CAM AREA A QUALIFYING EXAM
May 25, 2000, 9:00 a.m.–12:00 noon

Work any 6 of the following 8 problems.

1. For an infinitesimally thin spherical shell $S$ of radius $R$ centered at the origin with uniform surface density $\rho$, compute the moment of inertia with respect to the $z$ axis. That is, compute the surface integral

$$I = I_z = \rho \int_S (x^2 + y^2) \, d\sigma.$$ 

2. Consider the two-dimensional wave equation describing vibrations of a membrane

$$\alpha^2 \nabla^2 w = w_{tt},$$

where $w$ is the deflection of the membrane and $\alpha^2 > 0$ is the tension to density ratio. The membrane is fixed on its boundary.

   (a) Use separation of variables $w = \phi(x, y)T(t)$ to arrive at the equations

$$T'' + k^2 \alpha^2 T = 0 \quad \text{and} \quad \nabla^2 \phi + k^2 \phi = 0.$$ 

   (b) Determine the eigenfrequencies and their corresponding mode shapes for a square membrane $0 < x < a$, $0 < y < a$.

3. Let $f$ be a Lebesgue measurable function from $(0, 1)$ into $[0, \infty]$. Prove that $f$ is Lebesgue integrable if, and only if, the series

$$\sum_{n=0}^{\infty} 2^n \, m(\{x : f(x) > 2^n\})$$

converges, where $m$ denotes the Lebesgue measure.

4. Consider the two-point boundary value problem

$$-(au')' + bu' = f, \quad \text{in } (0, 1),$$

$$u(0) = 0,$$

$$au'(1) + cu(1) = g.$$ 

   (a) Derive a variational formulation of the problem over an appropriate Hilbert space $H \subset H^1(0, 1)$.

   (b) Derive a “least restrictive” condition on the constants $a$, $b$, and $c$ to guarantee coercivity of the bilinear form.

   (c) Apply the Lax-Milgram Lemma to prove that the system is well posed.

5. Let $k \geq 0$ be an integer, let $P_k$ denote the set of polynomials of degree less than or equal to $k$ on $\mathbb{R}^d$, let $N$ denote the dimension of $P_k$, and let $\{f_i : i = 1, \ldots, N\}$ be a basis for the dual of $P_k$. Let $\Omega \subset \mathbb{R}^d$ be connected.
(a) Show that we can extend each $f_i$ to $H^{k+1}(\Omega)$. State carefully the general theorem that is used.
(b) If the seminorm
$$|v|_{H^{k+1}(\Omega)} = \left\{ \sum_{|\alpha|=k+1} \int_\Omega |D^\alpha v|^2 \, dx \right\}^{1/2}$$
and each $f_i(v)$ vanish for some $v \in H^{k+1}$, what can we conclude about $v$?
(c) Show that there is some constant $C$ such that for all $v \in H^{k+1}$,
$$\|v\|_{H^{k+1}(\Omega)} \leq C \left( |v|_{H^{k+1}(\Omega)} + \sum_{i=1}^N |f_i(v)| \right).$$

   (a) State the spectral theorem for $T$.
   (b) Show that if $H$ is infinite dimensional and $T^{-1}$ exists, then $T^{-1}$ is unbounded. What does this say about the Laplace operator $\nabla^2 : H \to H$, where $H = H^2(\Omega) \cap H_0^1(\Omega)$?
   (c) Prove that $T$ is also a positive operator if, and only if, the spectrum of $T$ lies in $[0, \infty)$.

7. Consider the functional
$$F(y) = \int_0^1 \left[ (y(x))^2 - y(x) y'(x) \right] \, dx,$$
defined for $y \in C^1([0, 1])$.
   (a) Write down the Euler-Lagrange equations and solve them to find all extremals.
   (b) If we require $y(0) = 0$, show by example that there is no minimum.
   (c) If we require $y(0) = y(1) = 0$, show that the extremal is a minimum. Hint: note that $y y' = (\frac{1}{2} y^2)'$.

8. Consider
$$y + \tanh y = x.$$
   (a) Show that for any $x \in \mathbb{R}$, there is a unique solution $y(x)$ to the equation.
   (b) Show that as $x \to \infty$,
$$y(x) = x - 1 + o(1).$$
1. Spherical coordinates: \((r, \psi, \Theta)\)

\[
\begin{align*}
\mathbf{r} & = \begin{cases} 
  x = r \sin \psi \cos \Theta \\
  y = r \sin \psi \sin \Theta \\
  z = r \cos \psi
\end{cases} \\
\frac{\partial \mathbf{r}}{\partial \psi} & = (r \cos \psi \cos \Theta, r \cos \psi \sin \Theta, -r \sin \psi ) \\
\frac{\partial \mathbf{r}}{\partial \Theta} & = (-r \sin \psi \sin \Theta, r \sin \psi \cos \Theta, 0 )
\end{align*}
\]

| \frac{\partial \mathbf{r}}{\partial \psi} \times \frac{\partial \mathbf{r}}{\partial \Theta} | = \left( r^4 \sin^4 \psi + r^4 \cos^2 \psi \sin^2 \psi \right)^{\frac{1}{2}} \\
= r^2 \sin \psi \\
x^2 + y^2 = r^2 \sin^2 \psi
\]

\[
\int_S (x^2 + y^2) \, dS = \int_S \int_{R^2 \sin^2 \psi} R^4 \sin \psi \, d\psi \, d\Theta
\]

\[
= 2\pi \int_0^\pi \sin^3 \psi \, d\Theta
\]

\[
= 2\pi R^4 \left[ \left. \frac{\cos^3 \psi}{3} \right|_0^\pi \right]
\]

\[
= 2\pi R^4 \left[ \left( -\frac{1}{3} + 1 \right) - \left( \frac{1}{3} - 1 \right) \right] = \frac{8}{3} \pi R^4 \rho
\]
\( w = \varphi(x, y) \ T(t) \)

\( \varphi_{tt} = \varphi \ T'' \)

\( \Delta w = \Delta \varphi \cdot T \)

\[ \varphi_{tt} - \alpha^2 \Delta w = \varphi T'' - \alpha^2 \Delta \varphi \ T = 0 \]

\[ \varphi T'' = \alpha^2 \Delta \varphi \ T \]

\[ \frac{T''}{\alpha^2 T} = \frac{\Delta \varphi}{\varphi} = \lambda \quad (\text{separation constant}) \]

Since operator \(-\Delta\) with Dirichlet BC is self-adjoint and positive definite, we can conclude that \(\lambda < 0\). Set \(\lambda = -k^2\), \(k > 0\). We get

\[-\Delta \varphi = -\lambda \varphi = k^2 \varphi \]

\[ T'' + k^2 \alpha^2 T = 0 \]

Eigenpairs for a square membrane

\[ \varphi(x, y) = \chi(x) \ \chi(y) \]

\[-\Delta \varphi = k^2 \varphi \quad \Rightarrow \quad -\chi'' \chi - \chi \chi'' = k^2 \chi \chi \]

\[-\frac{\chi''}{\chi} = \frac{\chi''}{\chi} + k^2 = \mu \]
Again, operate $-X''$ with Dirichlet BC is self-adjoint and positive definite, so $\mu = N^2$, $N > 0$. We get

$$-X'' = \mu X = N^2 X$$

$$X'' + N^2 X = 0$$

\[ X(x) = A \cos Nx + B \sin Nx \]

\[ X(0) = 0 \Rightarrow A = 0 \]

\[ X(l) = 0 \Rightarrow N \cdot l = n \pi \]

\[ \therefore N = N_n = n \pi, \quad n = 1, 2, \ldots \]

\[ X = X_n = B \sin n \pi x \]

So:

\[ \frac{X''}{X} = \mu - k^2 = N^2 - k^2 = n^2 \pi^2 - k^2 \]

\[ -X'' = \left( k^2 - n^2 \pi^2 \right) X \]

By the same reasoning, $k^2 - n^2 \pi^2 = m^2 \pi^2$, so $k^2 = k_{(n,m)}^2 = (n^2 + m^2) \pi^2$ and

\[ \phi_{(n,m)}(x, y) = C_{nm} \sin n \pi x \sin m \pi y \]
3. **Necessity**

Denote $A_n = \{ x : 2^n < f(x) \leq 2^{n+1} \}$  
\[ n = 0, 1, \ldots \]

\[ A_{n-1} = \{ x : 0 \leq f(x) \leq 1 \} \]

Thus

\[ 0 \cdot m(A_{n-1}) + \sum_{n=0}^{\infty} 2^n m(A_n) < \int f < \infty \]

This implies that

\[ 2 \cdot \sum_{n=0}^{\infty} 2^n m(A_n) = \sum_{n=0}^{\infty} 2^{n+1} m(A_n) < \infty \]

Now

\[ 2^n m(\{ x : f(x) > 2^n \}) \leq \sum_{K=n}^{\infty} 2^K m(A_K) \]

So

\[ \sum_{n=0}^{\infty} 2^n m(\{ x : f(x) > 2^n \}) \leq \sum_{n=0}^{\infty} \sum_{K=n}^{\infty} 2^K m(A_K) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{K=n}^{\infty} 2^K \right) m(A_K) \]

\[ \leq \sum_{n=0}^{\infty} \frac{1}{2^{k+1}} 2^{k+1} m(A_k) < \infty \]
Sufficiency

\[ \int f \leq 1 \cdot m(A_{-1}) + \sum_{n=0}^{\infty} 2^n m(A_n) \]

\[ \leq 1 + 2 \sum_{n=0}^{\infty} 2^n m(\{x : f(x) > 2^n \}) \]

\[ < \infty. \]
4.

\[-(au')' + bu' = f', \quad \int_{0}^{1} f' \]

\[-\int_{0}^{1} (au')'v + bu'v = \int_{0}^{1} f v\]

\[-\int_{0}^{1} (au')'v = \int_{0}^{1} au'v - au'v = -a u'(1)v(1)\]

Set \(v(0) = 0\)

\[-au'(1) = cu(1) - g \implies \]

\[-\int_{0}^{1} (au'v' + bu'v) dx + cu(1)v(1) = \int_{0}^{1} f v dx + g v(1)\]

Setting \(H = \{ v \in H'(0,1) : v(0) = 0 \}\), we have

\[
\begin{cases}
  u \in H \\
  \int_{0}^{1} (au'v' + bu'v) dx + c u(1)v(1) = \int_{0}^{1} f v dx + g v(1)
\end{cases}
\]

\(\forall v \in H\)
\[ \int_0^1 (au'^2 + bu'v) + cu^2(1) \]
\[ = \frac{1}{2} \int_0^1 (u^2)' = \frac{1}{2} b u^2(1) \]
\[ = \frac{1}{2} b u(1)^2 \]

So:
\[ \int_0^1 (au'^2 + bu'v) + cu^2(1) \]
\[ = \int_0^1 a(u'^2) + (\frac{1}{2} b + c) u^2(1) \]

Thus, sufficient conditions for coercivity might be:
\[ a > 0, \quad \frac{1}{2} b + c \geq 0 \]

Then:
\[ \int_0^1 a(u')^2 + (\frac{1}{2} b + c) u^2(1) \geq a \int_0^1 u'^2 \]

Remark: Due to the BC at \( x = 0 \), first order norms and seminorms on \( H \) are equivalent.

Use Cauchy - Schwarz inequality and continuous embedding of \( H \) into \( C([0,1]) \) to show coercivity of both linear and bilinear forms.
5. (a) We extend \( f_i \) to \( \mathcal{H}^{k+1}(\Omega) \supset \mathcal{P}_k \) by the nontrivial Hahn-Banach Theorem: Let \( Z \) be a subspace of a normed linear space \( X \) and \( f \) a continuous linear functional on \( Z \). Then there exists a continuous linear functional \( f^* \) on \( X \) which extends \( f \) (i.e., \( f^*|_Z = f \)) such that

\[
\|f^*\|_{X^*} = \|f\|_{Z^*}.
\]

(b) If \( |v|_{\mathcal{H}^{k+1}} = 0 \), then

\[
D^k v = 0 \quad \forall \quad |\alpha| = k+1
\]

\[\Rightarrow \quad v \in \mathcal{P}_k\]

If each \( f_i(v) = 0 \), then \( v \equiv 0 \).

(c) Proof by contradiction. Assume \( \exists \{v_n\} \subseteq \mathcal{H}^{k+1}(\Omega) \) such that

\[
\|v_n\|_{\mathcal{H}^{k+1}(\Omega)} \leq 1
\]

\[
|v_n|_{\mathcal{H}^{k+1}(\Omega)} + \sum_{i=1}^{N} |f_i(v_n)| \to 0.
\]

By the Rellich Theorem, \( \mathcal{H}^k \subset \subset \mathcal{H}^{k+1} \), so for some subsequence, again denoted as \( \{v_n\} \),

\[
v_n \rightharpoonup v \quad \text{in} \quad \mathcal{H}^k.
\]
But $|v_n|_{H^{k+1}} \to 0$, so $\sum_{n=1}^{\infty} v_n \in H^{k+1}$ is Cauchy, and $v_n \to v$

So $|v|_{H^{k+1}} = 0$. Moreover

$0 = \lim_{n \to \infty} f_i(v_n) = f_i(v)$

So $v = 0$. However, $\|v_n\|_{H^{k+1}} \to \|v\|_{H^{k+1}}$ is now contradicted, as $1 \not\to 0$. 
6. (a) Let $T$ be a self-adjoint, compact linear operator on a Hilbert space $H$. The spectrum of $T$, $\sigma(T)$, consists of a countable (or finite) number of real values $\lambda$ that may accumulate only at 0. The residual spectrum is empty, and, in fact if $\lambda \in \sigma(T)$ is nonzero, then $T - \lambda I$ is an eigenvalue with a finite dimensional eigenspace. Moreover, we can find a complete orthonormal basis $\{\chi_i\}$ for $H$ consisting of eigenvectors of $T$. In this basis, if

$$x = \sum \alpha_i \chi_i = \sum (x, \chi_i) \chi_i$$

then

$$Tx = \sum \lambda_i \chi_i$$

where $T\chi_i = \lambda_i \chi_i$ are paired.

(b) $H$ is infinite dimensional, so $0 \in \sigma(T)$. Since the residual spectrum is empty, $0$ is an eigenvalue (and $T^{-1}$ does not exist) or $T^{-1}$ is unbounded. The latter must hold since $T^\dagger$ exists.

The Laplace operator $\Delta^2 : H \to H$ is not defined on all of $H$: however $D(H) = H^2 \cap H_0$ is dense in $H$. Now $\Delta^2$ is symmetric, and $(\Delta^2)^{-1}$ is compact so we expect $\Delta^2$ to be unbounded; there is no $C > 0$ s.t. $\|D^2 u\|_H \leq C \|u\|_H$ (which is clear).
(\Rightarrow) \quad \text{Since } \sigma(T) \subseteq \mathbb{R}, \text{ consider any } \gamma \in \sigma(T) \text{ st. } \gamma_i \neq 0. \text{ Then } 0 \leq (Tx_i, x_i) = (\gamma_i x_i, x_i) = \gamma_i \|x_i\|^2 \Rightarrow \gamma_i \geq 0.

(\Leftarrow) \quad \text{If } \pi \in \mathcal{H}, \text{ let } \pi = \sum \alpha_i x_i \text{ then } \langle T\pi, \pi \rangle = \langle \sum \alpha_i x_i, \sum \alpha_i x_i \rangle = \sum i \alpha_i \|x_i\|^2 \text{ by orthornormality } \geq 0
7. (a) \[ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \]

where \( f(x, y, y') = y^2 - x y' \)

\[ \Rightarrow \quad 2y - y' = (-y')' = -y' \]

\[ \Rightarrow \quad 2y = 0 \quad \Rightarrow \quad y = 0 \]

(b) \( y(0) = 0 \)

\( F(0) = 0 \), so want \( y(x) \) st. \( F(y) < 0 \)

Try \( y(x) = x \Rightarrow y' = 1 \)

\( F(x) = \int_0^1 (x^2 - x) = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} < 0 \).

(c) \( y(0) = y(1) = 0 \)

\[ F(y) = \int_0^1 \left[ y^2 - \left( \frac{1}{2} y^2 \right) \right] \, dx \]

\[ = \int_0^1 y^2 - \frac{1}{2} y^2 \, \bigg|_0^1 = \int_0^1 y^2 \, dx \geq 0. \]

So \( F(0) = 0 \) is a min.
8. (a) Let \( g(y) = y + \tanh y \)

Note:
\[
\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}
\]

So,
\[
\tanh(-y) = -\tanh y \quad \text{and} \quad |\tanh y| < 1
\]

Now, \( g(y) \to \infty \) as \( y \to \infty \),
\[
g(0) = 0, \quad g'(y) = 1 - (1 - \tanh^2 y) = \tanh^2 y > 0
\]
for \( y \neq 0 \).

Thus, unique \( y \to \infty \) and so \( y = g(x) \) solving \( g(y) = x \).

(b) As \( x \to \infty \),
\[
y = x - \tanh y \geq x - 1 \to \infty
\]

Now,
\[
\tanh y = 1 - 2 \frac{e^{-y}}{e^y + e^{-y}} = 1 + o(1)
\]
as \( y \to \infty \).

(i.e., \( x \to \infty \),
\[
y = x - 1 + o(1)
\]
as \( x \to \infty \).
CAM AREA A QUALIFYING EXAM
May 30, 2001, 9:00 a.m.–12:00 noon

Work any 6 of the following 7 problems.

1. Let $V$ denote a cone given in cylindrical coordinates,

$$V = \{(r, \theta, z) : 0 < z < a, \ 0 < r < z\},$$

where $a > 0$. Let $u$ be a velocity field given in the cylindrical coordinates:

$$u = u_r e_r + u_\theta e_\theta + u_z e_z,$$

where $u_r = 1$, $u_\theta = 0$, and $u_z = 1$. Compute the net flow through the surface of the cone

$$\int_{\partial V} u \cdot n \, dS$$

($n$ is the outward normal unit vector to the cone surfaces $\partial V$).

2. Consider the sequence of polynomial spaces in $\mathbb{R}^2$ and the corresponding linear operators,

$$\mathbb{R} \xrightarrow{i} \mathcal{P}^{p+1} \xrightarrow{\nabla} \mathcal{P}^p \xrightarrow{\nabla \times} \mathcal{P}^{p-1} \xrightarrow{0} \{0\}.$$

Here $\mathcal{P}^p$ denotes scalar-valued polynomials of order $p$, $\mathcal{P}^p$ denotes vector-valued polynomials of order $p$, and $\nabla \times E = \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}$. Demonstrate that this is an exact sequence, i.e., that the null space of each of the operators (starting with gradient $\nabla$) coincides with the range of the previous operator in the sequence. Hint: Recall the fundamental relation between the nullity and rank of a linear operator defined on a finite-dimensional space.

3. Let $X$ and $Y$ be two Banach spaces, and $D$ a dense subset of $X$. Prove that every continuous linear operator $A$ from $D$ into $Y$ has a unique continuous and linear extension $\hat{A}$ taking $X$ into $Y$. Show that the norms of $A$ and $\hat{A}$ are identical.

   (a) State the Closed Graph Theorem.
   (b) Let $H$ be a Hilbert space, and $A : H \to H$ a linear operator. If for all $x, y \in H$

   $$(x, Ay)_H = (Ax, y)_H,$$

   prove that $A$ is bounded.

5. Find the infimum of

$$\frac{\int_{-1}^{1} (u'(x))^2 \, dx}{\int_{-1}^{1} (u(x))^2 \, dx}$$

among all non-zero functions in $H^1_0(-1, 1)$. 
6. Show that $H^1(-\pi, \pi)$ is compactly contained in $L^2(-\pi, \pi)$. Hint: Use Fourier series and reduce the problem to the finite dimensional case (finite number of coefficients) where there is compactness.

7. Use the Contraction Mapping Theorem to prove local existence and uniqueness for the initial-value problem

\[
\begin{cases}
q(0) = 1, \\
\frac{dq}{dt} = q^2 + t, \quad t \in (0, T).
\end{cases}
\]

Give a lower bound for $T$, the length of the time interval for which the solution is guaranteed to exist.
1. By the Gauss' Theorem:

\[ \int_{\partial V} \mathbf{u} \cdot d\mathbf{s} = \int_{V} \text{div} \mathbf{u} \, dV \]

In cylindrical coordinates \(\varphi\):

\[ \text{div} \mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial y_i} \mathbf{e}_i \]

Cylindrical coordinates \((r, \theta, z)\):

\[ \mathbf{e}_r = (\cos \theta, \sin \theta, 0) \]

\[ \mathbf{e}_\theta = \frac{\partial}{\partial \theta} = (-\sin \theta, \cos \theta, 0) \]

\[ \mathbf{e}_z = \mathbf{e}_z = (0, 0, 1) \]

So

\[ \text{div} \mathbf{u} = \frac{\partial}{\partial r} \left( u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \right) \cdot \mathbf{e}_r \\
+ \frac{2}{r} \mathbf{e}_\theta \left( \begin{array}{c} -1 \\
\cos \theta, \sin \theta, 0 \end{array} \right) \cdot \mathbf{e}_\theta \\
+ \frac{1}{r} \mathbf{e}_z \left( \begin{array}{c} \cos \theta, \sin \theta, 0 \end{array} \right) \cdot \mathbf{e}_z \]

\[ = \frac{1}{r} \left( \cos \theta, \sin \theta, 0 \right) \cdot \mathbf{e}_\theta = \frac{1}{r} \]
Parameterization of the cone:

\[ 0 < z < a \]
\[ 0 < \theta < 2\pi \]
\[ 0 < r < z \]

\[
2\pi \int_0^a \left( \int_0^z \frac{1}{r} \, dr \right) \, dz = 2\pi \int_0^a \frac{1}{2} \, dz = \pi a^2
\]
2. Facts:

1. \( \mathcal{N}(\nabla) = \text{constants, trivial} \)

2. \( \nabla = \nabla q \iff \nabla \times \nabla = 0 \)

   is true in any simply-connected domain, including \( \mathbb{R} \)

   Differentiation lowers the order of polynomials, so the operators are well-defined.

3. We need to show that \( \nabla x \) is a surjection.

   One way to show it is to compare the dimensions of the spaces:

   \[
   \dim \mathcal{R}(\nabla x) = \dim \mathcal{P}^b - \dim \mathcal{N}(\nabla x)
   = \dim \mathcal{P}^b - \dim \mathcal{R}(\nabla)
   = \dim \mathcal{P}^b - (\dim \mathcal{P}^{b+1} - \dim \mathcal{N}(\nabla))
   = (p+1)(p+2) - \frac{p+2}{2} \left( \frac{p+3}{2} \right) + 1
   = \frac{p+2}{2} \left( 2p+2 - p+3 \right) + 1
   = \frac{(p+2)(p-1) + 2}{2} = \frac{p^2 + p}{2} = \frac{p(p+1)}{2}
   \]

   \( \dim \mathcal{P}^{p-1} = \frac{p(p+1)}{2} \) \( \text{OK!} \)
3. For $x \in X$, let $\{x_n\}_{n=1}^{\infty} \subset D$ be such that $x_n \to x$. Now

$$\|A x_n - A x_m\|_Y = \|A(x_n-x_m)\|_Y \leq \|A\|\|x_n-x_m\|_Y,$$

so we conclude $\{A x_n\}_{n=1}^{\infty}$ is Cauchy in $Y$. Thus $\exists y \in Y$ s.t.

$$\lim_{n \to \infty} A x_n = y$$

Let $\hat{A}x = y$. We need to check that $\hat{A}$ is well-defined, i.e., independent of the sequence chosen. Let $\{x_m\}_{m=1}^{\infty} \subset D$.

Then, since norms are continuous,

$$\|A x_m - y\|_Y = \lim_{n \to \infty} \|A x_m - A x_n\|$$

$$\leq \lim_{n \to \infty} \|A\| \|x_m - x_n\| = \|A\| \|x_m - x_n\|$$

$$\to 0$$ as $m \to \infty$.

Thus $\hat{A}$ is well-defined, and $\hat{A}/D = A$. Trivially $\hat{A} : X \to Y$ is linear: For $x, \xi \in X$ with $x_n \to x, \xi_n \to \xi$, $x_n, \xi_n \in D$ and $\alpha, \beta \in \mathbb{R}$

$$\hat{A}(\alpha x + \beta \xi) = \lim_{n \to \infty} A(\alpha x_n + \beta \xi_n)$$

$$= \alpha \lim_{n \to \infty} A x_n + \beta \lim_{n \to \infty} A \xi_n$$

$$= \alpha \hat{A}x + \beta \hat{A}\xi.$$

Now

$$\|\hat{A}\| = \sup_{\|x\|=1} \|\hat{A}x\| \geq \sup_{\|x\|=1} \|Ax\| = \|A\|,$$
and for \( x \in X \), \( \|x\|=1 \), and \( c > 0 \), there exists \( \bar{x} \in D \) such that
\[
\|\hat{A}x - A\bar{x}\| < \varepsilon \quad \text{and} \quad \|x - \bar{x}\| < \varepsilon
\]

Thus
\[
\|\hat{A}\| \leq \|A\bar{x}\| + \varepsilon \leq \|A\| \|\bar{x}\| + \varepsilon \leq \|A\| (1 + \varepsilon) + \varepsilon
\]
and
\[
\|\hat{A}\| = \sup_{\|x\|=1} \|\hat{A}x\| \leq \|A\| (1 + \varepsilon) + \varepsilon
\]

Thus \( \|\hat{A}\| = \|A\| \) and \( \hat{A} \) is continuous. Finally \( \hat{A} \) is unique since if \( B : X \to Y \) satisfies the properties,
\[
\|\hat{A}x - Bx\| = \lim_{n \to \infty} \|A\alpha_n - B\alpha_n\|
\]
\[
= \lim_{n \to \infty} \|A\alpha_n - A\alpha_n\| = 0,
\]
so \( B = \hat{A} \).
4. (a) Let \( X \) and \( Y \) be Banach spaces and \( A : X \rightarrow Y \) a linear operator. Then

\[ A \text{ is continuous } \iff A \text{ is closed.} \]

(b) By (a), we need only show \( A \) is closed. That is, if \( \lim_{n \to \infty} x_n = x \) and \( A x_n \to y \in H \), then we must show \( y = A x \).

But for \( z \in H \),

\[ ( y, z )_H = \lim_{n \to \infty} ( A x_n, z )_H = \lim_{n \to \infty} ( x_n, A z )_H = ( x, A z )_H = ( A x, z )_H, \]

so \( A x = y \).
5. Let
\[ q = \inf_{u \in H_0^1} \frac{\int (u')^2}{\int u^2} = \inf_{u \in H_0^1} \frac{\int (u')^2}{\|u\|_{L^2}^2} \]

Thus we solve the constrained minimization problem. Let
\[ H(u; \lambda) = \int [(u')^2 - \lambda (u^2 - 1)] \, dx \equiv \int h(u) \]
for the Lagrange multiplier \( \lambda \). The Euler-Lagrange equations lead to
\[ \frac{d}{du} \left( \frac{d}{du} \right) (2u') = 2u'' - 2u = (2u')' = 2u'' \]

\[ \Rightarrow \quad u'' = \lambda u \]

The e-values of \(-\Delta\) are \( \geq 0 \), so \( \lambda \geq 0 \) and, with \( \mu = \frac{\pi}{2} \)
\[ u(x) = a \sin(\mu x) + b \cos(\mu x) \]

Now \( u \in H_0^1(-1, 1) \), so
\[ 0 = u(\pm 1) = \pm a \sin \mu + b \cos \mu \]

\[ \Rightarrow \quad b \cos \mu = 0 \quad \text{and} \quad a \sin \mu = 0. \]

So \( b = 0 \) or \( \mu = \frac{\pi}{2} + n\pi, \ n = 0, 1, 2, \ldots \)
and \( a = 0 \) or \( \mu = m\pi, \ m \geq 1, 3, \ldots \)

Now note that
\[ -u'' = \lambda u \iff \quad (u')^2 = 2uv \quad \forall v \in H_0^1 \]

\[ \Rightarrow \quad (u')^2 = 2uv \Rightarrow \quad \rho = \inf q = \inf \mu^2 = \left[ \frac{\pi^2}{4} \right] \]
Problem 6:

1. Since \( H^1(-\pi, \pi) = \{ f \in L^2(-\pi, \pi), Df \in L^2(-\pi, \pi) \} \), then \( H^1(-\pi, \pi) \subset L^2(-\pi, \pi) \) by def.

2. To see that \( H^1(-\pi, \pi) \) is compactly contained in \( L^2(-\pi, \pi) \) it is enough to show that any bounded set \( K \) on \( H^1(-\pi, \pi) \) is precompact in \( L^2(-\pi, \pi) \). That is, \( K \) is compact in \( L^2 \).

Since \( L^2 \) and \( H^1 \) on \( (-\pi, \pi) \) are metric spaces then \( K \) is compact iff \( K \) is sequentially compact: every sequence in \( K \) has a convergent subsequence in \( K \) in \( L^2(-\pi, \pi) \).

Let \( \{ f_n \} \) be a sequence in \( K \), a bounded set in \( H^1(-\pi, \pi) \). Using the identification of the Thomson and Fourier coeff.

there exists a constant \( C \), depending on \( K \), such that

\[
\| f_n \|_{H^1(-\pi, \pi)} = \sum_{n \geq 0} n^2 (a_n^2 + b_n^2) \leq C \| K \|
\]

then, given an \( \epsilon > 0 \), there exists an \( N_0 \) such that

\[
n^2 (a_n^2 + b_n^2) \leq \frac{\epsilon}{2k}
\]

with \( k = \sum_{n \geq N_0} \frac{1}{n^2} \), a finite positive number.
so \[ a_n^2 + b_n^2 \leq \frac{\varepsilon}{2k} \frac{1}{n^2} \quad \forall n \geq 1. \]

Now, we take \( S = \text{Span} \{ \sin nx, \cos nx, 0 \leq n < \infty \} \), which is a finite dimensional subspace of \( L^2(-\pi, \pi) \).

If any bounded sequence of \( S \) has a bounded convergent subsequence, then take \( \{ g_n^2 \} = \{ \sum_{n=0}^{N_0} a_n \sin nx + b_n \cos nx \} \).

Then \( g_n \in S \) and, because of the above estimate, we obtain that \[ \| g_n \|_{L^2(-\pi, \pi)} \]

\[ \| g_n \|_{L^2(-\pi, \pi)} \leq \sum_{n=0}^{N_0} a_n^2 + b_n^2 \leq N_0^2 \| f \|_{L^2(-\pi, \pi)}. \]

is bounded in \( S \). Therefore, there exists a convergent subsequence \( g_{n_k} \rightarrow g \) in \( L^2(-\pi, \pi) \), s.t.

\[ \| g_{n_k} - g \|_{L^2} \leq \frac{\varepsilon}{2} \quad \forall n \geq n_0. \]

Then taking \( f_n = z \cdot \), we estimate

\[ \| f_n - g \|_{L^2} \leq \| f_n - g_{n_k} \|_{L^2(-\pi, \pi)} + \| g_{n_k} - g \|_{L^2(-\pi, \pi)} \]
\[
\sum_{n > \tilde{\eta}_0} \frac{e}{\tilde{\eta}^2} \leq \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon.
\]

So. \( f_{\tilde{n}} \to g \) in \( L^2(-\pi, \pi) \).

Thus \( K \) is compact in \( L^2(-\pi, \pi) \).
7. Convert to an equivalent integral equation

\[ q = 1 + \int_0^t (q^2(s) + s) \, ds = 1 + \frac{t^2}{2} + \int_0^t q^2(s) \, ds \]

Define a map

\[ q \rightarrow Aq, \quad Aq = 1 + \frac{t^2}{2} + \int_0^t q^2(s) \, ds \]

We shall identify a domain \( D(A) \subset C[0, T] \) in which the map is contractive.

• Natural candidate for \( D(A) \) — a ball in \( C[0, T] \) centered at \( q \equiv 1 \) with some radius \( \varepsilon \).

Estimate:

\[ \| Aq - 1 \| = \sup_{t \in [0, T]} \left\| \int_0^t (q^2(s) + s) \, ds + \frac{t^2}{2} \right\| \]

\[ \left( \frac{1}{q(s)} - 1 \right) < \varepsilon \Rightarrow \]

\[ \frac{1}{q(s)} \leq \frac{1}{q(s)} - 1 + 1 \leq 1 + \varepsilon \]

\[ \frac{1}{q^2(s)} \leq (1 + \varepsilon)^2 \]

\[ \leq T (1 + \varepsilon)^2 + \frac{T^2}{2} \]

For operator \( A \) to map the ball into itself, it is necessary that

\[ T (1 + \varepsilon)^2 + \frac{T^2}{2} < \varepsilon \]
**Contraction**

\[
\| Aq_1 - Aq_2 \| = \sup_{[0, T]} \int_0^T (q_i - q_{i+1}) (q_i + q_{i+1}) ds \leq 2(1+\varepsilon) \| q_1 - q_2 \|
\]

Now, given any \( \varepsilon > 0 \), say e.g. \( \varepsilon = 1 \), we can determine \( T \) that satisfies both conditions:

\[
T (1 + \varepsilon)^2 + \frac{T^2}{2} = 4T + \frac{T^2}{2} < 1
\]

\[
2T (1 + \varepsilon) = 4T < 1
\]

\[
\int \frac{T^2}{2} + 4T - 1
\]

\[
\frac{1}{4} \rightarrow T
\]

\[
T = \sqrt{8} - 4
\]

provides a lower bound for interval \((0, T)\) in which the solution exists and is unique.
Choose six out of the following nine problems.

1. Consider the standard spherical coordinates \((r, \psi, \theta)\),

![Diagram of spherical coordinates]

and a vector field \(E\) prescribed in the spherical coordinates,

\[ E_r = r, \quad E_\psi = \psi, \quad E_\theta = \theta. \]

(a) State the Stokes Theorem.
(b) Verify the Stokes Theorem for the vector field \(E\) and the hemispherical cap

\[ S = \{ (r, \psi, \theta) : r = 1, \quad \theta \in [0, 2\pi], \quad \psi \in [0, \frac{\pi}{2}] \} \]

by computing and comparing the relevant surface and line integrals.

2. Use the Laplace transform and Residue Theorem to solve the following Initial-Value Problem.

\[
\begin{cases}
  \dot{x} - 2x = H(t - 1), & t > 0, \\
  x(0) = 2,
\end{cases}
\]

wherein \(H\) is the usual Heaviside function.

3. Use the separation of variables to solve the boundary-value problem \(-\Delta u = 1\) on the unit square \(0 < x < 1, 0 < y < 1\), where \(u(x, y) = 0\) on the boundary of the domain.

4. Use the Banach Contractive Map Theorem to determine a bound for \(|\lambda|\) for which the integral equation

\[ y(t) = 1 + \lambda \int_1^t \frac{y(s)}{s^2} \, ds, \quad t \in (1, \infty), \]

has a unique solution in an appropriate function space.
5. Consider the complex contour integral

\[ \oint_C \frac{dz}{(z^2 + 1)(e^z - 1)(z - 1)^2}, \]

where \( C \) is the counter-clockwise circle \(|z| = 3\).

(a) State the Cauchy Integral Theorem.

(b) Identify all complex poles of the integrand \( f(z) = \frac{1}{(z^2 + 1)(e^z - 1)(z - 1)^2} \), and classify their order.

(c) Use the Cauchy Integral Theorem to evaluate the integral.

6. Consider the Telegrapher’s equation

\[ \frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} + bu = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0, \]

\[ u(x, 0) = f(x), \]

\[ \frac{\partial u}{\partial t}(x, 0) = g(x), \]

where \( a \) and \( b \) are constants.

(a) Find two fundamental solutions to this problem (one each for the influence of \( f \) and \( g \)). You may express your solution in terms of the inverse Fourier Transform.

(b) In terms of the fundamental solutions from (a), write the solution to the full problem above in terms of convolutions.

7. Consider the following functions on \( \mathbb{R} \times \mathbb{R} \):

\[ d_1(x, y) = \sqrt{|x - y|}, \quad d_2(x, y) = |x - y|, \quad d_3(x, y) = 1 - e^{-|x-y|}. \]

(a) Define what is meant by a metric on \( \mathbb{R} \).

(b) Show that \( d_1, d_2, \) and \( d_3 \) are metrics.

(c) Do these generate the same topology on \( \mathbb{R} \)? Why or why not?

8. Let \( T \) be an \( n \times n \) matrix and \( b \in \mathbb{R}^n \). Given \( x_0 \in \mathbb{R}^n \), define for \( i \geq 0 \)

\[ x_{i+1} = Tx_i + b. \]

Prove that \( x_i \) converges as \( i \to \infty \) if, and only if, \( \rho(T) < 1 \). Here, \( \rho(T) \) is the spectral radius of \( T \), i.e., the maximum of the absolute values of the eigenvalues of \( T \). You may assume that \( T \) is diagonalizable.

9. Let \( X \) be a nonempty set.

(a) Define what is meant by a topology \( \mathcal{X} \) on \( X \).

(b) If \( A \subseteq X \), define what is meant by \( \text{int} A \) (the interior of \( A \)).

(c) Prove that for \( A, B \subseteq X \),

\[ \text{int} A \cup \text{int} B \subset \text{int} (A \cup B). \]

(d) Show by example that equality may fail in (c).
Consider the standard spherical coordinates \((\tau, \psi, \theta)\).

and a vector field \(E\) prescribed in the spherical coordinates

\[
\begin{align*}
E_\tau &= r \\
E_\psi &= \psi \\
E_\theta &= \theta
\end{align*}
\]

State the Stokes' Theorem.

Verify the Stokes' theorem for the vector field \(E\) and a hemispherical cap

\[
S = \{(r, \theta, \psi); r = 1, \theta \in [0, \pi], \psi \in [0, \pi]\}
\]

by computing and comparing the relevant surface and line integrals.
\[ \begin{align*}
\begin{cases}
x = r \sin \psi \cos \theta \\
y = r \sin \psi \sin \theta \\
z = r \cos \psi 
\end{cases}
\end{align*} \]

\[
\begin{align*}
x_\rho &= \frac{\partial x}{\partial \rho} = ( \sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi ) \\
| x_\rho |^2 &= \sin^2 \psi \cos^2 \theta + \sin^2 \psi \sin^2 \theta + \cos^2 \psi = 1 \\
x_\varphi &= x_\rho \\
x_\varphi &= x_\rho \\
\end{align*} \]

\[
\begin{align*}
x_\psi &= \frac{\partial x}{\partial \psi} = ( \cos \psi \cos \theta, \cos \psi \sin \theta, -\sin \psi ) \\
| x_\psi |^2 &= r^2 \\
x_\theta &= \frac{1}{r} x_\psi = ( \cos \psi \cos \theta, \cos \psi \sin \theta, -\sin \psi ) \\
x_\theta &= \frac{1}{r} x_\psi \\
\end{align*} \]

\[
\begin{align*}
x_\theta &= \frac{\partial x}{\partial \theta} = (-r \sin \psi \sin \theta, r \sin \psi \cos \theta, 0 ) \\
| x_\theta | &= r \sin \psi \\
x_\theta &= (-r \sin \psi \sin \theta, r \sin \psi \cos \theta, 0 ) \\
x_\theta &= \frac{1}{r \sin \psi} x_\theta \\
\end{align*} \]

\[
\begin{align*}
E &= E_r x_r + E_\psi x_\psi + E_\theta x_\theta 
\end{align*} \]
the Stokes Flux

\[ \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, dS' = \int_{\partial S} \mathbf{E} \cdot d\mathbf{r} \]

\[ \partial S = \{(r, \psi, \theta) : r = 1, \psi = \frac{\pi}{2}, \theta \in [0, 2\pi] \} \]

\[ \int_S \mathbf{E} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{E}_r \frac{dr}{\psi} \, d\psi = \int_0^{2\pi} \mathbf{E}_\theta \, d\theta \]

\[ = \int_0^{2\pi} \psi \, d\theta = \frac{\theta^2}{2} \bigg|_0^{2\pi} = 2\pi^2 \]

In a general, curvilinear system of coordinates,

\[ \nabla \times \mathbf{E} = \sum_{i=1}^{3} \mathbf{e}^i \times \frac{\partial \mathbf{E}}{\partial u^i} \]

curvilinear coordinates
In the spherical coordinates

\[ \nabla \times E = e_r \times \frac{\partial E}{\partial r} + \frac{1}{r} e_\theta \times \frac{\partial E}{\partial \theta} + \frac{1}{r \sin \theta} e_\phi \times \frac{\partial E}{\partial \phi} \]

\[ E = r e_r + \gamma e_\theta + \Theta e_\phi \]

\[ \frac{\partial E}{\partial r} = e_r \]

\[ \frac{\partial E}{\partial \theta} = r \frac{\partial e_r}{\partial \theta} + e_\theta + \gamma \frac{\partial e_\theta}{\partial \theta} \]

\[ \frac{\partial e_r}{\partial \theta} = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) = e_\theta \]

\[ \frac{\partial e_\theta}{\partial \theta} = (-\sin \phi \sin \theta, -\sin \phi \cos \theta, -\cos \phi) = -e_\phi \]

So:

\[ \frac{\partial E}{\partial \theta} = r e_\theta + e_\theta - \gamma e_r = -e_\phi + (r+1) e_\phi \]

\[ \frac{\partial E}{\partial \phi} = r \frac{\partial e_r}{\partial \phi} + e_\phi + \phi \frac{\partial e_\phi}{\partial \phi} + e_\theta \]

\[ \frac{\partial e_r}{\partial \phi} = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0) = \sin \phi e_\theta \]

\[ \frac{\partial e_\phi}{\partial \phi} = (-\cos \theta, -\sin \theta, 0) = -e_\phi \]

So:

\[ \frac{\partial E}{\partial \phi} = r \sin \phi e_\theta + \gamma \cos \phi e_\theta + e_\phi - \Theta (e_\phi \sin \phi + e_\theta \cos \phi) \]

\[ = -\Theta \sin \phi e_r - \Theta \cos \phi e_\theta + (r \sin \phi + e_\theta \cos \phi) e_\phi \]
$\nabla \times E = \frac{1}{r} \frac{\partial}{\partial r} \left( r E_r \right)
+ \frac{1}{r \sin \psi} \frac{\partial}{\partial \theta} \left( -\theta \sin \psi E_\theta - \theta \cos \psi E_\phi + (r \sin \psi + \rho \cos \rho r \sin \psi) \phi \right)
+ \frac{1}{r \sin \psi} \frac{\partial}{\partial \phi} \left( -\theta \cos \psi E_\phi - \theta \sin \psi E_\theta + (r \sin \psi + \rho \cos \rho r \sin \psi) \phi \right)

= \frac{\psi}{r} E_\theta - \frac{\theta}{r} E_\phi + \frac{\theta \cos \psi}{r \sin \psi} E_r

Outward unit normal vector \( \mathbf{n} = \frac{\mathbf{e}_r}{r} \)

\((\nabla \times E) \cdot \mathbf{e}_r = \frac{\theta \cos \psi}{r \sin \psi}\)

The surface integral \((r = 1)\)

$$\int_{S} (\nabla \times E) \cdot n \, ds = \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \frac{\theta \cos \psi}{r \sin \psi} \frac{r^2 \sin \psi}{r \sin \psi} \, d\theta \, d\psi$$

$$= \int_{0}^{\frac{\pi}{2}} \cos \psi \, d\psi \int_{0}^{2\pi} \theta \, d\theta$$

$$= \sin \psi \left[ \frac{\theta^2}{2} \right]_{0}^{2\pi} = 2\pi \frac{\theta^2}{r}$$
2. Use the Laplace transform and Residue Theorem to solve the following IVP.

\[
\begin{cases}
  \dot{x} - 2x = H(t-1) & ; t > 0 \\
x(0) = 2
\end{cases}
\]

**Elementary solution:**

- \(0 < t < 1\)
  
  \[
  \dot{x} - 2x = 0 \quad x(t) = C e^{2t}
  \]
  
  \[
  x(0) = 2 \quad \Rightarrow \quad x(t) = 2 e^{2t} \quad x(1) = 2 e^2
  \]

- \(1 < t < \infty\)
  
  \[
  \dot{x} - 2x = 1
  \]
  
  **Particular solution:** \(x = -\frac{1}{2}\)
  
  \[
  x(t) = C e^{2t} - \frac{1}{2}
  \]
  
  \[
  x(1) = C e^2 - \frac{1}{2} = 2 e^2
  \]
  
  \[
  C = \left(2 e^2 + \frac{1}{2}\right) \frac{1}{e^2} = 2 + \frac{1}{2 e^2}
  \]

  \[
  x(t) = -\frac{1}{2} + (2 + \frac{1}{2 e^2}) e^{2t}
  \]

Laplace transform:

\[
\int_0^\infty H(t-1) e^{-st} dt = \int_0^\infty e^{-st} dt \\
= -\frac{1}{s} e^{-st} \bigg|_0^\infty = \frac{e^{-s}}{s} \quad (\text{Re } s > 0)
\]

\[L(x) = sx - x(0) = sx - 2\]

\[sx - 2x - 2 = \frac{e^{-s}}{s} \]

\[x(s-2) = 2 + \frac{e^{-s}}{s} \]

\[x = \frac{2}{s-2} + \frac{e^{-s}}{s(s-2)}\]

Inverse Laplace transform:

\[
\frac{1}{2\pi i} \int_{q-i\infty}^{q+i\infty} \frac{2e^{st}}{s-2} ds = \lim_{R \to \infty} \left\{ \frac{1}{2\pi i} \int_{q-i\infty}^{q+i\infty} - \frac{1}{2\pi i} \int_{C}^{0} \right\} \\
= \text{Res} \left( \frac{2e^{st}}{s-2} \right) - \lim_{R \to \infty} \int_{C}^{0} \\
= 2e^{2t}
\]
\[ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-1)}}{s(s-2)} \, ds \]

\[ t < 1 \quad \text{arc} \quad 0 \to 2 \quad \text{D} \]

To conclude that \( \int_{\gamma-i\infty}^{\gamma+i\infty} \, ds = 0 \)

\[ t > 1 \quad \text{arc} \quad 0 \to 2 \quad \text{R} \]

\[ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-1)}}{s(s-2)} \, ds = \text{Res}_0 \frac{e^{s(t-1)}}{s(s-2)} + \text{Res}_2 \frac{e^{s(t-1)}}{s(s-2)} \]

\[ = -\frac{1}{2} + \frac{e^{2(t-1)}}{2} \]
3. Use the separation of variables to solve the following BVP.

\[ \begin{align*}
\frac{\partial u}{\partial x} &= a \left( 2x - 1 \right) \\
\frac{\partial^2 u}{\partial x \partial y} &= 2a = -1 \quad \Rightarrow \quad a = -\frac{1}{2} \\
u &= -\frac{1}{2} x (x-1)
\end{align*} \]

- A particular solution

\[ u = ax(x-1) = a \left( x^2 - x \right) \]

- Look for \( v \) such that

\[ v = \frac{\partial}{\partial y} u \]

\[ \begin{align*}
v &= 0 \\
\Delta v &= 0 \\
v &= 0 \\
v &= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} u \right) \]

\[ \begin{align*}
v &= k(x) \psi(y) \\
\Delta v &= \psi'' + k \psi = 0 \\
-\frac{k''}{k} &= \frac{\psi''}{\psi} = \lambda
\end{align*} \]
Sturm–Liouville problem

\[- x'' = \lambda x, \quad x(0) = 0, \quad x(1) = 0\]

The operator is positive-definite, so \( \lambda = k^2 > 0 \)

\[- x'' + k^2 x = 0\]

\[x(t) = A \sin k x + B \cos k x\]

\[x(0) = 0 \Rightarrow B = 0\]

\[x(1) = 0 \Rightarrow \sin k = 0 \Rightarrow k = k_n = n \pi, \quad n = 1, 2, \ldots\]

\[- y'' - k_n^2 y = 0\]

\[y = A_n e^{k_n x} + B_n e^{-k_n x}\]

General solution:

\[v(x, y) = \sum_{n=1}^{\infty} \sin n \pi x \left( A_n e^{n \pi y} + B_n e^{-n \pi y} \right)\]

BC:

\[y = 0 \quad \sum_{n=1}^{\infty} \sin n \pi x \left( A_n + B_n \right) = \frac{1}{2} x (x-1)\]

\[y = 1 \quad \sum_{n=1}^{\infty} \sin n \pi x \left( A_n e^{n \pi} + B_n e^{-n \pi} \right) = \frac{1}{2} x (x-1)\]

Multiply by \( \sin n \pi x \) and integrate in \( x \) to obtain:

\[
\begin{cases}
(A_n + B_n) = \int_0^1 x (x-1) \sin n \pi x \, dx \\
(A_n e^{n \pi} + B_n e^{-n \pi}) = \int_0^1 x (x-1) \sin n \pi x \, dx
\end{cases}
\]

Solve for \( A_n, B_n \).
4. Use the Banach Contraction Map Theorem to determine an upper bound for $h$ for which the following integral equation has a unique solution in an appropriate function space.

$$y(t) = 1 + \lambda \int \frac{y(s)}{s^2} \, ds$$

Seek $y \in C(1, \infty)$, $y$ bounded

Define $(Ay)(t) = 1 + \lambda \int \frac{y(s)}{s^2} \, ds$

It is sufficient to show that $A$ has a fixed point.

- $A$ is well-defined

$$|Ay(t)| \leq 1 + |\lambda| \int \frac{|y(s)|}{s^2} \, ds$$

$$\leq 1 + c |\lambda| \int \frac{ds}{s^2}$$

$$\leq 1 + c |\lambda| \int_1^{\infty} \frac{ds}{s^2} \leq 1 + c |\lambda|$$

- $Ay_i - Ay_2 = A(y_i - y_2)$

$$= \lambda \int \frac{y_i(s) - y_2(s)}{s^2} \, ds$$

$$|Ay_i - Ay_2| \leq |\lambda| \int \frac{|y_i(s) - y_2(s)|}{s^2} \, ds$$

$$\leq |\lambda| \|y_i - y_2\| \int_1^{\infty} \frac{ds}{s^2}$$

$$\leq |\lambda| \|y_i - y_2\|$$

For $|\lambda| < 1$, $A$ is a contraction $\Rightarrow A$ has a unique fixed point.
Consider the complex contour integral

\[ \oint_{C} \frac{dz}{(z^2+1)(e^z-1)(z-1)^2} \]

where \( C \) is the counter-clockwise circle \( |z|=3 \).

(a) State the Cauchy Integral Thm.
(b) Use the CIT to evaluate the integral.

Identify all complex poles of

\[ f(z) = \frac{1}{(z^2+1)(e^z-1)(z-1)^2} \]

and classify their order.
Consider the Telegrapher's eqn
\[ \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial t} + b u = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \]
\[ \frac{\partial u}{\partial t} (x, 0) = g(x), \]
where \(a\) and \(b\) are constants.

(c) Find 2 fundamental solutions to this equation (one each for the influence of \(f\) and \(g\)). You may express your solution in terms of the inverse Fourier transform.

(b) In terms of the fundamental solutions to (c), write the solution to the full problem in terms of convolutions.
Consider the following functions on $\mathbb{R} \times \mathbb{R}$

\[
\begin{align*}
    d_1(x, y) &= \sqrt[3]{|x - y|} \\
    d_2(x, y) &= |x - y| \\
    d_3(x, y) &= 1 - e^{-|x - y|}
\end{align*}
\]

(a) Define what is meant by a metric on $\mathbb{R}$.

(b) Show that $d_1, d_2,$ and $d_3$ are metrics.

(c) Do these generate the same topology on $\mathbb{R}$? Why or why not.
Let $T \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix, and $b \in \mathbb{R}^n$. Given $x_0 \in \mathbb{R}^n$, define

$$x_{n+1} = Tx_n + b$$

Prove that $x^n$ converged iff $\rho(T) = \max |\lambda_i| < 1$,

where $\rho(T)$ is the spectral radius of $T$. You may assume that $T$ is diagonalizable.
(a) Define what it means by a topology $\tau$ on a set $X$.

(b) If $A \subseteq X$, define what it means by $\text{int} A$.

(c) Prove that for $A, B \subseteq X$, $\text{int} (A \cup B) \supseteq \text{int} A \cup \text{int} B$.

(c) Show by example that equality may fail in (c).
CAM AREA A PRELIMINARY EXAM (CAM 385C–D)

May 29, 2007, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Prove the Mazur Separation Lemma, which says that if $X$ is a normed linear space, $Y$ a linear subspace of $X$, $w \in X$ but $w \notin Y$, and

$$d = \text{dist}(w, Y) = \inf_{z \in Y} \|w - z\|_X > 0,$$

then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: work in $Z = Y + \mathbb{R}w$, and extend to $X$ using the Hahn-Banach Theorem.]

2. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose $K \in C(\bar{\Omega} \times \bar{\Omega})$ and $T : L_2(\Omega) \to L_2(\Omega)$ is defined by

$$Tf(x) = \int_\Omega K(x, y) f(y) \, dy.$$

(a) Show that $T$ is well defined.
(b) State the Ascoli-Arzelà Theorem about continuous functions defined on a compact metric space.
(c) Prove that $T$ is a compact operator. [Hint: use the density of $C(\bar{\Omega})$ in $L_2(\Omega)$ and apply Ascoli-Arzelà.]

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = f(x),$$

on the domain $(x, t) \in \mathbb{R} \times [0, T]$ where $f$ is a continuous function that vanishes (i.e., is zero) outside the closed interval $[-1, 1]$.

(a) Find a solution to this equation. [Hint: use the Fourier transform.]
(b) Show that the initial data controls the solution in the sense that

$$\int_{-\infty}^{\infty} |u(x, t)|^2 \, dx \leq \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

(c) By looking at the form of your solution, explain why the solution is infinitely smooth in the space variable and that heat dissipates with time.

4. Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. For $y \in \mathbb{R}^d$, the translation operator $\tau_y$ is defined by $\tau_y \phi(x) = \phi(x - y)$.

(a) Show that

$$u(\tau_y \phi) - u(\phi) = \int_0^1 \sum_{j=1}^d y_j \frac{\partial u}{\partial x_j} (\tau_{ty} \phi) \, dt.$$

(b) Apply this to

$$f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) = \left\{ f \in L_{1,\text{loc}}(\mathbb{R}^d) : \frac{\partial f}{\partial x_j} \in L_{1,\text{loc}}(\mathbb{R}^d) \text{ for all } j \right\}$$

to show that

$$f(x + y) - f(x) = \int_0^1 y \cdot \nabla f(x + ty) \, dt.$$
5. The symmetric gradient of a vector function \( v \in (H^1(\Omega))^d \) is a \( d \times d \) matrix of the form

\[
\varepsilon(v)_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

It arises often in structural mechanics problems. Assuming that \( \Omega \in \mathbb{R}^2 \) is smooth and bounded, prove Korn’s inequality, which says that there is some \( C > 0 \) such that

\[
\|v\|_{H^1(\Omega)} \leq C\|\varepsilon(v)\|_{L^2(\Omega)} \quad \text{for all } v \in (H^1_0(\Omega))^2.
\]

[Hint: You will need to show that all rigid motions, i.e., vector fields of the form \( v = \alpha(x_2, -x_1) + (\beta, \gamma) \) vanish in \((H^1_0(\Omega))^2\).]

6. Consider a stream between the lines \( x = 0 \) and \( x = 1 \), with speed \( v(x) \) in the \( y \)-direction. A boat leaves the shore at \((0, 0)\) and travels with constant speed \( c > 0 \), relative to the water, to the given terminal point \((1, \beta)\). The problem is to find the path \( y(x) \) of minimal crossing time.

(a) Let \( \frac{dx}{dt} = u \) and \( \frac{dy}{dt} = w \). Then \( y'(x) = w/u \) and \( c^2 = u^2 + (w - v)^2 \). Show that the crossing time is

\[
t = \int_0^1 \frac{dx}{u} = \int_0^1 \frac{\sqrt{c^2[1 + (y')^2] - v^2} - vy'}{c^2 - v^2} \, dx.
\]

[Hint: you will need to use the quadratic formula.]

(b) Find the Euler-Lagrange equations, and reduce them to a single equation for \( y' \).

(c) If \( v \) is constant, find \( y \).
Work any 5 of the following 6 problems

1. Consider a hollow sphere \( a^2 \leq x^2 + y^2 + z^2 \leq b^2 \). Suppose the mass density function \( \sigma(x, y, z) \) over the sphere is constant.
   (a) Show that the total mass of the sphere is
   \[
   M = \frac{4}{3} \pi (b^3 - a^3) \sigma.
   \]
   (b) The moment of inertia about the \( x \) axis for a volume \( V \) is defined as
   \[
   I_x = \int \int \int_V (y^2 + z^2) \sigma dV.
   \]
   Show that for the hollow sphere
   \[
   I_x = \frac{2}{5} M \frac{b^5 - a^5}{b^3 - a^3}.
   \]
   (c) In the limit as \( a \to b \), the hollow sphere becomes a thin shell. Compute \( I_x \) for a thin shell of radius \( b \).

2. Consider the problem of finding the steady-state temperature distribution \( \psi(x, y) \) in the cut-off disk \( D \) in the figure below, with boundary conditions as shown.
   (a) Show that the conformal map
   \[
   w(z) = \frac{z - 1}{z + 1}
   \]
   sends the domain \( D \) into the domain \( D' \) in \( (u, v) \) space.
   (b) Changing to polar coordinates \( \rho \) and \( \phi \) in \( (u, v) \) space, find the solution \( \psi(\rho, \phi) \).
   (c) Finally, express \( \psi \) in terms of the original variables \( x \) and \( y \).
3. (a) Explain how to solve the problem

\[ f(x_1, \ldots, x_n) = \text{extremum} \]
\[ g_j(x_1, \ldots, x_n) = 0, \quad j = 1, \ldots, m \]

where \( m < n \), using the Lagrange multiplier method.

(b) Apply the method to the problem

\[ f = x + y + z = \text{extremum} \]
\[ g_1 = x^2 + y^2 + z^2 - 1 = 0 \]
\[ g_2 = x + y = 0 \]

4. Consider the following Initial-Value Problem (IVP),

\[ \begin{cases} \dot{x} = x^2 t & t \in (0, T) \\ x(0) = 1 \end{cases} \]

(a) Use elementary means to find the solution. Determine the maximum \( T \) for which the solution is defined. This is your “sanity check”.

(b) Convert the IVP to an equivalent integral operator equation,

\[ x = Ax \]

where \( A \) is an integral operator defined on Chebyshev space \( C([0, T]) \).

(c) Define a set \( \Omega \subset C([0, T]) \) such that:

- operator \( A \) maps set \( \Omega \) into itself, i.e.

\[ A(\Omega) \subset \Omega \]

- operator \( A \) is contractive.

Find the largest \( T \) for which the two conditions are satisfied.
(d) Formulate Banach Contractive Map Theorem and use it to conclude that there exists a unique solution to the IVP.

5. Let $X$ be an arbitrary set. Suppose we are given two topologies on the set with the corresponding families of open sets $\mathcal{X}^i$, bases of neighborhoods $\mathcal{B}^i_x$, $x \in X$ and filters of neighborhoods $\mathcal{F}^i_x$, $x \in X$, $i = 1, 2$. Prove that the following conditions are equivalent.

\[
\mathcal{B}^1_x \supset \mathcal{B}^2_x, \quad \forall x \in X
\]

\[
\mathcal{F}^1_x \supset \mathcal{F}^2_x, \quad \forall x \in X
\]

\[
\mathcal{X}^1 \supset \mathcal{X}^2
\]

identity map $\text{id} : (X, \mathcal{X}^1) \to (X, \mathcal{X}^2)$ is continuous

Recall that we say then that the first topology is stronger than the second. Provide an example of a set and two different topologies illustrating the discussion.

6. Consider $\mathbb{R}^N$ with the $l^1$-norm,

\[
x = (x_1, \ldots, x_N), \quad \|x\|_1 = \sum_{i=1}^{N} |x_i|
\]

Let $\|x\|$ be now any other norm defined on $\mathbb{R}^n$.

a. Show that there exists a constant $C > 0$ such that,

\[
\|x\| \leq C\|x\|_1, \quad \forall x \in \mathbb{R}^N
\]

b. Use [a] to demonstrate that function,

\[
\mathbb{R}^N \ni x \to \|x\| \in \mathbb{R}
\]

is continuous in $l^1$-norm.

c. Use Weierstrass Theorem to conclude that there exists a constant $D > 0$ such that

\[
\|x\|_1 \leq D\|x\|, \quad \forall x \in \mathbb{R}^N
\]

Therefore, the $l_1$ norm is equivalent to any other norm on $\mathbb{R}^N$. Explain why the result implies that any two norms defined on an arbitrary finite-dimensional vector space must be equivalent.
CAM AREA A PRELIMINARY EXAM (CAM 385C–D)
May 30, 2008, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Hahn-Banach Theorem.
   (a) State the Hahn-Banach Theorem.
   (b) Let $X$ be a NLS and $x, y \in X$. Prove that $f(x) = f(y)$ for all $f \in X^*$ if and only if $x = y$.
   (c) Prove the Mazur Separation Lemma: Let $X$ be a NLS and $Y$ a linear subspace. If $w \in X \setminus Y$ and
      
      $$d = \text{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0,$$
      
      then there is $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(y) = 0$ for all $y \in Y$. [Hint: work in $Z = Y + \mathbb{F}w$, and extend to $X$ using the Hahn-Banach Theorem.]

2. Let $H$ be a Hilbert space and $Z \subset H$ a closed linear subspace. Recall that we define the Hilbert space of cosets of $Z$ as $H/Z = \{x + Z : x \in H\}$ with the norm $\|\cdot\|_{H/Z} : H/Z \to \mathbb{R}$ given by

   $$\|\hat{x}\|_{H/Z} = \inf_{y \in \hat{x}} \|y\|_H.$$

   (a) If $T \in H^*$, let $Z = \ker(T)$ and define the function $\hat{T} : H/Z \to \mathbb{R}$ by $\hat{T}(\hat{x}) = T(x)$, where $\hat{x} = x + Z$. Show that this function is well defined and injective.
   (b) Find a continuous linear map that takes $H/Z$ one-to-one and onto $Z^\perp = \{x \in H : \langle x, z \rangle = 0 \text{ } \forall z \in Z\}$ such that norms are preserved.

3. Let the underlying field be real and let $V \in L_2((0,1) \times \Omega)$, where $\Omega$ is some domain. Suppose that we define $T : L_2(0, 1) \to L_2(0, 1)$ by

   $$Tf(x) = \int_0^1 \int_\Omega V(x, \omega) V(y, \omega) f(y) \, d\omega \, dy.$$

   (a) Justify that $T$ is well defined, compact, symmetric, and positive (semi-definite).
   (b) State the Spectral Theorem for compact, positive, self-adjoint linear operators.
   (c) Apply the theorem to $T$ to show that we can express

   $$V(x, \omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha(\omega) v_\alpha(x),$$

   for some orthonormal basis $\{v_\alpha(x)\}_{\alpha \in \mathcal{I}}$ of $L_2(0, 1)$. Also give an expression for the coefficients $a_\alpha(\omega)$.
4. Suppose that \( f \) is a real, periodic, and continuous function on \([-\pi, \pi]\). Define the Fourier coefficients for \( n \) an integer by
\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.
\]
(a) Show that
\[
|\hat{f}(n)| \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(t + \pi/n) - f(t)| \, dt.
\]
[Hint: \( \cos(x + \pi) = -\cos(x) \) and \( \sin(x + \pi) = -\sin(x) \).]
(b) Explain how this formula implies that the Fourier coefficients of \( f \) must decay to zero for large \(|n|\).

5. Suppose that \( \Omega \subset \mathbb{R}^d \) is bounded and has a Lipschitz boundary. Suppose also that \( \epsilon \in (0, 1) \), \( a_\epsilon \in L_\infty(\Omega) \) satisfies \( 0 < a_\ast \leq a_\epsilon \leq a^* < \infty \) on \( \bar{\Omega} \), and \( f \in L_2(\Omega) \). For each \( \epsilon \), consider the boundary value problem
\[
-\nabla \cdot (a_\epsilon \nabla u_\epsilon) = f \quad \text{in } \Omega,
\]
\[
u_\epsilon = 0 \quad \text{on } \partial \Omega.
\]
(a) State the Lax-Milgram Theorem (for Hilbert spaces).
(b) Prove that for each \( \epsilon \), there is a unique solution to the boundary value problem.
(c) Show that there exists some \( u \in H_0^1(\Omega) \) and a single subsequence \( u_{\epsilon_n} \) for which both \( u_{\epsilon_n} \rightharpoonup u \) weakly in \( H_0^1(\Omega) \) and \( u_{\epsilon_n} \rightarrow u \) strongly in \( L_2(\Omega) \). Be sure to state the theorems that you use to show these results.

6. Suppose that \( X \) is a Banach space and we wish to solve \( F(x) = 0 \), where \( F : X \rightarrow X \). Suppose there is at least one root \( x \). If we use Newton's Method, we are given \( x_0 \) and define
\[
x_{n+1} = G(x_n), \quad n = 0, 1, 2, \ldots, \quad \text{where} \quad G(y) = y - DF(y)^{-1}F(y).
\]
Assume that \( F \) is \( C^1 \) on all of \( X \), \( DF(y) \) is invertible for all \( y \in X \), and \( \|DF(y)^{-1}\| \) is uniformly bounded in \( y \in X \). Let the error be denoted by \( e_n = x_n - x \).
(a) Show that
\[
\|e_{n+1}\| = o(\|e_n\|).
\]
[Hint: note that \( (x_{n+1} - x) = (x_n - x) - DF(x_n)^{-1}[F(x_n) - F(x)] \), since \( x \) is a root, and apply the definition of the Fréchet derivative.]
(b) State the Mean Value Theorem for NLS's.
(c) Now assume that \( F \) is \( C^3 \) on all of \( X \). You are given that
\[
DG(y)(k) = DF(y)^{-1}D^2F(y)(DF(y)^{-1}F(y), k),
\]
and we assume that \( \|D^2G(y)\| \) is uniformly bounded in \( y \in X \). Show that \( DG(x) = 0 \), and use this fact to show that there is some \( C > 0 \) such that
\[
\|e_{n+1}\| \leq C\|e_n\|^2.
\]
That is, if Newton's method converges, it does so quadratically. [Hint: note that \( (x_{n+1} - x) = G(x_n) - G(x) \) and apply the Mean Value Theorem twice.]
(d) Under the extra hypothesis given in (c), give a reasonable condition on \( x_0 \) to insure that \( x_n \rightarrow x \).
Work any 5 of the following 6 problems

1. Consider the following Initial-Value Problem (IVP).
   \[
   \begin{cases}
   \dot{x} = x(x + 1) t & i \in (0, T) \\
   x(0) = 1
   \end{cases}
   \]
   (a) Use elementary means to find the solution. Determine the maximum \( T \) for which the solution is defined. This is your "sanity check".
   (b) Convert the IVP to an equivalent integral operator equation,
   \[
   x = A x
   \]
   where \( A \) is an integral operator defined on Chebyshev space \( C([0, T]) \).
   (c) Define a set \( \Omega \subseteq C([0, T]) \) such that:
       - operator \( A \) maps set \( \Omega \) into itself, i.e.
         \[
         A(\Omega) \subseteq \Omega
         \]
       - operator \( A \) is contractive on \( \Omega \).
       Find a concrete value of \( T \) for which the two conditions are satisfied.
   (d) Formulate Banach Contractive Map Theorem, and use it to conclude that there exists a unique solution to the IVP.

2. Show that any two norms \( \| \cdot \|_p \) and \( \| \cdot \|_q \) in \( \mathbb{R}^n \), \( 1 \leq p, q \leq \infty \), are equivalent, i.e. there exist constants \( C_1 > 0, C_2 > 0 \) such that
   \[
   \|x\|_p \leq C_1 \|x\|_q \quad \text{and} \quad \|x\|_q \leq C_2 \|x\|_p
   \]
   for any \( x \in \mathbb{R}^n \). Try to determine optimal (minimum) constants \( C_1 \) and \( C_2 \).

3. In process of computing the inverse of the Laplace transform,
   \[
   \hat{f}(s) = \frac{1}{s - a},
   \]
   we need to show that the integral
   \[
   \int \frac{e^{st}}{s - a} \, ds
   \]
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Figure 1: Contour integration for the inversion of Laplace transform.

over the semicircle shown in Fig. 1 vanishes as \( R \to \infty \). Use parametrization

\[
s = \gamma + Re^{i\theta} = \gamma + R(\cos \theta + i \sin \theta), \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})
\]

to convert the integral to a real integral over interval \((\frac{\pi}{2}, \frac{3\pi}{2})\), and use the Lebesgue Dominated Convergence Theorem to show that this integral vanishes as \( R \to \infty \) (you can think of \( R \) as integer).

4. Find the solution to the vibrating string problem with a nonzero source and damping proportional to the velocity:

\[
\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - 2 \frac{\partial y}{\partial t} + x, \quad 0 < x < 1, \ t > 0
\]

\[
y(0, t) = y(1, t) = 0, \quad t > 0,
\]

\[
y(x, 0) = f(x), y_t(x, 0) = 0, \quad 0 < x < 1
\]

5. Consider the real integral

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 6} \, dx
\]

(a) Without evaluating it, prove that this integral is finite.

(b) Evaluate it using any means you wish.

6. Compute the surface integral (flux across the surface)

\[
\int_{\Sigma} \mathbf{F} \cdot \mathbf{N} \, d\sigma
\]

where \( \mathbf{F} = xi + yj + zk \), \( \Sigma \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) lying between the planes \( z = 1 \) and \( z = 2 \), and \( \mathbf{N} \) is the unit normal pointing from the interior of the sphere outward.

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CAM AREA A PRELIMINARY EXAM (CAM 385C–D)

May 29, 2009, 9:00 a.m.-12:00 noon

Work any 5 of the following 6 problems.

1. Suppose \( f \in L_p(\mathbb{R}^d) \) and \( g \in L_q(\mathbb{R}^d) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( 1 < p < \infty \). Show that

\[
    f \ast g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \, dy
\]

is a continuous function with

\[
    \lim_{|x| \to \infty} f \ast g(x) = 0.
\]

[Hint: Recall that if \( h \in L_1(\mathbb{R}^d) \), then \( \int_{|x|>R} h(x) \, dx \to 0 \) as \( R \to \infty \).]

2. Suppose that \( H \) is a Hilbert space and \( V \subset H \) is a nonempty, closed convex set. Let \( x \in H \) such that \( x \notin V \).

   (a) Define what it means for \( V \) to be convex.
   (b) Prove that there is a unique \( y \in V \) such that \( \|x - y\| \) is minimal.
   (c) If \( V \) is a linear subspace, prove that \( x - y \) is orthogonal to \( V \).

3. Consider an operator \( A : L^2[0,1] \to L^2[0,1] \) defined by

\[
    Af(x) = \int_0^x f(t) \, dt.
\]

   (a) Show that \( A^* f(x) = \int_x^1 f(t) \, dt \) and \( A^* Af(x) = \int_0^1 [1 - \max(x,t)] f(t) \, dt \).
   (b) Show that \( A^* A \) is self-adjoint, positive, and compact on \( L^2[0,1] \).
   (c) Show that if \( \lambda \neq 0 \) is an eigenvalue of \( A^* A \) with eigenfunction \( f \), then \( \lambda f'' = -f \) almost everywhere on \([0,1]\), and also \( f(0) = 0 \) and \( f'(1) = 0 \).
   (d) Show that \( \|A^* A\| = 4/\pi^2 \) and \( \|A\| = 2/\pi \).

4. Suppose \( X \) is a Banach space and \( T \in B(X,X) \) has norm \( \|T\| < 1 \). For \( n \) a positive integer, define \( T^n \) to be the composition of \( T \) with itself \( n \) times.

   (a) If \( R \in B(X,X) \), show that \( \|RT\| \leq \|R\| \|T\| \).
   (b) Show that

\[
    S = I + T + T^2 + \ldots = \sum_{n=0}^{\infty} T^n = \lim_{N \to \infty} \sum_{n=0}^{N} T^n
\]

    is a well defined element of \( B(X,X) \). We call \( S \) the Neumann series of \( T \).

   (c) Show carefully that \( I - T \) is invertible and that \( (I - T)^{-1} = S \).
5. Consider the partial differential equation

\[(\nabla \cdot a \nabla)(\nabla \cdot a \nabla)u = f \quad \text{in } \Omega \subset \mathbb{R}^d,\]

where \(\Omega\) is bounded with a smooth boundary, scalar function \(a \in C^\infty_0(\bar{\Omega})\), \(a(x) \geq a_\ast > 0\), and \(f \in L^2(\Omega)\).
(a) What are the homogeneous essential boundary conditions for this problem?
(b) Prove that there is \(C > 0\) such that, if \(u \in H^1_0(\Omega) \cap H^2(\Omega)\), then

\[\|u\|_{H^2} \leq C\|\nabla \cdot a \nabla u\|_{L^2}.\]

[Hint: Use the Elliptic Regularity Theorem.]
(c) If we impose the boundary conditions \(u = 0\) and \(\nabla \cdot a \nabla u = g\) on \(\partial \Omega\), prove that there is a unique solution to the weak or variational form of the problem.

6. Let \(X\) be a Banach space and \(F : X \to X\) be a smooth map. Suppose that \(x_\ast\) is a simple root of \(F\) in the sense that \(F(x_\ast) = 0\) and the derivative \(DF(x_\ast)\) is invertible. Given any starting point \(x_0\), consider the full Newton iteration scheme

\[x_{k+1} = G(x_k) \quad \text{where} \quad G(x) = x - DF(x)^{-1}F(x).\]

Here we prove that if \(x_0\) is sufficiently close to \(x_\ast\), then \(x_k \to x_\ast\) as \(k \to \infty\).
(a) Show that \(G(x_\ast) = x_\ast\), \(DG(x_\ast) = 0\) and that there is a closed ball \(B\) about \(x_\ast\) such that

\[\|DG(x)\| \leq \frac{1}{2}\] for all \(x \in B\). [Hint: You do not need to compute \(DG(x)\), only \(DG(x_\ast)\).]
(b) Show that \(G(x) \in B\) for all \(x \in B\).
(c) Show that \(G : B \to B\) is a contraction.
(d) Prove that \(x_k \to x_\ast\) as \(k \to \infty\) for any \(x_0 \in B\).
Choose 7 out of the following 8 problems.

1. Consider vectors
   \[ a_1 = (1, 1), \quad a_2 = (0, 2) \]
   in \( \mathbb{R}^2 \). Verify that \( a_1, a_2 \) is a basis for \( \mathbb{R}^2 \). Let \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear transformation such that
   \[ Aa_1 = 2a_1 - a_2, \quad Aa_2 = -a_1 + \frac{1}{2}a_2 \]  
   (0.1)
   (a) Do conditions (0.1) define uniquely a linear transformation? Explain.
   (b) Is \( A \) a monomorphism, epimorphism, isomorphism? Why?
   (c) Write down the matrix representation of \( A \) with respect to basis \( a_1, a_2 \).
   (d) Compute the adjoint transformation \( A^* \) with respect to the canonical scalar product in \( \mathbb{R}^2 \).
   (e) Consider linear problem,
   \[ Ax = y \]  
   (0.2)
   and find necessary and sufficient condition for \( y \) for problem (0.2) to have a solution. Is the solution unique?

2. Consider open interval \((0, 1)\) and a sequence of functions \( f_n : (0, 1) \rightarrow \mathbb{R} \).
   \[ f_n(x) = \begin{cases} 
   n^2x & 0 < x < \frac{1}{n} \\
   -n^2(x - \frac{2}{n}) & \frac{1}{n} < x < \frac{2}{n} \\
   0 & \frac{2}{n} < x 
   \end{cases} \]
   (a) Determine pointwise limit \( f_0 \) of \( f_n(x) \).
   (b) State the Lebesgue Dominated Convergence Theorem.
   (c) Does \( f_n \) converge to \( f_0 \) in an \( L^p \)-norm? For what range of \( p \)? Explain.

3. Consider the initial-value problem,
   \[ \frac{dx}{dt} = x^3(t - 1), \ t \in (0, T) \quad x(0) = 1 \]
   (a) Use elementary means to solve the problem analytically. Determine the maximum value of \( T \) for which the solution exists.
   (b) State the Banach Contractive Map Theorem.
   (c) Use the theorem to demonstrate that, for sufficiently small \( T \), the problem has a unique solution in \((0, T)\). Compare the results obtained using both methods.
4. Consider space $C[0, 1]$ equipped with two topologies:

- **uniform convergence** topology implied by the Chebyshev metric,
  \[ d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \]

- **pointwise convergence topology** implied by bases of neighborhoods:
  \[ B(x_1, \ldots, x_N, \epsilon) = \{ g \in C[0, 1] : |f(x_i) - g(x_i)| < \epsilon, i = 1, \ldots, N \} \] (0.3)
  where $x_1, \ldots, x_N$ is an arbitrary, finite sequence in $[0, 1]$ (number $N$ of elements in the sequence may vary), and $\epsilon > 0$. In other words, the family of sets is parametrized with finite sequences and $\epsilon > 0$.

(a) Verify that the family (0.3) satisfies axioms for a base of neighborhoods (of $f$).

(b) Define the notion of stronger and weaker topologies and provide different equivalent criteria.

(c) Which of the two considered topologies is stronger? Explain.

5. Consider the heat equation defined on the simply connected, open domain $D$ in two space dimensions:

\[
\begin{align*}
t_u - \Delta u &= f(x, y, t), \quad (x, y) \in D, \quad t > 0, \\
u|_{\Gamma_D} &= g, \quad t > 0, (x, y) \in \Gamma_D \\
\nabla u \cdot \mathbf{n}|_{\Gamma_N} &= h, \quad t > 0, (x, y) \in \Gamma_N \\
u(x, y, 0) &= i(x, y), \quad (x, y) \in D.
\end{align*}
\]

where $\Gamma_D$ is the Dirichlet portion of the boundary $\partial D$, $\Gamma_N$ is the Neumann portion of the boundary, and $\mathbf{n}$ is the unit outward normal to the boundary of $D$.

Prove that if $f = g = h = i = 0$, the unique solution is $u \equiv 0$. by showing that

\[ \int_D u^2(x, y, T) dx dy = 0 \]

for any time $T > 0$. State how this result relates to the uniqueness of solutions to the general system above.

6. Give a closed form solution for the heat equation in the last problem if $D$ is the unit square $(0, 1) \times (0, 1), f = 1, \Gamma_D = \partial D$ (the entire boundary is Dirichlet), $g = 0$ and $i = \sin(\pi x) \cos(\pi y)$.

7. Prove the Cauchy-Goursat Theorem: If $f(z)$ is a complex function which is analytic in a simply connected domain $D$. Then for every simple closed contour $C$ in $D$.

\[ \int_C f(z) dz = 0. \]
8. Evaluate \( \int_C zdx + xdy + ydz \), where \( C \) is the counterclockwise curve obtained by intersecting the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2 \).
CSEM Area A-CAM Preliminary Exam (CAM 385C-D)
June 1, 2010, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Consider the operator $T : C([0, 1]) \to C([0, 1])$ defined by

$$Tf(x) := \int_0^x f(t) \, dt$$

(a) Determine the spectral radius of $T$.
(b) Determine the norm of $T$.
(c) Let $M := \{ f \in C([0, 1]) : f(0) = 0 \}$. Prove that the Banach quotient space $C([0, 1])/M$ is isomorphic to $\mathbb{C}$.

2. Let $X$ be an infinite dimensional Banach space over $\mathbb{C}$, and assume that $f, f_1, \ldots, f_n \in X^*$. Assume that there exists a constant $C > 0$ such that whenever $|f_j(x)| < C$ holds for all $j = 1, \ldots, n$ with $x \in X$, then $|f(x)| < 1$.

(a) Prove that $\bigcap_{j=1}^n \ker(f_j) \subset \ker(f)$.
(b) Prove that $f$ is a linear combination of the $f_j$. [Hint: Using (a), study the range $R(F) \subset \mathbb{C}^{n+1}$ of the map $F : X \to \mathbb{C}^{n+1}$ defined by $F(x) := (f_1(x), \ldots, f_n(x), f(x))$.]

3. Let $\mathcal{D}(\mathbb{R})$ denote the set of test functions on $\mathbb{R}$, and $\mathcal{D}'(\mathbb{R})$ the space of distributions.
(a) Assume that $\psi \in \mathcal{D}(\mathbb{R})$ is a test function such that $\psi(0) \neq 0$. For $n \in \mathbb{N}$, consider the functions

$$n \psi(nx), \quad \psi(x/n), \quad \text{and} \quad \sum_{k=0}^n \psi(x-k).$$

If they exist, find their limits as $n \to \infty$ pointwise and in $L^\infty(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$.
(b) Assume that $u \in \mathcal{D}'(\mathbb{R})$ satisfies $u(x^2 \phi) = 0$ for all $\phi \in \mathcal{D}(\mathbb{R})$. Prove that

$$u(\phi) = c_0 \phi(0) + c_1 \phi'(0) \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

4. Let the field be complex but $m \in L^\infty(\mathbb{R}^d)$ be real; in fact, let $0 < m_\ast \leq m(x)$ for a.e. $x \in \mathbb{R}^d$. Define the multiplier operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $Tu := (m \tilde{u})^\gamma$. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary, we can define a similar operator $\tilde{T} : L^2(\Omega) \to L^2(\Omega)$ by $\tilde{T}u := (m \tilde{E}_0u)^\gamma|_\Omega$, where $E_0$ is extension by zero and we restricted the result back to $\Omega$.
(a) Consider the differential equation

$$-\Delta u + Tu = f \quad \text{in } \mathbb{R}^d.$$

Prove that if $f \in H^{-1}(\mathbb{R}^d)$, then there is a unique solution $u \in H^1(\mathbb{R}^d)$.
(b) Show that $\tilde{T}$ is a bounded, symmetric, and strictly positive definite operator.
(c) Consider the boundary value problem

$$-\Delta u + \tilde{T}u = f \quad \text{in } \Omega \quad \text{and} \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Prove that if $f \in H^{-1}(\Omega)$, then there is a unique solution $u \in H^1(\Omega)$. 
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, and define

$$H(\text{div}; \Omega) := \{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \}.$$ 

(a) Show that $H(\text{div}; \Omega)$ is a Hilbert space with the inner-product

$$(u, v)_{H(\text{div})} := (u, v)_{(L^2(\Omega))^d} + (\nabla \cdot u, \nabla \cdot v)_{L^2(\Omega)}.$$ 

(b) The trace theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial \Omega$. Nonetheless, show that $\partial_\nu : H(\text{div}; \Omega) \to H^{-1/2}(\partial \Omega) = (H^{1/2}(\partial \Omega))^*$ is a well defined bounded linear operator in the sense of integration by parts:

$$\int_{\partial \Omega} v \cdot \nu \phi d\sigma(x) = \int_{\Omega} \nabla \cdot v \phi dx + \int_{\Omega} v \cdot \nabla \phi dx.$$ 

[Hint: In what space must $\phi$ lie?]

(c) Prove the following inf-sup condition: there exists $\gamma > 0$ such that

$$\inf_{w \in L^2(\Omega)} \sup_{v \in H(\text{div}; \Omega)} \frac{(w, \nabla \cdot v)_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)} \|v\|_{H(\text{div}; \Omega)}} \geq \gamma > 0.$$ 

6. Let the field be real and $G : C_0^B(\mathbb{R}) \to C_0^B(\mathbb{R})$ be defined by

$$G(u)(x) := \int_{\mathbb{R}} e^{-|x-y|} \frac{u^2(y)}{1 + u^2(y)} dy + \cos x.$$ 

(a) Show that $G$ has at least one fixed point.

(b) Find $DG(u)$ [You do not need to justify your result.]

(c) Let $G_0(u) := G(u)(0)$, so $G_0 : C_0^B(\mathbb{R}) \to \mathbb{R}$. Find all critical points of $G_0$. 
CSEM Area A-CSE Preliminary Exam

June 1, 2010

1. Suppose $K$ is a compact set in $\mathbb{R}^n$, $n > 1$.

   (a) State the Bolzano-Weierstrass Theorem for bounded sequences in $\mathbb{R}$.

   (b) Prove that every sequence $x_k$ in $K$ has a subsequence converging to a point in $K$.

   (c) Prove that if $f$ is a continuous function defined on $K$ taking on values in $\mathbb{R}$, then $f$ attains its supremum on $K$.

2. (a) Let $V$ denote the vector space $\mathbb{R}^3$ with basis vectors $a_1 = (1, 0, 0)$, $a_2 = (1, 1, 0)$ and $a_3 = (1, 1, 1)$, and $W$ the vector space $\mathbb{R}^2$ with canonical basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $T : V \rightarrow W$ be a linear transformation defined by $Ta_1 = -e_2$, $Ta_2 = 2e_2$ and $Ta_3 = e_1 + 3e_2$.

   - Determine the matrix representation of $T$.
   - Compute the adjoint of $T$ with respect to the canonical scalar inner product.
   - Determine the null space of $T$ and the rank of $T$.

   (b) Let $V$ be the vector space spanned by functions $\cos(nx)$ and $\sin(nx)$, $n = 1, \ldots, N$, where $N$ is finite (with the usual definitions of addition and scalar multiplication). Let $Tv = -v''$ be the linear transformation that maps $v \in V$ to its second derivative. Prove that if $y \in V$, then there exists an $x \in V$ with $T^2x = y$.

3. Let $X$ and $Y$ be two topological spaces and $f : X \rightarrow Y$. Prove that $f$ is continuous iff $f^{-1}(B) \subset f^{-1}(\overline{B})$ for every $B \subset Y$. 
4. Consider the following initial-value problem.

\[
\begin{align*}
\dot{x} + x &= \delta(t - 2) \\
x(0) &= \dot{x}(0) = 0
\end{align*}
\]

where \(\delta\) denotes the Dirac's delta.

(a) Explain how the presence of the delta functional translates into an interface condition, and solve the problem using elementary calculus.

(b) Define the Laplace transform for the delta distribution and compute it.

(c) Compute the Laplace transform of the solution to the initial-value problem.

(d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the "Laplace domain" and compare it with the solution obtained using the elementary calculus.

5. Use whatever means you know to solve the Laplace boundary-value problem in a half-circular domain shown in Fig. 1.

![Figure 1: Laplace equation in a half-circular domain.](image)

6. (a) Use the general formula for the curl in a curvilinear system of coordinates \(x = x(u_j)\).

\[
\nabla \times E = -\sum_{j=1}^{3} \frac{\partial E}{\partial u_j} \alpha^j
\]

(\(\alpha^j\) denote the co-basis vectors) to derive the formula for the curl in the standard cylindrical system of coordinates \((r, \theta, z)\).

(b) State the Stokes' theorem.

(c) Verify the Stokes' theorem by computing explicitly the involved volume and surface integrals for a cylindrical domain.

\[
D = \{(r, \theta, z) : r < 1, 0 < z < 2\}
\]

and the vector field \(E = re_r + ze_z\), where \(e_r, e_z\) denote the unit vectors of the cylindrical system of coordinates.

*Hint:* The use of cylindrical coordinates makes the problem easier.
CSEM Area A-CAM Preliminary Exam (CAM 385C–D)
May 31, 2011, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let \( f_n \) converge weakly to \( f \) in \( L^2([0,1]) \) and define

\[
F_n(x) := \int_0^x f_n(\xi) \, d\xi \quad \text{and} \quad F(x) := \int_0^x f(\xi) \, d\xi.
\]

(a) Show that \( F_n(x) \) converges to \( F(x) \) pointwise.
(b) Show that \( F_n(x) \) converges to \( F(x) \) weakly in \( H^1([0,1]) \).
(c) Show that \( F_n(x) \) converges uniformly on \([0,1]\) to \( F(x) \). [Hint: Given \( \epsilon > 0 \) and fixed \( x \in [0,1] \), consider the points \( x_j = \epsilon j \) and \( x^* = \epsilon j^* \) nearest \( x \).]

2. Let \( X \) be a Banach space and \( T : X \to X \) a bounded linear operator. Suppose further that \( \|T\| < 1 \).
   (a) State the meaning of \( \|T\| \) and show that \( \|T^2\| \leq \|T\|^2 \).
   (b) Prove carefully that \( I - T \) has an inverse, given by the Neumann series

\[ I + T + T^2 + T^3 + \cdots. \]

(c) For \( S : X \to X \) a bounded linear operator, show that the spectrum of \( S \) is contained in the closed circle of radius \( \|S\| \) in the complex plane.

3. For \( f \in L^2(\mathbb{R}^d) \), consider a solution \( u \in H^2(\mathbb{R}^d) \) to the problem

\[ u - \Delta u = f. \]

(a) Show that the solution is unique using the Fourier Transform.
(b) If \( f \in H^s(\mathbb{R}^d), \ s \geq 0 \), find \( r \) such that \( u \in H^r(\mathbb{R}^d) \). Justify your answer.
(c) For what values of \( s \) can you be sure that a fundamental solution is continuous?

4. Suppose that \( u \in H^1(\Omega) \), where \( \Omega \subset \mathbb{R}^d \) is a bounded domain. Define the \( H^1(\Omega) \)-seminorm by

\[ |u|_{H^1(\Omega)} = \left\{ \sum_{|\alpha|=1} \|D^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2}. \]

(a) Show that there is some constant \( C_\Omega \), depending on \( \Omega \) but not on \( u \), such that

\[ \inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\Omega)} \leq C_\Omega |u|_{H^1(\Omega)}. \]

[Hint: You may take \( c = \bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) \, dx \) and argue by contradiction.]
(b) Let \( \Omega = (0,h)^d \). Show that there is a constant \( C \), independent of \( \Omega \) and \( u \), such that

\[ \inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\Omega)} \leq C \, h \, |u|_{H^1(\Omega)}. \]

[Hint: Change variables to integrate over \((0,1)^d\), and use (a).]
5. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. The equations of linear elasticity can be formulated for the displacement vector $\mathbf{u} = (u_1, u_2)^T$ using the symmetric gradient tensor $\mathbf{e}(\mathbf{u})$ defined by

$$
\epsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

as

$$
\sum_i \frac{\partial}{\partial x_i} \left[ 2\mu \epsilon_{ij}(\mathbf{u}) + \lambda \nabla \cdot \mathbf{u} \delta_{ij} \right] = f_j \quad \text{in } \Omega,
$$

where $\mu$ and $\lambda$ are positive constants. We assume the boundary condition $\mathbf{u} = 0$ on $\partial \Omega$.

(a) Develop a weak form for the equations that involve the integrals ($\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}$) and $\sum_{i,j} (\epsilon_{ij}(\mathbf{u}), \epsilon_{ij}(\mathbf{v}))$. For the latter, you will need to use the fact that $\mathbf{e}(\mathbf{u})$ is indeed symmetric.

(b) In which spaces should $\mathbf{u}$ and $\mathbf{f} = (f_1, f_2)^T$ lie?

(c) Show that there is a unique solution to the problem using the Lax-Milgram Theorem.

You will need to use Korn's Inequality, which says that there is $\gamma > 0$ such that

$$
\|\mathbf{e}(\mathbf{u})\|_{(L^2(\Omega))^{2\times2}} \geq \gamma \|\mathbf{u}\|_{(H^1(\Omega))^2} \quad \forall \mathbf{u} \in (H^1_0(\Omega))^2.
$$

6. Set up and apply the contraction mapping principle to show that the boundary value problem

$$
\begin{align*}
\epsilon u'' - \epsilon u^2 &= f(x), \quad x \in (0, 1), \\
u(0) &= \alpha \text{ and } u'(1) = \beta,
\end{align*}
$$

has a unique, continuous solution if $\epsilon > 0$ is small enough, where $f(x)$ is a smooth function on $[0, 1]$. 

1. Let $V$ and $W$ be finite dimensional vector spaces and $A : V \to W$ a linear transformation. State the Rank and Nullity Theorem relating the dimension of $V$ to the nullity and rank of $A$. Use this theorem to prove:

(i) $A$ is a monomorphism if and only if the rank $A = \dim V$.

(ii) $A$ is an epimorphism if and only if rank $A = \dim W$.

(iii) Give an example of a matrix $A$ which is left-invertible but not right-invertible.

2. Let $f$ be a nonnegative, real-valued, Lebesgue integrable function which is summable over measurable set $E$.

(i) Prove that if $f$ is bounded, then given $\epsilon > 0$, there exists a $\delta > 0$ such that for every set $A \subset E$, with meas $(A) < \delta$, we have

$$\int_A f < \epsilon.$$ 

(ii) Prove the result if $f$ is unbounded. Hint: consider a sequence of functions $f_n(x)$ with $f_n(x) = f(x)$ when $f(x) \leq n$ and $f_n(x) = n$ otherwise.

3. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \rho\}$.

(i) Give an example of a nontrivial topology on $\mathcal{X}$.

(ii) Consider the set $A = \{\beta, \gamma\}$. In your topology, determine the interior of $A$ and the closure of $A$.

4. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \dot{x} = -H(t - 2) \\ x(0) = \dot{x}(0) = 0, \end{cases}$$

where $H$ denotes the Heaviside function.

(a) Solve the problem using elementary means.

(b) Define the Laplace transform. Apply it to both sides of the equation and find the solution in the Laplace domain.

(c) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the Laplace domain and compare it with the solution obtained using the elementary calculus.
5. (a) Define the Green function for the boundary-value problem shown in Fig. 1.

![Laplace equation in a quarter-space domain](image)

Figure 1: Laplace equation in a quarter-space domain.

(b) Use the free-space Green function

$$\Phi = \frac{1}{2\pi} \ln r$$

and the method of images to determine the Green function for the problem.

6. (a) Use the general formulas for curl and div in a curvilinear system of coordinates $\xi_i$,

$$\nabla \times E = -\frac{\partial E}{\partial \xi_i} \times a^i, \quad \nabla \cdot v = \frac{\partial v}{\partial \xi_i} \cdot a^i$$

where $a^i$ are the co-basis vectors, to derive particular formulas for the standard cylindrical coordinates.

(b) Use the formulas to verify that the following vector field is solenoidal.

$$v_r = -\theta, \quad v_\theta = r, \quad v_z = \frac{\theta z}{r}$$

(c) Define vector potential $A$. Find a vector potential for field $v$.  


CSEM Area A-CAM Preliminary Exam (CSE 386C/D)
May 31, 2012, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.
Remark: \(\mathbb{N}\) is the set of natural numbers \(\{1, 2, 3, \ldots\}\).

(a) Show that \(Y_1 = \{x = (x_1, x_2, x_3, \ldots) \in \ell^2 \mid x_{2n} = 0, n \in \mathbb{N}\}\) is a closed subspace of \(\ell^2\), and find \(Y_1^\perp\).
(b) What is \(Y_2^\perp\) if \(Y_2 = \text{span}\{e_1, \ldots, e_n\} \subset \ell^2\), where \(e_{j,k} = \delta_{j,k}\)?
(c) Take \(x = (1, 2, 3, 4, 0, 0, 0, \ldots)\). What are the orthogonal projections \(P_{Y_1}(x)\) and \(P_{Y_2}(x)\), and the distances \(d(x, Y_1)\) and \(d(x, Y_2)\)?

2. Let \(T_t\) be the operator \(T_t(\varphi)(x) = \varphi(x + t)\) on \(L^2(\mathbb{R})\).
(a) What is the norm of \(T_t\)?
(b) Show that \(T_t\) does not converge as \(t \to \infty\) in \(B(L^2(\mathbb{R}), L^2(\mathbb{R}))\).
(c) To what operator does \(T_t\) converge as \(t \to \infty\) if the Hilbert space is \(L^2(\mathbb{R}, e^{-x^2}dx)\)?

3. Let \(H\) be a Hilbert space and \(A\) a bounded linear operator on \(H\). Recall that \(|A| = (A^*A)^{1/2}\) is a self-adjoint, bounded linear operator. A bounded linear operator \(U\) on \(H\) is a partial isometry if \(\|Ux\| = \|x\|\) for all \(x \in N(U)^\perp\) (i.e., \(U\) is an isometry except on its nullspace, where it is zero).
(a) Show that \(\|A^*A\| = \|A\|^2\) for all \(x \in H\).
(b) Show that \(H = R(|A|) \oplus N(|A|)\) and that \(N(|A|) = N(A)\).
(c) Show that there exists a partial isometry \(U\) such that \(A = U|A|\). [Hint: define \(U : R(|A|) \to R(A)\) by \(U(|A|x) = Ax\) (is this well defined?) and extend \(U\) first to \(R(|A|)\) and then to all of \(H\).]

4. Consider \(x \in \mathbb{R}^{n \times n}\) with associated \(\ell^2\)-norm \(\|x\|_2 = \left(\sum_{j=1}^{n} \sum_{k=1}^{n} |x_{j,k}|^2\right)^{1/2}\) and total-variation semi-norm
\[
|x|_{TV} = \sum_{j=1}^{n} \sum_{k=1}^{n-1} |x_{j,k+1} - x_{j,k}| + \sum_{j=1}^{n-1} \sum_{k=1}^{n} |x_{j+1,k} - x_{j,k}|.
\]
(a) Why is \(|\cdot|_{TV}\) not a norm?
(b) Prove directly the following Sobolev inequality: if \(x_{1,j} = x_{j,1} = 0\) for all \(1 \leq j \leq n\), then \(\|x\|_2 \leq \frac{1}{2} |x|_{TV}\). [Hint: Find two ways to write \(x_{j,k}\) as a sum of discrete differences.]
(c) Does the result in (b) extend to \(\ell^2(\mathbb{N} \times \mathbb{N})\)?
5. Consider the following problem in $\Omega \subset \mathbb{R}^d$, a $C^1$ domain. Given functions $f$, $g$, and $c \geq 0$, find a solution pair $(u, w)$ to the partial differential boundary value problem (BVP)

$$\begin{align*}
-\Delta u &= c(w - u) + f \quad \text{in } \Omega, \\
-\Delta w &= c(u - w) + g \quad \text{in } \Omega, \\
w &= 0 \quad \text{and} \quad u = 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(a) Determine appropriate Hilbert spaces within which the solution pair $(u, w)$ and functions $f$, $g$, and $c$ should lie, and formulate an appropriate variational problem for the BVP.

(b) Show that the BVP and your variational problem are equivalent.

(c) Show that there is a unique solution to the variational problem.


(a) State the Implicit Function Theorem.

(b) Let $X$ and $Y$ be Banach spaces. Let both $F : X \to Y$ and $G : X \to Y$ be $C^1$ on $X$, and $H(x, \epsilon) = F(x) + \epsilon G(x)$, for $\epsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \epsilon) = 0$ for $\epsilon$ sufficiently close to $0$.

(c) For small $\epsilon$, prove that there is a solution $w \in H^2(0, \pi)$ to

$$w'' = w + \epsilon w^2, \quad w(0) = w(\pi) = 0.$$
Solve the following six problems in 3 hours.

1. It is known (you may assume) that if \( f \) and \( g \) are nonnegative, measurable functions, then

\[
\int_{E} (f + g) = \int_{E} f + \int_{E} g
\]

Use this result to prove the following two propositions:

- Let \( f \) and \( g \) be nonnegative measurable functions, with \( f \) summable over \( E \) and \( g(x) < f(x) \). Prove

\[
\int_{E} (f - g) = \int_{E} f - \int_{E} g
\]

and \( g \) is also summable over \( E \).
- If \( f \) and \( g \) are summable over \( E \), then

\[
\int_{E} (f + g) = \int_{E} f + \int_{E} g
\]

2. Let \( T : V \rightarrow W \), where \( T \) is a linear transformation and \( V \) and \( W \) are finite dimensional vector spaces of dimension \( n \) and \( m \), respectively. Discuss the solvability (existence and uniqueness) of the equation \( Tx = y \), if

- \( m = n \)
- \( m < n \)
- \( m > n \)

3. Let \( X \) be a metric space, define the following:

- A metric on \( X \)
- An open set on \( X \)
- A closed set on \( X \)
- A compact set on \( X \)

Prove that compact subsets of metric spaces are closed. Hint: Prove that the complement of the subset is open.
4. Consider the following initial-value problem.

\[
\begin{align*}
\dot{x} - \dot{\dot{x}} &= -\delta(t - 2) \\
x(0) &= 1, \quad \dot{x}(0) = 0,
\end{align*}
\]

where $\delta$ denotes the Dirac's delta "function".

(a) Define precisely delta functional and reinterpret its action in terms of appropriate jump conditions.

(b) Solve the problem using elementary means.

(c) Define the Laplace transform. Apply it to both sides of the equation and find the solution in the Laplace domain.

(d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the Laplace domain and compare it with the solution obtained using the elementary calculus.

5. (a) Recall the formula for the gradient in polar coordinates,

\[
\nabla u = \frac{\partial u}{\partial r} e_r + \frac{1}{r} \frac{\partial u}{\partial \theta} e_\theta
\]

and use integration by parts to derive the corresponding formula for the Laplacian.

(b) Use separation of variables and whatever means you need to solve the boundary-value problems shown in Fig. 1. Symbol $u_n$ stands for the normal derivative of function $u$.

![Figure 1: Steady-state heat conduction in a wedge domain.](image)
6. (a) Recall elementary Green's formula (integration by parts) in $\mathbb{R}^2$.

(b) Use the formula to derive the integration by parts formula for the curl operator in 2D:

$$\iint_D (\text{curl} \mathbf{f}) g \, dA = \iint_D f \cdot (\nabla \times g) \, dA + \int_{\partial D} (\mathbf{n} \times f) g \, ds$$

where $\mathbf{n}$ is the outward normal unit vector. Fill in the appropriate definitions of scalar-valued curl of a vector-valued function $\mathbf{f}(x, y)$, vector-valued $\nabla \times$ of scalar-valued functions $g(x, y)$ and scalar-valued vector product of vectors $\mathbf{n}$ and $\mathbf{f}$.

(c) Verify the formula by computing all involved integrals for a quadrant of a circle shown in Fig 2 and

$$\mathbf{f} = (-y, x), \quad g = 1$$

![Figure 2: Domain $D$ (quadrant of a circle with radius 2).](image-url)
Solve the following six problems in 3 hours.

1. Linear algebra sanity check.

   Consider rotation $A$ (about the origin) about angle $\alpha$ in space $\mathbb{R}^2$.

   (i) Write down matrix representation of the rotation in the canonical basis $e_1 = (1, 0)$, $e_2 = (0, 1)$.

   Matrix representation of a linear map $A$ is defined by
   \[
   Ae_j = \sum_i A_{ij}e_i
   \]

   Consequently, in the canonical basis:
   \[
   A_{ij} = \begin{pmatrix}
   \cos \alpha & -\sin \alpha \\
   \sin \alpha & \cos \alpha
   \end{pmatrix}
   \]

   (ii) Write down matrix representation of the rotation in the basis basis $a_1 = (1, 0)$, $a_2 = (1, 1)$.

   A bit more work.

   \[
   Aa_1 = Ae_1 = \cos \alpha e_1 + \sin \alpha e_2 = \cos \alpha a_1 + \sin \alpha (a_2 - a_1) = (\cos \alpha - \sin \alpha) a_1 + \sin \alpha a_2
   \]

   and,

   \[
   Aa_2 = A(e_1 + e_2) = \cos \alpha e_1 + \sin \alpha e_2 + (-\sin \alpha e_1 + \cos \alpha e_2) = (\cos \alpha - \sin \alpha) e_1 + (\cos \alpha + \sin \alpha) e_2 = (\cos \alpha - \sin \alpha) a_1 + (\cos \alpha + \sin \alpha) (a_2 - a_1) = -2 \sin \alpha a_1 + (\cos \alpha + \sin \alpha) a_2
   \]

   So, the matrix representation is:

   \[
   A_{ij} = \begin{pmatrix}
   \cos \alpha - \sin \alpha & -2 \sin \alpha \\
   \sin \alpha & \cos \alpha + \sin \alpha
   \end{pmatrix}
   \]

   (iii) Consider the canonical inner product in $\mathbb{R}^2$:

   \[(x, y) := x_1y_1 + x_2y_2\]
and determine the cobasis $\alpha^1, \alpha^2$ (with respect to the canonical inner product). By inspection, 

$$\alpha^1 = (1, -1), \alpha^2 = (0, 1).$$

(iv) Determine the adjoint $A^*$ of map $A$ with respect to the inner product and find its representation, first in basis $e_1, e_2$, and then in basis $\alpha_1, \alpha_2$. The adjoint in the canonical basis is simply the transpose of the matrix, 

$$A^*_{ij} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix}$$

which reveals that the adjoint is rotation by angle $-\alpha$, the inverse of $A$, so $A^* = A^{-1}$ (rotation is an orthonormal map). Consequently, matrix representation of the adjoint in basis $\alpha_1, \alpha_2$ can be obtained by replacing $\alpha$ with $-\alpha$ in the matrix representation of $A$, 

$$A_{ij} = \begin{pmatrix} \cos \alpha + \sin \alpha & 2 \sin \alpha \\ -\sin \alpha & \cos \alpha - \sin \alpha \end{pmatrix}$$

(v) Determine matrix representation of $A^*$ in cobasis $\alpha^1, \alpha^2$.

Simple. Matrix representation of the adjoint in the cobasis, for any linear map $A$, equals the transpose of the matrix representation of the map in the original basis, 

$$A^*_{ij} = \begin{pmatrix} \cos \alpha - \sin \alpha & \sin \alpha \\ -2 \sin \alpha & \cos \alpha + \sin \alpha \end{pmatrix}$$
2. Prove that any two norms in a finite dimensional vector space are equivalent. Explain why the result implies that we can talk about the topology for finite dimensional vector spaces.
3. Consider the following Initial-Value Problem (IVP):

\[
\begin{align*}
\frac{dq}{dt} &= t \ln^2(q(t)), \quad t > 0 \\
q(0) &= 1
\end{align*}
\]

- State Banach Contractive Map Theorem.

Let \((X, d)\) be a complete metric space. Let \(D \subset X\) (then \((D, d)\) is itself a metric space, too...), and \(A : D \to D\) is a contraction, i.e.

\[d(A(f), A(g)) \leq k d(f, g), \quad \forall f, g \in D, \quad k < 1\]

Then function \(A\) has a unique fixed point in set \(D\).

- Use the theorem to prove local existence and uniqueness of solution to the IVP, i.e. that there exists an interval \((0, T)\) in which the equation is satisfied. Provide a concrete value of \(T\).

The problem is equivalent to the solution of the integral equation:

\[q(t) = 1 + \int_0^t s \ln^2(q(s)) \, ds\]

Consider the Chebyshev space \(C[0, T]\) (with an unknown \(T\) at this point...) and define the map \(A\) using the right-hand side of the equation above:

\[(Aq)(t) = 1 + \int_0^t s \ln^2(q(s)) \, ds\]

First of all, we need to define a set \(D \subset C[0, T]\) such that map \(A\) sets the set \(D\) into itself. Assume that \(q(t)\) will vary in the box:

\[D = \{q \in C[0, T] : e^{-2} \leq q(t) \leq e^2, \quad 0 \leq t \leq T\}\]

(notice that the box includes the initial value \(q = 1\)). Then \(-2 \leq \ln q(t) \leq 2\), i.e. \( |\ln q(t)| \leq 2\), so \(\ln^2 q(t) \leq 4\). Consequently,

\[|\int_0^t s \ln^2 q(s) \, ds| \leq 4 \int_0^t s \, ds = 2t^2\]

so,

\[|(Aq)(t) - 1| \leq 2T^2\]

This gives two bounds for \(T\). From the right:

\[(Aq)(t) \leq 1 + 2T^2 \leq e^2 \quad \Rightarrow \quad T \leq \sqrt{\frac{e^2 - 1}{2}},\]

and from the left:

\[e^{-2} \leq 1 - 2T^2 \leq (Aq)(t) \quad \Rightarrow \quad T \leq \sqrt{\frac{e^2 - 1}{2e^2}}.\]
Now, map $A$ must be a contraction. With flux $F(s, q) = s \ln^2 q$, 

$$\left| \frac{\partial F}{\partial q} \right| = 2s |\ln q| \leq 4s$$

so the flux satisfies the Lipschitz condition:

$$|F(s, q_1) - F(s, q_2)| \leq 4s |q_1 - q_2| .$$

This leads to the estimate;

$$|(Aq_1)(t) - (Aq_2)(t)| \leq \int_0^t 4s \ ds \|q_1 - q_2\|_{C[0,T]} \leq 2T^2 \|q_1 - q_2\|_{C[0,T]} .$$

Consequently, a sufficient condition for contraction is

$$T < \sqrt{\frac{1}{2}} .$$

Out of the three conditions for the maximum $T$ the most restrictive is the second one, so it is sufficient to assume that

$$T \leq \sqrt{\frac{e^2 - 1}{2e^2}} .$$
4. Use the Residue Theorem to find the inverse Fourier Transform

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega \]

for \( \hat{f}(\omega) = \frac{1}{(2-i\omega)^2} \).

Same idea as in (a). This time the only root \( \omega = 2/i = -2i \) is in the lower half plane, so for \( x > 0 \) (choosing above) we get \( f(x) = 0 \).

And for \( x < 0 \) (choosing below) we get what? The integrand is

\[ \frac{1}{2\pi i} \frac{e^{i\omega x}}{(2-i\omega)^2} = -\frac{1}{2\pi i} \frac{e^{i\omega x}}{\omega + 2i} \]

So for \( x < 0 \) we have

\[ f(x) = -2\pi i (\text{Res} @ -2i) = -2\pi i \frac{d}{d\omega} \left( -\frac{1}{2\pi i} \frac{e^{i\omega x}}{\omega + 2i} \right) \bigg|_{\omega = -2i} \]

\[ = -xe^{i2x} \]
5. Verify Stokes Theorem for the function $v = yz k$, where $S$ is the surface $z = 1 - x^2 - y^2$ cut off by the planes $x = 0$, $y = 0$ and $z = 0$, and $C$ is oriented as shown in Fig. 1.

![Figure 1: A surface $S$ with curved boundary $C$.]

$S: z = 1 - x^2 - y^2$ is a paraboloid of revolution, not a sphere.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} yz \, dz + \int_{C_2} z \, dx + \int_{C_3} z \, dy$$

$$= \int_{C_1} yz \, dz,$$ on which $z = 1 - y^2$. Take $y$ as the parameter for $C_1$.

$$= \int_0^1 y(1-y^2)(-2y \, dy) = \frac{4}{15}$$

Surface integral:

$$\mathbf{F} = \nabla (x^2 + y^2 + z) = 2x \mathbf{i} + 2y \mathbf{j} + z \mathbf{k}, \quad \mathbf{n} = \frac{(2x \mathbf{i} + 2y \mathbf{j} + z \mathbf{k})}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\mathbf{v} \cdot \mathbf{n} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & 2y & z \\ \frac{2x \mathbf{i} + 2y \mathbf{j} + z \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}} \end{array} \right| = z \mathbf{k}$$

$$\iint_S (2x \mathbf{i} + 2y \mathbf{j} + z \mathbf{k}) \cdot \frac{\mathbf{n} \, dA}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$= \iint_R 2x(1-x^2-y^2) \, dxdy = \int_0^1 \int_0^{\pi} 2 \pi \rho^2 (1-\rho^2) \rho \, d\theta \, d\rho = \frac{4}{15}. \quad \checkmark$$
6. Solve the Laplace equation $\Delta u = 0$ on the unit square in $\mathbb{R}^2$ subject to the boundary conditions that $u = 1$ on all sides but at $x = 1$, where $u_x(1,y) = f(y)$.

$$\nabla = \bar{u} + 1, \quad \bar{u} = X Y$$

$$\frac{\bar{X}''}{\bar{X}} = -\frac{\bar{Y}''}{\bar{Y}} = k^2$$

$$\bar{Y}'' + k^2 \bar{Y} = 0$$

$$\bar{Y}(0) = B = 0$$

$$\bar{Y}(\pi) = A \sin k \pi = 0 \quad k = n \pi \quad n, 1, 2, \ldots$$

$$\bar{X}'' - k^2 \bar{X} = 0$$

$$\bar{X}(0) = C + D = 0$$

$$\bar{X}(\pi) = C (e^{k\pi} - e^{-k\pi})$$

$$\bar{U}(x) = \sum A_n (e^{k_n x} - e^{-k_n x}) \sin k_n y$$

$$\bar{U}_x(1,y) = \sum A_n k_n (e^{k_n} + e^{-k_n}) \sin k_n y = f(y)$$

$$= \sum A_n k_n \sin k_n y$$

solve for $D_n = \left( \frac{f(y) \sin k_n y}{(e^{k_n} + e^{-k_n}) \sin k_n y} \right)$

$$A_n = \sum D_n \frac{k_n (e^{k_n} + e^{-k_n})}{(e^{k_n} + e^{-k_n}) \sin k_n y}$$

$$\Rightarrow \quad U = \sum A_n (e^{k_n x} - e^{-k_n x}) \sin k_n y + 1$$
CSEM Area A-CAM Preliminary Exam (CSE 386C / 386D)

May 30, 2013, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let $H$ be a real Hilbert space and suppose that $P$ is a bounded linear projection on $H$. Let $Q = I - P$ and define $M = P(H)$ and $N = Q(H)$. Suppose that $M$ and $N$ are closed.
   (a) Show that there exists $C > 0$ such that
   \[ \| x - Px \| \leq C \inf_{y \in M} \| x - y \| \quad \text{for all } x \in H. \]
   [Hint: Relate this to the orthogonal projection $P_M$.]
   (b) Prove that $P$ is an orthogonal projection if and only if
   \[ \inf_{y \in N, \| y \| = 1} \inf_{x \in M} \| y - x \| = 1. \]
   [Hint: For the converse, it is enough to show that for any $z \in H$, $z - Pz = Qz \perp M$. Consider $y = Qz/\| Qz \|$.]

2. Let $X$ be a Banach space with dual $X^*$. Let $\{L_n\}_{n=1}^{\infty} \subset X^*$ and $\{x_n\}_{n=1}^{\infty} \subset X$. Assume that $L_n \to L \in X^*$ in the weak-* sense, and $x_n \to x$ in the norm of $X$.
   (a) State the Uniform Boundedness Principle.
   (b) Show that if $X$ is a reflexive Banach space, then $L_n(x_n) \to L(x)$.

3. Let $\Omega = (-1,1)^2$ and define
   \[ H = \{ u \in H^2(\Omega) : u(x,0) = u_y(0,y) = 0 \text{ for a.e. } x, y \in (-1,1) \}. \]
   (a) Why is $H$ a well defined, complete linear subspace of $H^2(\Omega)$?
   (b) Prove that there is some $C > 0$ such that for $u \in H$,
   \[ \| u \|_{H^1(\Omega)} \leq C \{ \| u_{xx} \|_{L^2(\Omega)} + \| u_{xy} \|_{L^2(\Omega)} + \| u_{yy} \|_{L^2(\Omega)} \}. \]

4. Consider the Telegrapher’s equation
   \[ u_{tt} + 2u_t + u = c^2 u_{xx} \quad \text{for } x \in \mathbb{R} \text{ and } t > 0; \]
   with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ given in $L^2(\mathbb{R})$.
   (a) Use the Fourier transform (in $x$ only) and its inverse to find a representation of the solution. [Hint: The solution to $y'' + 2y' + (1 + \alpha^2)y = 0 \text{ is } y(t) = e^{-\alpha t}(A \cos(\alpha t) + B \sin(\alpha t))$.]
   (b) Justify that your representation is indeed a solution. You may assume that $f$ and $g$ are in the Schwartz space.
5. Contraction mappings.
(a) State the contraction-mapping theorem.
(b) Consider $K \in C^0([0,1]^2)$. Show that for $\lambda$ small enough, for any $g \in C^0([0,1])$, there exists a unique solution $f \in C^0([0,1])$ to

$$f(x) = g(x) + \lambda \int_0^1 K(x,y) f(y) \, dy.$$ 

6. Let $\Omega$ be a bounded smooth domain with $\nu$ being the normal vector on its boundary. Consider the solution $(u,v)$ of the differential problem

$$u - \Delta u = f + au, \quad \text{in } \Omega,$$

$$-\Delta w = g - au, \quad \text{in } \Omega,$$

$$\nabla u \cdot \nu = \gamma \text{ and } w = 0, \quad \text{on } \partial \Omega,$$

where $a \in L^\infty(\Omega)$.
(a) Provide an appropriate weak form for the problem. In what Sobolev spaces should $f$, $g$, and $\gamma$ lie?
(b) Prove that there exists a unique solution to the problem.
1. \( H \ni P : H \rightarrow M \), \( Q = I - P : H \rightarrow N \)

(a) \[
\inf_{y \in N, \|y\| = 1} \|y - x\| = \|x - P_m x\| = \|P_m^\perp x\|
\]
\[
x = P_m x + P_m^\perp x
\]
\[
P_m x = P_m x + PP_m^\perp x
\]
\[
x - P_m x = x - P_m x + PP_m^\perp x = (I + PP_m^\perp) PP_m^\perp x
\]
\[
\Rightarrow \|x - P_m x\| \leq (1 + \|P\|) \|PP_m^\perp x\|
\]

(b) \( P = P_m \iff \inf_{y \in N, \|y\| = 1} \|y - x\| = 1 \).

\( \Rightarrow \) \( P = P_m \Rightarrow N = M^\perp \)
\[
\Rightarrow \|y - x\|^2 = \|y\|^2 + \|x\|^2 = 1 + \|x\|^2
\]
\[
\Rightarrow \inf_{x \in M} \text{ over } x \in M \text{ is } 1.
\]

\( \Leftarrow \) For any \( z \in H \), \( Qz \neq 0 \),

Let \( y = \frac{Qz}{\|Qz\|} \)
\[
\Rightarrow 1 \leq \inf_{x \in M} \frac{\|Qz\|}{\|Qz\| - x} = \|P_m^\perp \left( \frac{Qz}{\|Qz\|} \right)\|
\]
\[
\leq \frac{\|Qz\|}{\|Qz\|} = 1
\]
\[
\Rightarrow \frac{\|Qz\|}{\|Qz\|} = 1 \Rightarrow Qz \in M^\perp
\]
Thus \( z - P_m z = Qz \perp M \)

If \( Qz = 0 \), then \( z = P_m z \in M \)

In general, \( z - P_m z \perp M \), so \( P = P_m \).
2. \( \exists L_n \in X \), \( \exists x_n \in X \)

\[ L_n \xrightarrow{w^*} L, \quad x_n \rightarrow x \]

\( \text{(b) } \quad X = x^{**} \)

\[ L_n \xrightarrow{w^*} L \iff L_n(x) \rightarrow L(x) \quad \forall x \in X \]

\[ L_n(x_n) = L_n(x) + L_n(x_n - x) \]

\[ \Rightarrow \]

\[ L(x) \]

UBP \( \Rightarrow \) \( \|L_n\| \leq M \) or

\[ \exists x \text{ s.t. sup } |L_n(x)| = \infty. \]

But \( \quad L_n(x) \rightarrow L(x) \Rightarrow L_n(x) \text{ bounded.} \]

Thus

\[ \|L_n (x_n - x)\| \leq M \|x_n - x\| \rightarrow 0. \]

\[ \Rightarrow \quad L_n(x_n) \rightarrow L(x). \]

(a) UB: \( X \) Banach, \( Y \) NLS

\[ \exists \|T\| < B(X,Y) \Rightarrow \]

(i) \( \exists M \) s.t. \( \|T\| \leq M \) \( \forall \lambda \)

or (ii) \( \exists x \in X \) s.t. \( \sup_{\alpha} \|T_\alpha(x)\| = \infty. \)
3. \( \Omega = (-1,1)^2, \ H = \{u \in H^2: u(x,0) = u_y(0,y) = 0 \text{ a.e.} \} \)

(a) The trace operator is well defined down 1/2 dimension, so \( u(x,0) \) and \( u_y(0,y) \) exist and are cont. That is, \( \sigma = u(x,0) \), \( \tau = u_y(0,y) \) are well-defined and cont.

\[
H = \{u \in H^2: \nabla u = \begin{pmatrix} \sigma \\ \tau \end{pmatrix}, u = 0 \} = \mathcal{R}(\sigma) \cap \mathcal{R}(\tau) \text{ is closed, so a complete lin. subsp.}
\]

(b) Suppose not. Then \( \exists\) a seq. \( u_n \in H \) such that

\[
\|u_n\|_{H^2}^2 + \|u_{nx}\|_{L^2}^2 + \|u_{nxy}\|_{L^2}^2 \leq \frac{1}{n}
\]

But \( \|u_n\|_{H^2}^2 = 1 \).

But then \( u_{nx}, u_{nxy}, u_{nxy} \to 0 \)

Now \( \|u_n\|_{L^2}^2 \leq \frac{1}{n} \Rightarrow \exists\) subseq. \( u_{n_k} \to u \in H^2 \) and \( u_{n_k} \to u \) in \( H \).

Thus \( u \) is linear: \( u = a + \beta x + \gamma y \)

So \( u \in H \Rightarrow a + \beta x = 0 \text{ a.e.} \), \( \gamma = 0 \)

\( \Rightarrow a = -\beta \Rightarrow u = 0 \), contradicting that \( u_n \to u \)

\( \|u_n\|_{L^2} = 1 \), \( \|u\|_{L^2} = 0 \). ❌
4. \[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = c^2 \frac{\partial^2 u}{\partial x^2}, & t > 0 \\
u(x,0) = f, \quad u_t(x,0) = g
\end{cases}
\]

(a) \[
\hat{u}_{tt} + 2\hat{u}_t + \hat{u} = -c^2 |\hat{v}|^2 \hat{v}
\]

\[
\Rightarrow \hat{u}_{tt} + 2\hat{u}_t + (1+c^2 |\hat{v}|^2)\hat{u} = 0.
\]

\[
\Rightarrow \hat{u} = e^{-t}(A(3)\cos(c5t) + B(3)\sin(c5t))
\]

\[
\hat{u}(3,0) = \hat{f} = A(3),
\]

\[
\hat{u}_t(-3,0) = -\hat{u} + e^{-t}(-3A\sin(c5t) + 3B\cos(c5t))
\]

\[
\hat{u}_t(3,0) = -\hat{f} + 3B(3) = \hat{g}
\]

\[
\Rightarrow \hat{u}(3,t) = e^{-t}(\hat{f}(3)\cos(c5t) + \frac{\hat{g} + \hat{f}}{3} \sin(c5t))
\]

\[
\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int f(x - c3t) \cdot (2\pi)^{-1/2}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int [f * \frac{1}{c3} \int \cos(c5t) + (g+f) * \frac{1}{c3} \int \frac{\sin(c5t)}{c3}] d\tau
\]

(b) Since \( \cos(c3t) \) \( \frac{\sin(c5t)}{c3} \) \( \in C^\infty \cap L^\infty \),

the Fourier inverses exist.
5. (a) $(X, d)$ complete metric, $g : X \to X$ contraction
(i.e., $d(g(x), g(y)) \leq \theta d(x, y)$ for all $x, y \in X$
for some $\theta < 1$.
\[ \Rightarrow \exists ! \text{ fixed pt } \quad g(\xi) = \xi. \]

(b) $f(x) = g(x) + \lambda \int_0^1 k(x, y) f(y) \, dy$

$X = C^0([0, 1])$ is complete metric.

$\Phi(f) = g(x) + \lambda \int_0^1 k(x, y) f(y) \, dy$

$\Phi : X \to X$

$\| \Phi(f) - \Phi(f_2) \|_{L^\infty}$

$= \| \lambda \int_0^1 k(x, y) (f(y) - f_2(y)) \, dy \|$

$\leq \lambda \| f - f_2 \|_{L^\infty} \int_0^1 |k(x, y)| \, dx \, dy$

$\leq C_0 \| f - f_2 \|_{L^\infty}$

small for $|f| \text{ small}$. 

\[ \text{small}, \quad \text{small for } |f| \text{ small}. \]
6. \[ \begin{cases} u - \Delta u = f + aw, & \Omega \\ -\Delta w = g - au, & \Omega \\ \nabla w = \phi, & \partial \Omega \\ w = 0, & \partial \Omega \end{cases} \]

(a) \[ \langle u, \phi \rangle + (\nabla u, \nabla \phi) - \langle \nabla w, \phi \rangle - (aw, \phi) \]

\[ = (f, \phi) + (aw, \phi) - (aw, \phi) \]

\[ = (f, \phi) \]

Let \( u \in H^1(\Omega), \ w \in H^1_0(\Omega), \ \phi \in H^1(\Omega), \ \psi \in H^1(\Omega), \ f \in (H^1(\Omega))^*, \ g \in H^{-1}(\Omega), \ \gamma \in H^{-1/2}(\partial \Omega) \)

(b) Lax-Milgram

\[ a((u, w), (\phi, \psi)) = \langle u, \phi \rangle + (\nabla u, \nabla \phi) + (\nabla w, \nabla \psi) \]

\[ + (au, \psi) - (aw, \phi) \]

\[ |a| \leq (\|u\|_{H^1} + \|w\|_{H^1}) (\|\phi\|_{H^1} + \|\psi\|_{H^1}) \]

\[ \Rightarrow \text{bounded} \]

\[ a((u, w), (u, w)) = \|u\|_{H^1}^2 + \|w\|_{H^1}^2 \geq c(\|u\|_{H^1}^2 + \|w\|_{H^1}^2) \]

since \( w \in H^1_0(\Omega) \).

\[ \Rightarrow \text{coercive} \]

\[ b(\phi, \psi) = (f, \phi) + \langle g, \phi \rangle + (g, \psi) \]

\[ \leq \|f\|_{H^1} \|\phi\|_{H^1} + \|a\|_{H^{-1/2}} \|\phi\|_{H^1} + \|\psi\|_{H^{-1/2}} \]

\[ \Rightarrow \text{bounded} \]

\[ \Rightarrow \text{solv} \].
CSEM Area A-CAM Preliminary Exam (CSE 386C / 386D)
May 29, 2014, 9:00 a.m.–12:00 noon

Work any 5 of the following 6 problems.

1. Let $H$ be a separable Hilbert space, $\{v_i\}_{i=1}^{\infty}$ a countable orthonormal base, and $L : H \to H$ the operator such that $Lv_i = \sum_{j=1}^{\infty} 2^{-(i+j)}v_j$. Show that $L$ is compact.

2. Let $w$ be a distribution on $\mathbb{R}$ such that
   (i) $|\langle w, \phi \rangle| \leq \|\phi\|_{L^2}$ for any test function $\phi$;
   (ii) $|\langle w, \phi' \rangle| \leq \|\phi\|_{L^1}$ for any test function $\phi$.

   (a) Assuming only (i), show that $w$ can be represented by a function in $L^2$ with norm less than or equal to 1. [Hint: Riesz Representation Theorem.]
   (b) Assuming both (i) and (ii), show that $w$ can be represented by a Lipschitz function $u$ with Lip seminorm ($L^\infty$ norm of the derivative) less than or equal to 1.

3. Let $\Omega = [-1,1]^2$ and define $\bar{u}$ as the local average in each quadrant, i.e.,

   $\bar{u}(x) = \int_{b}^{b+1} \int_{a}^{a+1} u(y) \, dy_1 \, dy_2$ when $x \in (a,a+1) \times (b,b+1)$ and $a,b = -1,0$.

   (a) Prove that there is a constant $C > 0$ such that

   \[ \|u\|_{H^2} \leq C \left\{ \|\bar{u}\|_{L^\infty} + \sum_{|\alpha|=2} \|D_\alpha u\|_{L^2} \right\} \quad \text{for all } u \in H^2(\Omega). \]

   (b) Will the result in (a) hold in spatial dimension $d > 2$? Why or why not?

4. Let $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary and consider the variational problem: Find $u \in V$ such that

   $(au,v) + (\nabla \cdot u, \nabla \cdot v) = (f, \nabla \cdot v)$ for all $v \in V,$

where $u$ and $v$ are vectors in $\mathbb{R}^2$, $a \in L^\infty(\Omega)$, $a(x) \geq a_* > 0$ for some constant $a_*$, and

   $(au,v) = \int_{\Omega} a(x) (u_1(x)v_1(x) + u_2(x)v_2(x)) \, dx$, and $\nabla \cdot u = \text{div} \, u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}.$

   (a) For the problem to make sense, define $V$ and a space for $f$.
   (b) State the Lax-Milgram theorem for Hilbert spaces.
   (c) Show that the hypotheses of the Lax-Milgram theorem hold for this problem. What norm do we use for $V$?
   (d) Define $p = f - \nabla \cdot u$ and determine the strong form of the equation represented by the variational problem. What boundary condition should you impose on $p$?
5. Given $I = [0, b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$
\begin{align*}
  u'(t) &= g(t)f(u(t)), & \text{for a.e. } t \in I, \\
  u(0) &= \alpha,
\end{align*}
$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose that $f$ is Lipschitz continuous and satisfies $f(0) = 0$. Consider the functional

$$
F(u) = \alpha + \int_0^t g(s) f(u(s)) \, ds.
$$

(a) Show that $F$ maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ is solution to (1) if and only if it is a fixed point of $F$.

(b) Show that there exists $b$ small enough, not depending on $\alpha$, such that $F$ has a unique fixed point in $C^0(I)$.

(c) Show that (1) has a unique solution $u \in C^0(I) \cap W^{1,p}(I)$ for any $g \in L^p(I)$ and $b > 0$.

6. Let

$$
F(u) = \int_{-1}^5 [(u'(x))^2 - 1]^2 \, dx.
$$

(a) Find all extremals in $C^1([-1, 5])$ such that $u(-1) = 1$ and $u(5) = 5$.

(b) Decide if any extremal from (a) is a minimum of $F$. [You may consider $u(x) = |x|$]
1. \( H, \sum_{i=1}^{\infty} v_i^2, L v_i = \sum_{j} 2^{-(i+j)} v_j \)

**Note:** \( L: H \to H \) is well defined.

\( u \in H \Rightarrow u = \sum_{i} u_i v_i \)

\( L u = \sum_{i} (2^{-i} u_i) 2^{-j} v_j \), convergent.

Now if \( u_n \in H, \| u_n \| \leq 1 \) (say), then

\( L u_n = \sum_{i} (2^{-i} u_{n,i}) 2^{-j} v_j, u_{n,i} = \langle u_n, v_i \rangle \)

Let \( u_n \to u \in H \).

Then

\[ \| L(u_n - u) \|^2 = \sum_{i,j} 2^{-2(i+j)} \langle L u_n - u_j v_i \rangle^2 \]

**Note** \( \langle u_n - u_j, v_i \rangle \to 0 \) \( \forall i \) (as \( n \to \infty \)).

Let \( \varepsilon > 0 \) be given. Then choose

\[ M \text{ s.t. } \sum_{i=M+1}^{\infty} 2^{-2(i+j)} \cdot 2 \leq \varepsilon \]

and \( N \text{ s.t. } \| u_n - u_j, v_i \| \leq \varepsilon \) \( \forall i < M, n > N \).

Then

\[ \| L(u_n - u) \|^2 \leq \left( \sum_{i=M}^{\infty} 2^{-2(i+j)} \right) \| L u_n - u_j v_i \|^2 \]

\[ \leq \sum_{i=M}^{\infty} 2^{-2(i+j)} \cdot 2 + \sum_{i=M}^{\infty} 2^{-2(i+j)} \varepsilon \]

\[ \leq C \varepsilon \to 0. \]

Thus \( L u_n \to Lu \) and so \( L \)

is compact.
2. \( w \in \mathcal{D}' \)

(i) \( |\langle w, \varphi \rangle| \leq \|\varphi\|_2 \) \( \forall \varphi \in \mathcal{D} \)

(ii) \( |\langle w, \varphi' \rangle| \leq \|\varphi\|_2 \) \( \forall \varphi \in \mathcal{D} \)

(a) Define \( T: L^2 \rightarrow L^2 \) as

\[
T(f) = \lim_{n \to \infty} T(f_n) = \lim_{n \to \infty} \langle w, f_n \rangle
\]

where \( f_n \xrightarrow{L^2} f \), \( f_n \in \mathcal{D} \).

Then \( T \) is a bounded linear functional and so \( T(f) = \langle f, g \rangle \) for some \( g \in L^2 \). Hence

\( T(\varphi) = \langle w, \varphi \rangle = \langle g, \varphi \rangle \) \( \forall \varphi \in \mathcal{D} \)

\[\iff w = g \in L^2 \]

Moreover, \( \|T\| \leq 1 \implies \|g\| \leq 1 \).

(b) We know from (a) that \( \langle w, \varphi \rangle = \langle g, \varphi \rangle \)

Thus \( \langle w, \varphi' \rangle = \langle g, \varphi' \rangle \)

\[= -\langle g', \varphi \rangle \]

\[\Rightarrow |\langle g', \varphi \rangle| \leq \|\varphi\|_2 \] \( \forall \varphi \in \mathcal{D} \)

\[\Rightarrow \sup_{\varphi} \frac{|\langle g', \varphi \rangle|}{\|\varphi\|_2} = \|g'\|_\infty \leq 1 \] \( \Rightarrow L^1 = L^1 \)

Thus \( g \in W^{1,\infty} \) and \( \|g'\|_\infty \leq 1 \).
3. \( \overline{u} = \sum_{i=0}^{a+1} \int_{-1}^{1} u(x) \, dx, \quad x \in (a_i, a_{i+1}) \times (b_i, b_{i+1}), \quad a_i, b_i = 2p \)

(a) \( \| \overline{u} \|_H^2 \leq c \left( \| \overline{u} \|_H^2 + \sum_{|x| \leq 2} \| D_\overline{u} \|_2^2 \right) \)

Suppose not. Then there exists \( \overline{u}_n \in H^2 \)

s.t. \( \| \overline{u}_n \|_H^2 = 1 \) and

\( \| \overline{u}_n \|_L^2 + \sum_{|x| \leq 2} \| D_\overline{u}_n \|_2^2 \leq \frac{1}{n}, \quad n = b \)

Now \( \overline{u}_n \to \overline{u} \) in \( H^2 \)

and \( D_\overline{u}_n \to 0 \), \( \| \overline{u}_n \|_2 \to 0 \)

Thus \( \overline{u} \in \mathcal{P}_1 \) is a linear polynomial.

Now \( \overline{u}_n \to \overline{u} = 0 \)

So, e.g., \( \sum_{\mathcal{P}_2} \overline{u} = 0 \Rightarrow \overline{u} = \alpha x_1 + \beta x_2 \)

But each quadrant average \( = 0 \Rightarrow \overline{u} = 0 \).

However, \( \overline{u}_n \to \overline{u} \) in \( H^1 \) and \( D_\overline{u}_n \to D_\overline{u} \)

\( \Rightarrow \overline{u}_n \to \overline{u} \)

Thus \( \| \overline{u}_n \|_1 = 1 \Rightarrow \| \overline{u} \|_1 = 1 \), a contradiction.

Thus the result holds.

(b) Yes, if we define \( \overline{u} \) as the average over each unit cube in \( \mathcal{R} \) with one vertex at a vertex of \( L-1 \) th, since the argument above will continue to work.
4. \( \Omega \subseteq \mathbb{R}^2 \), \( \alpha \in L^\infty \), \( \alpha > 0 \)
Find \( u \in V \): \( (au,v) + (\nabla u, \nabla v) = (f, \nabla v) \) \( \forall v \in V \)

(a) \( V = \mathbb{R}^2 \), \( \forall v \in L^2(\mathbb{R}) \)
\( f \in L^2(\mathbb{R}) \).

(b) Let \( H \) be a real Hilbert space, \( B : H \times H \to \mathbb{R} \) be bilinear, and \( X \subseteq H \)
be a closed subspace. Assume
1. \( |B(u,v)| \leq M \|u\|_X \|v\|_H \) \( \forall u, v \in H \)
2. \( |B(u,v)| \geq \alpha \|u\|^2 \) \( \forall u \in X \).

If \( u_0 \in H \) and \( f \in H^* \), then \( \exists ! \) \( u \) solving
Find \( u = X + u_0 \): \( B(u,v) = \langle f, v \rangle \) \( \forall v \in H \).
and \( \|u\| \leq \frac{1}{\alpha} \|f\| + (\frac{M}{\alpha} + 1) \|u_0\| \)

(c) \( 0 = (au, v) + (\nabla u, \nabla v) \)
\leq \|a\|_{L^\infty} \|u\|_2 \|v\|_2 + \|\nabla u\|_2 \|\nabla v\|_2 \)
\leq \frac{C}{\alpha \|u\|_2} \|v\|_2 \^2 \)

where \( \|u\|_2 = \|u\|_2 + \|\nabla u\|_2 \)

(2) \( (au, v) + (\nabla u, \nabla v) \)
\geq \alpha \|u\|^2 + \|\nabla u\|^2 \)
\geq \min (\alpha, 1) \|u\|^2 \)
\( f \in L^2(\mathbb{R})^* \).

(d) \( p = -\nabla u + f \)
\( (au, v) - (p, \nabla v) = 0 \)
\( = (au, v) + (\nabla p, v) - \langle p, \nabla v \rangle \)
\Rightarrow \begin{cases} \nabla u + \nabla p = 0 & \Omega, p = 0, \partial \Omega \end{cases}
\left\{ \begin{array}{l} p + \nabla u = f, \Omega \end{array} \right. \)
5. \( I = [0, b] \), \( u: I \rightarrow \mathbb{R}, g \in L^p (p \geq 1) \), \( f: \mathbb{R} \rightarrow \mathbb{R} \)

\[
\begin{align*}
& (1) \quad \frac{d u}{d t} = g(t) \\ & u(0) = \alpha
\end{align*}
\]
\( f(\alpha) = 0, f \) Lipschitz

\[
F(u) = \alpha + \int_{0}^{t} g(s) f(u(s)) \, ds
\]

\( a) \quad u \in C^0(I) \Rightarrow f(u) \in C^0(I) \)

\[
\lim_{s \to t} \left| F(u(t)) - F(u(s)) \right| = \lim_{s \to t} \left| \int_{s}^{t} g(s) f(u(s)) \, ds \right|
\]

\[
\leq \lim_{s \to t} \| g \|_{L^p} \left( \int_{s}^{t} \| f(u(s)) \|_p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \lim_{s \to t} \| g \|_{L^p} \| f(u) \|_{L^\infty} (t-s) = 0
\]

since \( |f(u)| \leq \max_{-b \leq s \leq 0} |f(\alpha)| < \infty \).

Thus \( F(u) \) is continuous on \( I \).

\( F(u) \in L^p(I) \) since \( u \) is in \( C^0(I) \)

\[
\frac{d}{dt} F(u) = g(t) f(u(t)) \in L^p(I).
\]

Moreover,

\( u \in C^0 \cap W^{1,p} \) solves \( (1) \) \( \Rightarrow \)

\[
\int_{0}^{t} u' = \int_{0}^{t} g f(u) = u(t) - u(0) \Rightarrow F(u) = u.
\]

\( F(u) = u \Rightarrow \quad u' = \frac{d}{dt} F(u) = g f(u) \quad \checkmark \)

and \( u(0) = F(u(0)) = \alpha \).

(Continued)
(b) \( F : C^0(\mathbb{I}) \to C^0(\mathbb{I}) \). Need \( F \) is a contraction

\[
\| F(u) - F(v) \|_{C^0} = \left\| \int_0^t g(s)(f(u) - f(v)) \, ds \right\|_{C^0}
\]

\[
\leq L \int_0^1 g(s) \| u - v \|_{C^0} \, ds
\]

\[
L = \text{Lipschitz constant for } g.
\]

\[
\leq L \| g \|_{L^1(\mathbb{I})} \| u - v \|_{C^0}
\]

Now \( \| g \|_{L^1([0, b^*])} \to 0 \) as \( b \to 0 \), so

\( b^* \) is \( \frac{b}{2} \) for \( b \leq b^* \), \( L \| g \|_{L^1([0, b])} < \frac{1}{2} \)

Then \( F \) is a contraction

and \( \exists ! \) fixed point in \( C^0(\mathbb{I}) \).

(c) Suppose \( u, v \in C^0(\mathbb{I}) \cap W^{1,p}(\mathbb{I}) \) solve (1).

Then by (a) \( u = F(u) \) and \( v = F(v) \).

Thus

\[
(u - v)(t) = F(u)(t) - F(v)(t) = \int_0^t g(s)(f(u) - f(v)) \, ds
\]

\[
\leq L \| g \|_{L^1([0, b^*])} \| u - v \|_{C^0} \leq L \| g \|_{L^1([0, b^*])} \| u - v \|_{C^0}
\]

\[
\leq \frac{1}{2} \Rightarrow u(b) = v(b^*)
\]

Start from \( t = b^* \) and repeat the argument

to get \( u = v \) on \([b^*, 2b^*] \).

Etc. \( \Rightarrow u = v \) on \([0, 1] \).
6. \[ F(u) = \sum_{i=1}^{5} (u'(x))^2 - 1 \]

(a) Euler-Lagrange equations are:

\[ D_2 f = (D_3 f) \]

\[ f(x, u, u') = (u')^2 - 1 \]

\[ \Rightarrow 0 = (2((u')^2 - 1)2u')' \]

\[ u'(u')^2 - 1 = c \]

\[ c = 0 \Rightarrow u' = 0 \text{ or } u' = \pm 1. \]

\[ u(-1) = 1, \; u(5) = 5 \Rightarrow u \neq \text{const}. \]

\[ u \in C^1 \Rightarrow u' \neq 0. \]

(c) \[ 3 \text{ values solve } \frac{3t^2 - 5}{3} - c = 0 \]

\[ \Rightarrow u' = \text{const.} = \frac{4}{6} = \frac{2}{3} \]

\[ u(t) = \frac{2}{3}(t + \frac{5}{2}) \]

(b) \[ u(t) = \frac{4}{3}(t + \frac{5}{2}) \] is not a min.

For example

\[ u(t) = \begin{cases} -t, & -1 \leq t \leq 0 \\ t, & 0 \leq t \leq 5 \end{cases} \]

\[ \Rightarrow F(u) = 0. \text{ For } u \in C^1, \text{ we have } F(u) > 0. \]

\[ F(u) = \sum_{i=1}^{5} \left( \frac{2}{3} \right)^2 - 1 \right)^2 = \frac{25}{49} \gg 0. \]
1. A linear algebra "sanity check". Consider $\mathbb{R}^2$ with the inner product:

$$(x, y) = x_1y_1 + 2x_2y_2, \quad x = (x_1, x_2), \ y = (y_1, y_2). \quad (0.1)$$

- Recall definition of an inner product and (quickly) check that function (0.1) indeed satisfies the necessary properties.
- Is $\mathbb{R}^2$ with the inner product a Hilbert space? Explain why?
- Consider vectors $e_1 = (1, 0), e_2 = (1, 1)$ and prove that they provide a basis for $\mathbb{R}^2$.
- Determine the corresponding dual basis $e_1^*, e_2^* \in (\mathbb{R}^2)^*$.
- Define the Riesz operator $R$ corresponding to the inner product and find its matrix representation in the bases $e_i, e_j^*$.
- Determine the transpose operator to $R$ and its matrix representation with respect to the same bases. Discuss the result.

**Answers:**

- The form is bilinear, symmetric and positive definite and, therefore, it can be identified as an inner product.
- Yes, it is. Space $\mathbb{R}^n$ is complete with respect to the Euclidean metric and, due to the equivalence of any two norms, it is also complete with respect to the norm implied by our inner product.
- The two vectors are not collinear so they are linearly independent.
- Let $x \in \mathbb{R}^2$, We have:

$$x = (x_1, x_2) = (x_1 - x_2)(1, 0) + x_2(1, 1) = (x_1 - x_2)e_1 + x_2e_2$$

The dual basis is thus:

$$e_1^*(x) = x_1 - x_2, \quad e_2^*(x) = x_2.$$

- The Riesz operator $R$ corresponding to a particular inner product $(u, v)$ sets vector $u$ into the linear functional $(u, \cdot)$. More precisely,

$$(Ru, v) = (u, v)$$
Riesz operator is injective and, in the finite dimensional setting, automatically surjective as the
dual space is of the same dimension as the original space. In order to determine the matrix
representation of $R$, we consider vectors $Re_j$,

$$(Re_1)(y) = (e_1, y) = y_1 = (y_1 - y_2) + y_2 = e_1^*(y) + e_2^*(y)$$

$$(Re_2)(y) = (e_2, y) = y_1 + 2y_2 = (y_1 - y_2) + 3y_2 = e_1^*(y) + 3e_2^*(y)$$

The matrix representation of operator $R$ is thus:

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$  

- The transpose $R^T$ goes from the bidual to the dual space. In the finite dimensional case (in fact,
  for any Hilbert space), the bidual is identified (canonically isomorphic) with the original space.
  Due the symmetry of the inner product, transpose $R^T$ coincides with $R$. Indeed,

$$\langle Ru, v \rangle_{V' \times V} = (u, v)_{V} = (v, u)_{V} = \langle Rv, u \rangle_{V' \times V} = \langle u, Rv \rangle_{V'' \times V'}.$$  

Consequently, matrix representation of transpose $R^T$ coincides with that of $R$.  

2. A metric space problem. Let $X$ be a set and $\rho_1(x, y), \rho_2(x, y)$ two metrics on $X$. Define:

$$d(x, y) := \max\{\rho_1(x, y), \rho_2(x, y)\}.$$  

(0.2)

- Is $d$ also a metric on $X$? Prove or disprove.

- If the answer to the first question is positive, you have three topologies in $X$ corresponding to the three metrics. Discuss the relative strength of the corresponding topologies (which one is stronger or weaker than others?). *Hint: Recall the definition of bases of neighborhoods in a metric space.*

**Answers:**

- Yes, it is.

**Positive definiteness:** If $d(x, y) = 0$ then both $\rho_1(x, y) = \rho_2(x, y) = 0$ which implies that $x = y$.

**Symmetry:** We have:

$$\rho_i(x, y) = \rho_i(y, x), \quad i = 1, 2.$$  

Apply $\max_{i=1, 2}$ to both sides.

**Triangle inequality:** Start with:

$$\rho_i(x, y) \leq \rho_i(x, z) + \rho_i(z, y) \leq \max_{j=1, 2} \rho_j(x, z) + \max_{j=1, 2} \rho_j(z, y), \quad j = 1, 2,$$

and take maximum with respect to $i$ on both sides.

- Let $B^d(x, \epsilon)$ and $B^{\rho_i}(x, \epsilon)$ denote balls corresponding to metrics $d$ and $\rho_i$, resp. Inequality

$$\rho_i(x, y) \leq d(x, y), \quad i = 1, 2$$

implies that

$$B^d(x, \epsilon) \subset B^{\rho_i}(x, \epsilon), \quad i = 1, 2.$$  

Consequently, if $B^d, B^{\rho_i}$ denote the bases of neighborhoods in topologies generated by $d$ and $\rho_i$, resp., then

$$B^{\rho_i} \succ B^d$$

which demonstrates that metric topogoy corresponding to $d$ is stronger than both topologies corresponding to metrics $\rho_i$. We cannot draw any general conclusion about the relative strength of metric topologies corresponding to $\rho_i$, $i = 1, 2$.  

3. Contraction Maps. Consider the following Initial-Value Problem (IVP):

\[
\begin{aligned}
\frac{dq}{dt} &= t \ln(q(t)), \quad t > 0 \\
q(0) &= 1
\end{aligned}
\]

- State Banach Contractive Map Theorem.

**Answer:**

Let \((X, d)\) be a complete metric space. Let \(D \subset X\) (then \((D, d)\) is itself a metric space, too...), and \(A : D \to D\) is a contraction, i.e.

\[
d(A(f), A(g)) \leq k d(f, g), \quad \forall f, g \in D, \quad k < 1
\]

Then function \(A\) has a unique fixed point in set \(D\).

- Use the theorem to prove local existence and uniqueness of solution to the IVP, i.e. that there exists an interval \((0, T)\) in which the equation is satisfied. Provide a concrete value of \(T\).

**Solution:**

The problem is equivalent to the solution of the integral equation:

\[
q(t) = 1 + \int_0^t s \ln(q(s)) \, ds
\]

Consider the Chebyshev space \(C[0, T]\) (with unknown \(T\) at this point...) and define the map \(A\) using the right-hand side of the equation above:

\[
(Aq)(t) = 1 + \int_0^t s \ln(q(s)) \, ds
\]

First of all, we need to define a set \(D \subset C[0, T]\) such that map \(A\) sets the set \(D\) into itself. Assume that \(q(t)\) will vary in the box:

\[
D = \{ q \in C[0, T] : e^{-1} \leq q(t) \leq e, \quad 0 \leq t \leq T \}
\] (0.3)

(notice that the box includes the initial value \(q = 1\)). Then \(-1 \leq \ln q(t) \leq 1\), i.e. \(|\ln q(t)| \leq 1\). Consequently,

\[
| \int_0^t s \ln q(s) \, ds | \leq \int_0^t s \, ds = \frac{1}{2} t^2
\]

so,

\[
|(Aq)(t) - 1| \leq \frac{1}{2} T^2
\]

This gives two bounds for \(T\). From the right:

\[
(Aq)(t) \leq 1 + \frac{1}{2} T^2 \leq e \quad \Rightarrow \quad T \leq \sqrt{2(e - 1)},
\]
and from the left:
\[ e^{-1} \leq 1 - \frac{1}{2}T^2 \leq (Aq)(t) \quad \Rightarrow \quad T \leq \sqrt{2(1 - e^{-1})}. \]

Now, map \( A \) must be a contraction. With flux \( F(s, q) = s \ln q \),
\[
\left| \frac{\partial F}{\partial q} \right| = s \frac{1}{q} \leq es
\]
so, with \( q \) coming from box (0.3), the flux satisfies the Lipschitz condition:
\[ |F(s, q_1) - F(s, q_2)| \leq es|q_1 - q_2|. \]

This leads to the estimate;
\[
|(Aq_1)(t) - (Aq_2)(t)| \leq \int_0^t es \, ds \|q_1 - q_2\|_{C[0,T]} \leq \frac{e}{2}T^2 \|q_1 - q_2\|_{C[0,T]}.
\]

Consequently, a sufficient condition for a contraction is
\[ T < \sqrt{\frac{2}{e}}. \]

In conclusion, the IVP will have a unique solution for
\[ T < \min\{\sqrt{2(e - 1)}, \sqrt{2(1 - e^{-1})}, \sqrt{\frac{2}{e}}\}. \]
4. Consider the following initial-value problem.

\[
\begin{cases}
\ddot{x} + \dot{x} = \delta(t - 1), & t > 0 \\
x(0) = 1, \; \dot{x}(0) = 0,
\end{cases}
\]

where \(\delta\) denotes the Dirac’s delta “function”.

(a) Define precisely delta functional and reinterpret its action in terms of appropriate jump conditions at \(t = 1\) for the solution \(x(t)\) and its derivative \(\dot{x}(t)\).

Dirac’s delta at \(t = 1\) is a functional that assigns to every test function \(\phi\) its value at \(t = 1\),

\[
\mathcal{D}(\mathbb{R}) \ni \phi \rightarrow \phi(1) \in \mathbb{R}
\]

Delta is the distributional derivative of Heaviside function. Its presence translates into jump conditions at \(t = 1\),

\[
[x(1)] = 0, \quad [\dot{x}(1)] = 1
\]

(b) Utilize the jump conditions and solve the problem using elementary means.

For \(t \in (0, 1)\),

\[x(t) = A + Be^{-t}\]

Utilizing IC, we get

\[x(t) = 1\]

For \(t \in (1, \infty)\),

\[x(t) = A + Be^{-(t-1)}\]

Utilizing jump conditions at \(t = 1\) and the known value of \(x\) and \(\dot{x}\) at \(t = 1\), we get

\[x(t) = 2 - e^{-(t-1)}\]

(c) Define the Laplace transform. Apply it to both sides of the equation and find the solution in the Laplace domain.

\[
\mathcal{L}(f)(s) = \mathcal{F}(s) = \int_{0}^{\infty} f(t)e^{-st} \, dt
\]

\[
\int_{0}^{\infty} e^{-st}\delta(t - 1) \, dt = e^{-s}
\]

Recall the formulas resulting from integration by parts,

\[
\mathcal{F} = s\mathcal{F}(s) - x(0)
\]

\[
\mathcal{F} = s^2\mathcal{F}(s) - sx(0) - \dot{x}(0)
\]
Transforming both sides of the equation and accounting for the IC, we get:

\[(s^2 + s)x - s - 1 = e^{-s}\]

which gives the solution in the Laplace domain,

\[\bar{x}(s) = \frac{1}{s} + \frac{e^{-s}}{s(s + 1)}\]

(d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the Laplace domain and compare it with the solution obtained using the elementary calculus.

The following is just a sketch, look up your lecture notes for a detailed explanation.

**First term, case:** \(t < 0\). Use contour to the right to conclude that \(x = 0\).

**First term, case:** \(t > 0\). Simple pole at \(s = 0\). Use contour to the left to conclude that

\[x = \text{Re}_0 \frac{e^{st}}{s} = \lim_{s \to 0} e^{st} = 1\]

The argument showing that the integral over \(C_R\) vanishes in the limit, needs use of the Lebesgue Dominated Convergence Theorem.

**Second term, case:** \(t < 1\). Use contour to the right to conclude that \(x = 0\).

**Second term, case:** \(t > 1\). Simple poles at \(s = 0, -1\). Use contour to the left to conclude that

\[x = \text{Re}_0 (\frac{e^{s(t-1)}}{s(s + 1)}) + \text{Re}_{-1} (\frac{e^{s(t-1)}}{s(s + 1)}) = 2 - e^{-(t-1)}\]

The argument showing that the integral over \(C_R\) vanishes in the limit, is now easy; the denominator is \(O(R^2)\).

Summing up, we get the result coinciding with the elementary solution.
5. (a) Recall the formula for the gradient in polar coordinates,

\[ \nabla u = \frac{\partial u}{\partial r} e_r + \frac{1}{r} \frac{\partial u}{\partial \theta} e_\theta \]

and use integration by parts to derive the corresponding formula for the Laplacian.

Integrate by parts in \( r, \theta \),

\[
\int_{\Omega} \left( \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \right) r dr d\theta = \int_{\Omega} \nabla u \nabla v = - \int_{\Omega} \Delta u \, v + \text{boundary terms}
\]

to obtain:

\[
\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}
\]

(b) Use separation of variables and whatever means you need, to solve the boundary-value problems shown in Fig. 1.

![Figure 1: Steady-state heat conduction in a wedge domain.](image)

Boundary condition at \( \theta = \alpha \),

\[ u = 10 \]

prompts for looking first for a particular solution in the form

\[ u = 10 \Theta(\theta) \]

Substituting into the formula for the Laplacian, we learn that \( \Theta \) must satisfy the equation

\[ \Theta'' = 0 \]

Along with the BC,

\[ \Theta(0) = 0, \quad \Theta(\alpha) = 1 \]
this leads to the final form of the particular solution,

\[ u = \frac{10}{\alpha} \theta \]

Now, we look for the ultimate solution in the form,

\[ u(r, \theta) = \frac{10}{\alpha} \theta + v(r, \theta) \]

where \( v \) satisfies homogeneous BCs at \( \theta = 0, \alpha \) and the nonhomogeneous BC at \( r = a \),

\[ v(a, \theta) = -\frac{10}{\alpha} \theta \]

We can use now the standard separation of variables,

\[ v = R(r)\Theta(\theta) \]

to arrive at

\[ \frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta} = \lambda \]

Operator in \( \theta \) is self-adjoint and positive-definite, so \( \lambda = k^2, k > 0 \). We get

\[ \Theta = A \sin k\theta + B \cos k\theta \]

BC at \( \theta = 0 \) implies \( B = 0 \), and BC at \( \theta = \alpha \) implies that

\[ \sin k\alpha = 0 \implies k\alpha = n\pi, \quad n = 1, 2, \ldots \]

Thus,

\[ k = k_n = \frac{1}{\alpha} n\pi, \quad n = 1, 2, \ldots \]

This leads to Cauchy-Euler eqn in \( r \) and the final solution in the form,

\[ v = \sum_{n=1}^{\infty} A_n r^{k_n} \sin(k_n \theta) \]

Coefficients \( A_n \) are computed using \( L^2 \)-orthogonality of \( \sin(k_n \theta) \),

\[ A_n = -\frac{10}{\alpha a^{k_n}} \frac{\int_0^\alpha \theta \sin(k_n \theta) \, d\theta}{\int_0^\alpha \sin^2(k_n \theta) \, d\theta} \]
6. Let $D$ be the region shown in the Fig. 2, whose surface $S$ consists of three pieces.

Bottom: $S_1 : z = f_1(x, y), (x, y) \in R$

Top: $S_2 : z = f_2(x, y), (x, y) \in R$

Side: $S_3 : f_1(x, y) \leq z \leq f_2(x, y), (x, y)$ on $C$.

where $R$ is the projection of $D$ onto the $xy$-plane and $C$ is the boundary of $R$.

State and prove the Divergence Theorem for $\mathbf{F} = Q(x, y, z)\mathbf{k}$, where $Q$ is continuous and has continuous first partial derivatives on $D$. Assume that functions $f_1, f_2$ and curve $C$ are sufficiently regular, e.g. $C^1$.

Figure 2: A domain in $\mathbb{R}^3$.

**Proof:** See class notes. This problem is about 1D integration by parts and recalling the definition of surface integral.
1. Let $E'$ be the set of limit points of a set $E$ in a metric space. Prove that $E'$ is closed.

2. Suppose $A$ is the linear transformation $A(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_2 + x_3)$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

   * Compute the matrix of $A$ with respect to the canonical basis in $\mathbb{R}^3$ and the basis $\{(2, 1), (1, 0)\}$ of $\mathbb{R}^2$.
   
   * Determine the null space of $A$.
   
   * Determine whether or not $A$ is left or right invertible.
   
   * Determine the adjoint of $A$ with respect to the following inner products

     $$(x, y)_{\mathbb{R}^2} = 3x_1y_1 + 2x_2y_2$$

     $$(x, y)_{\mathbb{R}^3} = .5x_1y_1 + 4x_2y_2 + x_3y_3.$$ 

3. Give the definition of an $F_\sigma$ type set in $\mathbb{R}^n$. Suppose $E \in \mathbb{R}^n$ is Lebesgue measurable. Prove that $E = J - Z$ where $J$ is an $F_\sigma$-type set and $Z$ has measure zero.

4. Consider a car travelling from point $(x_1, y_1)$ to $(x_2, y_2)$. The velocity of the car is a function of $y$, i.e. $v = v(y)$. Determine, in details, the path that takes the car to finish its itinerary in the shortest time if $v(y) = 2$ and $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$.

5. Evaluate the following integral

   $$\int_{-\infty}^{\infty} \frac{2x}{x^3 + 27} \, dx$$

6. Recall the Laplacian in polar coordinates as

   $$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.$$ 

   The domain of interest has a pizza-slice shape determined by $0 < \theta < \alpha$ (where $\alpha$ is a given angle) and $0 < r < a$.

   * Let boundary conditions be $u(r, \alpha) = 0$, $u(r, 0) = 0$, and $u(a, \theta) = f(\theta)$. Solve the Laplace equation $\Delta u = 0$ for $u(r, \theta)$.

   * Now let us consider a new set of boundary conditions: $u(r, \alpha) = 50$, $u(r, 0) = 0$, and $u(a, \theta) = 0$. Again solve Laplace equation $\Delta u = 0$ for $u(r, \theta)$. 
Work any 5 of the following 6 problems.

1. Let $1 \leq p < \infty$ and define, for each $s > 0$, $T_s : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by $T_s(f)(x) = f(sx)$.
   (a) Verify that $T_s(f) \in L^p(\mathbb{R}^d)$ and that $T_s$ is bounded and linear. What is the norm of $T_s$?
   (b) Show that as $r \to s$, $\|T_r f - T_s f\|_{L^p} \to 0$. [Hint: Use that the set of continuous functions with compact support is dense in $L^p$ for $p < \infty$.]

2. Let $X$ and $Y$ be Banach spaces, and let $A : X \to Y$ be a linear operator.
   (a) Define the topological dual of the Banach space $X$.
   (b) Define the weak topology on the Banach space $X$.
   (c) Prove that the operator $A$ is (strongly) continuous if and only if it is weakly continuous (i.e., it is continuous when $X$ and $Y$ are equipped with their weak topologies).

3. Let $\tilde{H}^1 = \{ u \in H^1(0, 1) : u(0) = 0 \}$ and define the first order linear operator $A : \tilde{H}^1 \to L^2(0, 1)$ by $Au = u' - 2u$, where the derivative is understood in the sense of distributions.
   (a) Show that $A$ is bounded.
   (b) Show that the null space of the adjoint operator $A^*$ is trivial.
   (c) Prove that the operator $A$ is bounded below in $L^2(0, 1)$.
   (d) For an appropriate right-hand side $f$, discuss the well-posedness of the problem
      $$u \in \tilde{H}^1(0, 1), \quad Au = f.$$ 

4. Modify the previous problem to $Au = u' - 2u = f$, where now $u \in H^1_0(0, 1)$. Consider the "ultra-weak" variational formulation of this problem: Find $u \in U = L^2(0, 1)$ such that
   $$\int_0^1 u A^* v \, dx = \int_0^1 f v \, dx \quad \forall v \in V = H^1(0, 1),$$
   where $A^*$ denotes the formal adjoint of $A$. $A^* v = -v' - 2v$.
   (a) Define operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u, v)$.
   (b) State the Babuška-Nečas Theorem for Hilbert spaces.
   (c) Use this theorem to investigate the well-posedness of the variational formulation.

5. We want to prove that for any $f \in L^2(\mathbb{R}^d)$, there exists a unique solution $u \in H^1(\mathbb{R}^d)$ to
   $$-\Delta u + u = f, \quad \text{in } \mathbb{R}^d.$$ 
   (a) State the Lax-Milgram Theorem for Hilbert spaces.
   (b) Find the Variational problem associated to the PDE, and show carefully the equivalence for a function to be both the solution of the variational problem and a weak solution of the equation.
   (c) Show the existence and uniqueness of the solution of the variational problem.
   (d) Using the Fourier transform, show that the solution is actually bounded in $H^2(\mathbb{R}^d)$.

6. Let $X$ and $Y$ be Banach spaces, and let $F$ and $G$ take $X$ to $Y$ be $C^1$.
   (a) Let $H(x, \epsilon) = F(x) + \epsilon G(x)$ for $\epsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \epsilon) = 0$ for $\epsilon$ sufficiently close to 0.
   (b) For small $\epsilon$, prove that there is a solution $w \in H^2(0, \pi)$ to
      $$w'' = w + \epsilon w^2, \quad w(0) = w(\pi) = 0.$$