, \ldots$

(a) Which differential equation does the method approximate as $\Delta x, \Delta t \to 0$?

(b) Apply Von Neumann analysis to investigate stability and give conditions on for $\Delta x, \Delta t$ for $L^2$ stability.

(c) Prove a maximum principle for the scheme if $a = f = 0$. 
1 Exercise 1

A matrix A satisfies the restricted isometry property, RIP, if for any s-sparse x there exists a $\delta_s$ between (0, 1) such that with high probability

$$(1 - \delta_s)|x|^2 \leq |Ax|^2 \leq (1 + \delta_s)|x|^2$$

Let $A$ be a $n \times n$ Gaussian matrix where each element is a random Gaussian variable with zero mean and variance $\frac{1}{n}$. Prove with sufficient detail, that $A$ has the restricted isometry property.

2 Exercise 2

Suppose that the $X_i$ are independent random variables satisfying $X_i \leq M$ for $1 \leq i \leq n$. Let $X = \sum_{i=1}^{n} X_i$ and $\|X\| = \sqrt{\sum_{i=1}^{n} (E(X_i)^2)}$. Then prove that $\text{Probability}(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2\|X\|^2 + M^2}}$. 
