DISCRETE STABILITY, DPG METHOD AND LEAST SQUARES

Leszek Demkowicz
ICES, The University of Texas at Austin

Rensselaer Polytechnic Institute, MANE Dept.
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Collaboration:

U. Oregon: J. Gopalakrishnan

ICES: T. Bui, J. Chan, O. Ghattas, B. Moser, N. Roberts, J. Zitelli

Boeing: D. Young

Basque U: D. Pardo

KAUST: V. Calo, A. Niemi

IMA: W. Qiu

Sandia: P.N. Bochev, K.J. Peterson, D. Ridzal and Ch. M. Siefert

Los Alamos: M. Shaskov

C.U. Chile: I. Muga, N. Heuer

U. Nevada: J. Li

Outline

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- Babuška's Theorem.

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- Struggle with discrete stability.

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- Systematic choice of test norm.

Abstract Variational Problem

$$\begin{cases} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{cases} \Leftrightarrow \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \quad v \in V \end{array}$$

where

- U, V are Hilbert spaces,
- $b(u,v)$ is a continuous bilinear (sesquilinear) form on $U \times V$,

  $$|b(u,v)| \leq M \|u\|_U \, \|v\|_V$$

  that satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

  $$\inf_{\|u\|_U=1} \sup_{\|v\|_V=1} |b(u,v)| =: \gamma > 0 \quad \Leftrightarrow \quad \sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

- $l \in V'$ represents the load and satisfies the compatibility condition $l(v) = 0, \forall v \in V_0$ where

  $$V_0 := \{v \in V \, : \, b(u,v) = 0 \quad \forall u \in U\}$$

Banach Closed Range and Babuška Theorems

Let $b(u,v), u \in U, v \in V$ be a continuous bilinear form, $V_0 = \{\mathbf{0}\}$, $l \in V'$.
Consider the variational problem,

$$\left\{ \begin{array}{l} u \in U \\ b(u,v) = l(v), \quad \forall v \in V \end{array} \right.$$

The inf-sup condition

$$\sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

implies existence, uniqueness and stability*

$$\|u\|_U \leq \gamma^{-1} \|l\|_{V'}$$

*Oden, D, *Functional Analysis*, Chapman & Hall, 2nd ed., 2010, p.518

Banach Closed Range and Babuška Theorems

Let $b(u,v), u \in U, v \in V$ be a continuous bilinear form, $V_0 = \{\mathbf{0}\}$, $l \in V'$. Consider the approximate variational problem,

$$\begin{cases} u_{hp} \in U_{hp} \subset U \\ b(u_{hp}, v) = l(v), \quad \forall v \in V_{hp} \subset V \end{cases}$$

The discrete inf-sup condition

$$\sup_{v \in V_{hp}} \frac{|b(u_{hp}, v)|}{\|v\|_V} \geq \gamma_{hp} \|u_{hp}\|_U$$

implies existence, uniqueness and discrete stability

$$\|u_{hp}\|_U \leq \gamma_{hp}^{-1} \|l\|_{V'_{hp}}$$

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implies existence, uniqueness and discrete stability

$$\|u_{hp}\|_U \leq \gamma_{hp}^{-1} \|l\|_{V'_{hp}}$$

and convergence *

$$\|u - u_{hp}\|_U \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_U$$

*I. Babuska, "Error-bounds for Finite Element Method.", *Numer. Math*, **16**, 1970/1971.

Banach Closed Range and Babuška Theorems

(Uniform) discrete stability and approximability imply convergence.

A similar result for Finite Differences was proved by Peter Lax [†] who argued that proving discrete stability is more difficult that proving continuous stability.

[†] P. Lax, "Numerical Solution of Partial Differential Equations.". *Amer. Math. Monthly*, **72** 1965 no. 2, part II.

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Coercive Problems, Céa Lemma

If $U = V$, and the bilinear (sesquilinear) form is coercive [‡],

$$b(u,u,) \geq \alpha \|u\|_U^2$$

Then both continuous and discrete stability constants are bounded below by α,

$$\gamma, \gamma_{hp} \geq \alpha \quad \Longrightarrow \quad \frac{1}{\gamma_{hp}} \leq \frac{1}{\alpha}$$

Thus, for coercive problems, discrete stability is guaranteed automatically.
All strongly elliptic problems including linear elasticity, various plates and shells theories (static problems only) fall into this category.

[‡] Jean Céa, "Approximation variationnelle des problèmes aux limites". *Annales de l'Institut Fourier* **14.** 2. pp. 345-444.

Ritz and Bubnov-Galerkin Methods

FE classics:

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- If the bilinear form is symmetric (hermitian) and positive-definite,

$$b(u,v) = \overline{b(v,u)}, \quad b(v,v) > 0$$

$u, v \in$ a Hilbert space V,

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- then

$$\left\{ \begin{array}{l} u \in V \\ J(u) := \frac{1}{2} b(u,u) - l(u) \to \text{min} \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} u \in V \\ b(u,v) = l(v), \; v \in V \end{array} \right.$$

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- and, Bubnov-Galerkin method delivers the *best approximation error* in the energy norm,

$$\begin{cases} u_h \in V_h \subset V \\ b(u_h, v_h) = l(v_h), \ v_h \in V_h \end{cases} \quad \Leftrightarrow \quad \begin{cases} u_h \in V_h \\ \|u - u_h\|_E \to \text{min} \end{cases}$$

  where $\|v\|_E^2 = b(v,v)$.

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  where $\|v\|_E^2 = b(v,v)$.

- You cannot do better ! (in energy norm...)

Asymptotic Stability (Mikhlin)

Compact perturbation:

[§] D, *Computers & Mathematics with Applications*, **27**(12),69–84, 1994
D, J.T. Oden, *Comput. Methods Appl. Mech. Engrg.*,**133** (3-4), 287–318, 1996.

Asymptotic Stability (Mikhlin)

Compact perturbation:

- If we perturb $b(u,v)$ with a compact contribution,

$$b(u,v) + c(u,v)$$

$(|c(u,v)| \leq C\|u\|_H \|v\|_V,\ V \overset{c}{\hookrightarrow} H)$,

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- then the best approximation error property is achieved asymptotically[§],

$$\frac{\|u - u_{hp}\|_E}{\inf_{w_{hp}} \|u - w_{hp}\|_E} \to 0 \text{ as } \frac{h}{p} \to 0$$

We have an asymptotic discrete stability. To this class belong most of vibration and wave propagation problems.

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- Is h/p small enough to observe this in practice ?

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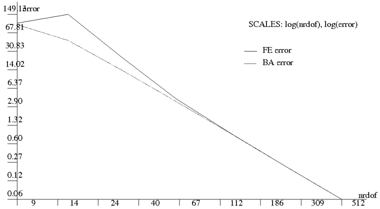
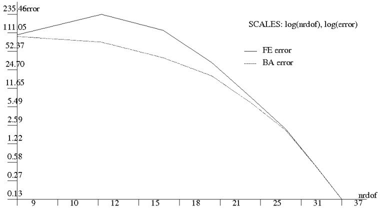
Pollution (Babuška, Ihlenburg)

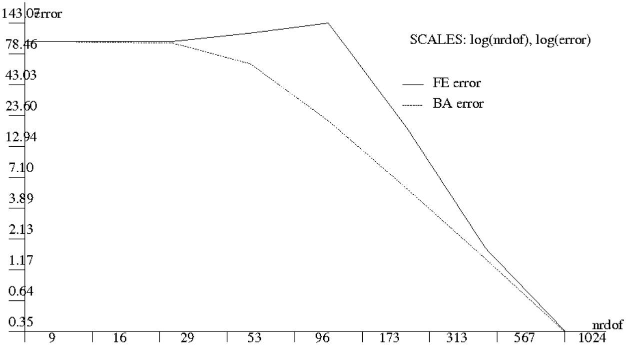




Vibrations of an elastic bar, $k = 32$ (5 wavelengths). FE and best approximation (BA) errors for uniform h- ($p = 2$) and p-refinements. ¶

¶See D., *Computing with hp Finite Elements*, Chapman & Hall, 2007, chap. 7

Pollution



Vibrations of an elastic bar, $k = 160$ (25 wavelengths). FE and best approximation (BA) errors for uniform h- ($p = 2$) refinements.

History of Discrete Stability by Demkowicz

1910 –– (Bubnov) Galerkin method

1954 –– numerical flux of P. Lax

1959 –– Petrov–Galerkin method

1964 –– Cea's lemma

1969 –– Mikhlin's asymptotic stability

1971 –– Babuska's theorem

1974 –– Brezzi's theory

1980 –– Barett and Morton use Petrov–Galerkin to symmetrize

1981 –– SUPG method of Brooks and Hughes, stabilized methods

1985 –– D and Oden use PG to change the norm of convergence

1986 –– Franca and Russo – bubble methods

1989 –– DPG method of Cockburn and Shu

2009 –– D and Gopalakrishnan – DPG method with optinal test functions

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Energy Norm

The supremum in the inf-sup condition defines an equivalent, problem-dependent *energy (residual) norm*,

$$\|u\|_E := \sup_{\|v\|=1} |b(u,v)| = \|Bu\|_{V'}$$

For the energy norm, $M = \gamma = 1$. Recalling that the Riesz operator is an isometry form V into V', we may characterize the energy norm in an equivalent way as

$$\|u\|_E = \|v_u\|_V$$

where v_u is the solution of the variational problem,

$$\begin{cases} v_u \in V \\ (v_u, \delta v)_V = b(u, \delta v) \quad \forall \delta v \in V \end{cases}$$

Optimal Test Functions

Select your favorite trial basis functions: $e_j,\ j = 1, \ldots, N$. For each function e_j, introduce a corresponding *optimal test (basis) function* $\bar{e}_j \in V$ that realizes the supremum,

$$|b(e_j, \bar{e}_j)| = \sup_{\|v\|_V = 1} |b(e_j, v)|$$

i.e. it solves the variational problem,

$$\begin{cases} \bar{e}_j \in V \\ (\bar{e}_j, \delta v)_V = b(e_j, \delta v) \quad \forall \delta v \in V \end{cases}$$

Define the discrete test space as $\bar{V}_{hp} := \text{span}\{\bar{e}_j,\ j = 1, \ldots, N\} \subset V$. It follows from the construction of the optimal test functions that the *discrete* inf-sup constant

$$\inf_{\|u_{hp}\|_E = 1} \sup_{\|v_{hp}\| = 1} |b(u_{hp}, v_{hp})| = 1$$

The Best Approximation

Consequently, Babuška's Theorem

$$\|u - u_{hp}\|_E \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

implies that

$$\|u - u_{hp}\|_E \leq \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

i.e., the method delivers the *best approximation error* in the energy norm. [‖]

[‖] D., J. Gopalakrishnan. "A Class of Discontinuous Petrov-Galerkin Methods. Part II: Optimal Test Functions." *Numer. Meth. Part. D. E.*, **27**, 70-105, 2011.

Stiffness Matrix Is Symmetric and Positive Definite

$$b(e_i, \bar{e}_j) = (\bar{e}_i, \bar{e}_j)_V = (\bar{e}_j, \bar{e}_i)_V = b(e_j, \bar{e}_i)$$

Energy Norm of FE Error $e_{hp} = u - u_{hp}$

can be computed *without* knowing the exact solution.

$$\begin{cases} v_{e_{hp}} \in V \\ (v_{e_{hp}}, \delta v)_V = b(u - u_{hp}, \delta v) = l(\delta v) - b(u_{hp}, \delta v) \quad \forall \delta v \in V \end{cases}$$

We have then

$$\|e_{hp}\|_E = \|v_{e_{hp}}\|_V$$

We shall call $v_{e_{hp}}$ *the error representation function*

Note: No need for an a-posteriori error estimation.

Least Squares (with a Twist)

$$\left\{\begin{array}{l} u \in U \\ b(u,v) = l(v) \quad v \in V \end{array}\right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \rightarrow V' \\ < Bu, v >= b(u,v) \end{array}$$

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- **Least squares:** $U_h \subset U$,

$$\frac{1}{2} \| Bu_h - l \|_{V'}^2 \to \min_{u_h \in U_h}$$

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- **Riesz operator:**

$$R_V : V \to V', < R_V v, \delta v >= (v, \delta v)_V$$

is an *isometry*, $\| R_V v \|_{V'} = \| v \|_V$.

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is an *isometry*, $\|R_V v\|_{V'} = \|v\|_V$.

- **Least squares reformulated:**

$$\frac{1}{2}\|Bu_h - l\|^2_{V'} = \frac{1}{2}\|R_V^{-1}(Bu_h - l)\|^2_V \to \min_{u_h \in U_h}$$

Least squares and optimal test functions

Taking Gâteaux derivative,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h$$

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$$< Bu_h - l, R_V^{-1}B\delta u_h >= 0 \quad \delta u_h \in U_h$$

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$$< Bu_h - l, v_h >= 0 \quad v_h = R_V^{-1}B\delta u_h$$

Least squares and optimal test functions

Taking Gâteaux derivative,

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or

$$< Bu_h, v_h >=< l, v_h > \quad v_h = R_V^{-1}B\delta u_h$$

Least squares and optimal test functions

Taking Gâteaux derivative,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h$$

or

$$b(u_h, v_h) = l(v_h)$$

where

$$\begin{cases} v_h \in V \\ (v_h, \delta v)_V = b(\delta u_h, \delta v) \quad \delta v \in V \end{cases}$$

Petrov-Galerkin Method with Optimal Test Functions is the least-squares method !

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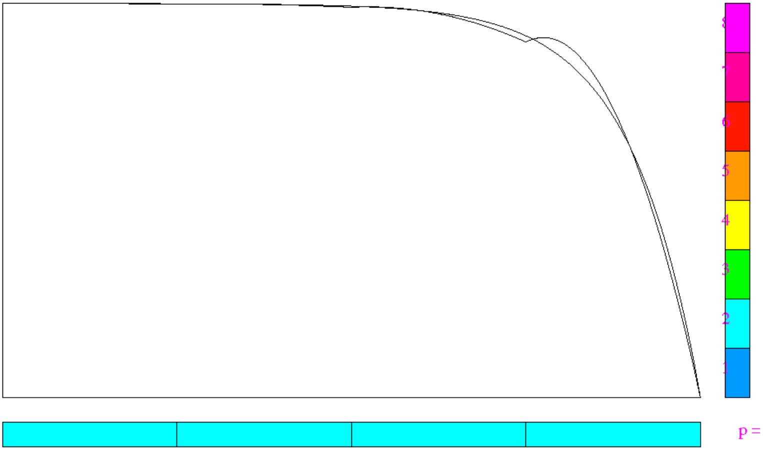
1D Convection–Dominated Diffusion

A reminder:

How does the usual Bubnov–Galerkin method perform for 1D Confusion ?

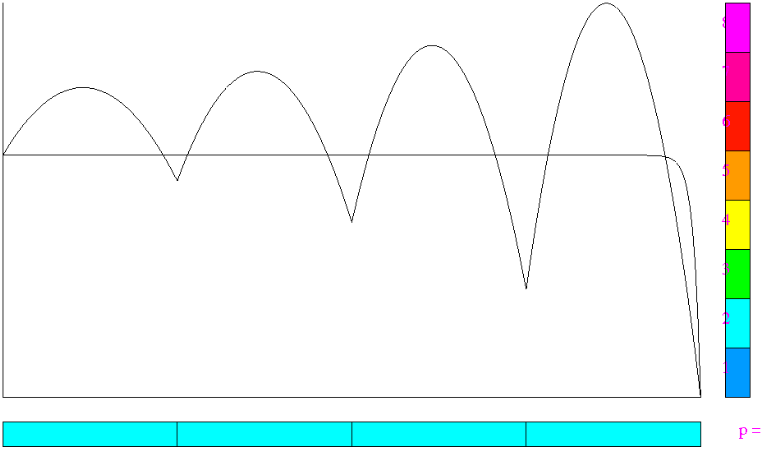
$$\begin{cases} -\epsilon u'' + u' = 0 & \text{in } (0,1) \\ u(0) = 1,\ u(1) = 0 \end{cases}$$

Bubnov-Galerkin Method



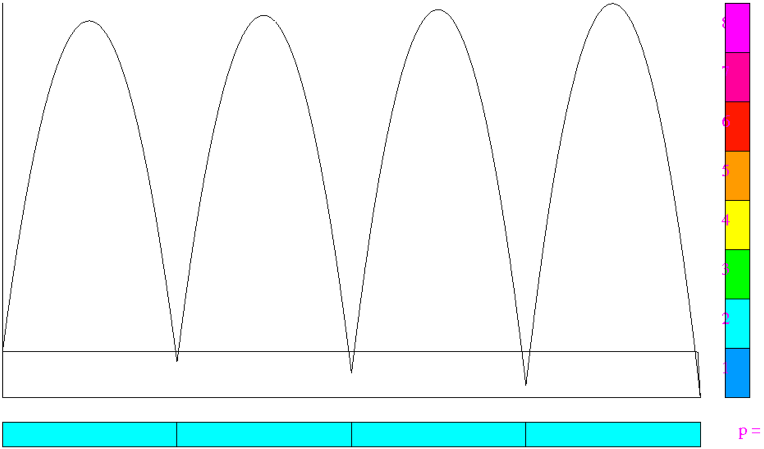
$$\epsilon = 10^{-1}$$

Bubnov-Galerkin Method



$$\epsilon = 10^{-2}$$

Bubnov-Galerkin Method



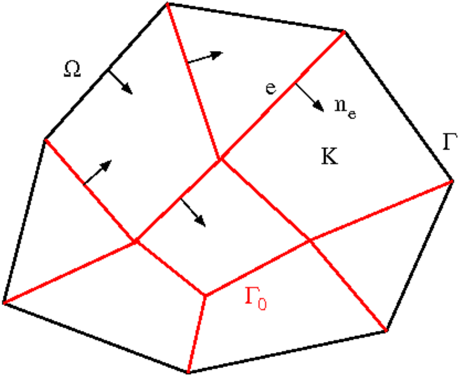
$$\epsilon = 10^{-3}$$

Ultraweak Variational Formulation and DPG Method for 2D Confusion Problem

2D Convection-Dominated Diffusion

$$\begin{cases} \frac{1}{\epsilon}\boldsymbol{\sigma} - \boldsymbol{\nabla} u &= 0 \quad \text{in } \Omega \\ -\text{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \partial\Omega \end{cases}$$

DPG Method



Elements:K
Edges:e
Skeleton:$\Gamma_h = \bigcup_K \partial K$
Internal skeleton:$\Gamma_h^0 = \Gamma_h - \partial\Omega$

DPG Method

Take an element K. Multiply the equations with test functions $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, K), v \in H^1(K)$:

$$\begin{cases} \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \boldsymbol{\nabla} u \cdot \boldsymbol{\tau} & = 0 \\ -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) v & = f v \end{cases}$$

DPG Method

Integrate over the element K:

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \boldsymbol{\nabla} u \cdot \boldsymbol{\tau} & = 0 \\ -\int_K \operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) v & = f v \end{cases}$$

Integrate by parts (relax) *both* equations:

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K} u\,\tau_n & = 0 \\ \int_K(\boldsymbol{\sigma}-\boldsymbol{\beta}u)\cdot\boldsymbol{\nabla}v - \int_{\partial K} q\,\mathrm{sgn}(\boldsymbol{n})\,v & = \int_K fv \end{cases}$$

where $q = (\boldsymbol{\sigma}-\boldsymbol{\beta}u)\cdot\boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{cases}$$

DPG Method

Declare traces and fluxes to be independent unknowns:

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K}\hat{u}\,\tau_n & = 0 \\ -\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\boldsymbol{\nabla} v + \int_{\partial K}\hat{q}\,\mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{cases}$$

DPG Method

Use BC to eliminate known traces

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathsf{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u}\,\tau_n & = \int_{\partial K\cap\partial\Omega} u_0\,\tau_n \\ -\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta}u)\cdot\boldsymbol{\nabla} v + \int_{\partial K} \hat{q}\,\mathsf{sgn}(\boldsymbol{n})v & = \int_K fv \end{cases}$$

Trace and Flux Spaces

$$
\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
\tilde{H}^{1/2}(\Gamma_h^0) &:= \{V|_{\Gamma_h^0} \,:\, V \in H_0^1(\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|v\|_{\tilde{H}^{1/2}(\Gamma_h^0)} &:= \inf\{\|V\|_{H^1} \,:\, V|_{\Gamma_h^0} = v\} \\
H^{-1/2}(\Gamma_h) &:= \{\sigma_n|_{\Gamma_h} \,:\, \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|\sigma_n\|_{H^{-1/2}(\Gamma_h)} &:= \inf\{\|\boldsymbol{\sigma}\|_{H(\mathrm{div},\Omega)} \,:\, \boldsymbol{\sigma n}|_{\Gamma_h} = \sigma_n\}
\end{aligned}
$$

DPG Method, a summary

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u}\,\tau_n & = \int_{\partial K\cap\partial\Omega} u_0\,\tau_n \\ -\int_K \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} v + \int_{\partial K} \hat{q}\,\mathrm{sgn}(\boldsymbol{n})v & = \int_K fv \end{cases}$$

Main points:

- Both equations have been integrated by parts (relaxed).
- Traces $\hat{u} \sim u$ and fluxes $\hat{q} \sim (\boldsymbol{\sigma} - \boldsymbol{\beta}u)\cdot\boldsymbol{n}_e$ are independent unknowns (DPG is a hybrid method).
- Boundary conditions have been built in.
- Test functions are discontinuous (come from “broken” Sobolev spaces). This is critical to enable the idea of using optimal test functions.

Functional Setting

Group variables:

Solution $\boldsymbol{U} = (u, \boldsymbol{\sigma}, \hat{u}, \hat{q})$:

$$u, \sigma_1, \sigma_2 \in L^2(\Omega_h)$$
$$\hat{u} \in \tilde{H}^{1/2}(\Gamma_h^0)$$
$$\hat{q} \in H^{-1/2}(\Gamma_h)$$

Test function $\boldsymbol{V} = (\boldsymbol{\tau}, v)$:

$$\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega_h)$$
$$v \in H^1(\Omega_h)$$

Variational problem:

$$b(\boldsymbol{U}, \boldsymbol{V}) = l(\boldsymbol{V}), \quad \forall \boldsymbol{V}$$

DPG Method, abstract notation

$$\begin{cases} \frac{1}{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\tau})_\Omega + (u, \mathrm{div}\boldsymbol{\tau})_{\Omega_h} - < \hat{u}, \tau_n >_{\Gamma_h^0} & = < u_0, \tau_n >_{\partial\Omega} \\ -(\boldsymbol{\sigma}, \boldsymbol{\nabla} v)_{\Omega_h} - < \hat{q}, v >_{\Gamma_h} & = (f, v)_\Omega \end{cases}$$

$$\begin{aligned} b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(\boldsymbol{\tau},v)) &= (u, \mathrm{div}\boldsymbol{\tau} + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_{\Omega_h} + (\boldsymbol{\sigma}, \tfrac{1}{\epsilon}\boldsymbol{\tau} - \boldsymbol{\nabla} v)_{\Omega_h} \\ &\quad - < \hat{u}, \tau_n >_{\Gamma_h^0} - < \hat{q}, v >_{\Gamma_h} \end{aligned}$$

DPG Method with Optimal Test Functions

Punchlines

Punchlines

- If the test norm is localizable, i.e.

$$(v, \delta v)_V = \sum_K (v, \delta v)_{V_K}$$

where $(v, \delta v)_{V_K}$ defines an inner product for test functions over element K,

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- then the determination of the optimal test functions is done locally. Given trial functions e_i, we compute on the fly corresponding optimal test functions $\hat{e}_i$ by solving element variational problems,

$$\begin{cases} \hat{e}_i \in V(K) \\ (\hat{e}_i, \delta v)_V = b(e_i, \delta v), \quad \forall \delta v \in V(K) \end{cases}$$

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$$\left\{ \begin{array}{l} \hat{e}_i \in V(K) \\ (\hat{e}_i, \delta v)_V = b(e_i, \delta v), \quad \forall \delta v \in V(K) \end{array} \right.$$

- Solution of the local problem above can still be only approximated using an "enriched space" and standard Bubnov-Galerkin method.

Sample test norm(s)

Mathematician's test norm:

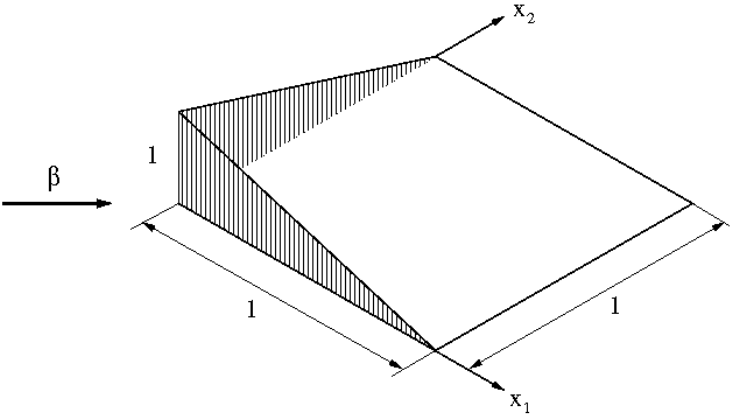
$$\|(v, \boldsymbol{\tau})\|_1^2 := \|v\|^2 + \|\boldsymbol{\nabla} v\|^2 + \|\boldsymbol{\tau}\|^2 + \|\mathsf{div}\boldsymbol{\tau}\|^2$$

Weighted norm:[**]

$$\|(v, \boldsymbol{\tau})\|_2^2 := \|v\|_w^2 + \|\boldsymbol{\nabla} v\|_w^2 + \|\boldsymbol{\tau}\|_w^2 + \|\mathsf{div}\boldsymbol{\tau}\|_w^2$$

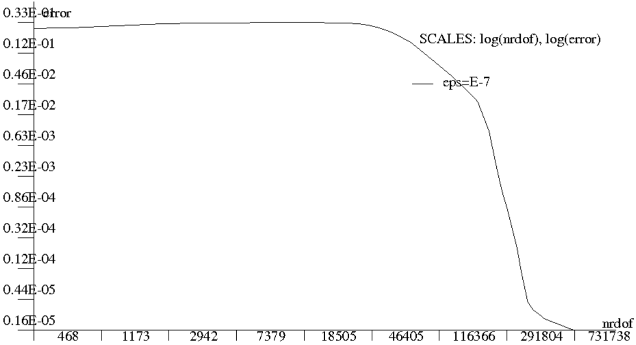
[**] D., J. Gopalakrishnan and A. Niemi, "A class of discontinuous Petrov-Galerkin methods. Part III: Adaptivity," *ICES Report* 2010-01, *App. Num Math.*, accepted.

2D Convection-Dominated Diffusion



Problem definition.

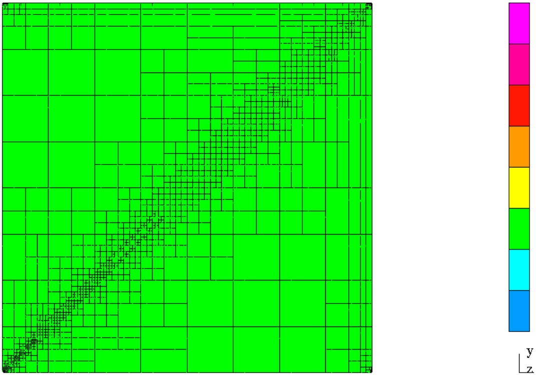
$\epsilon = 10^{-7}$



Convergence history in a (dynamically rescaled[††]) energy norm

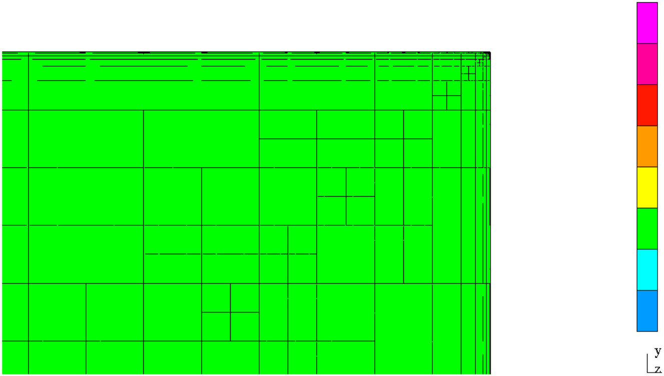
[††] To fight round off error

$\epsilon = 10^{-7}$



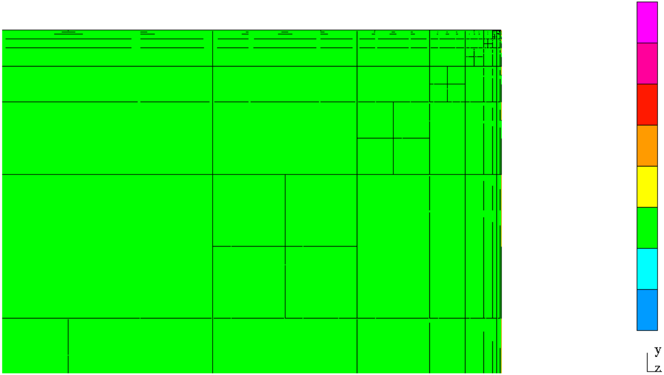
Optimal hp mesh after 45 mesh refinements.

$\epsilon = 10^{-7}$



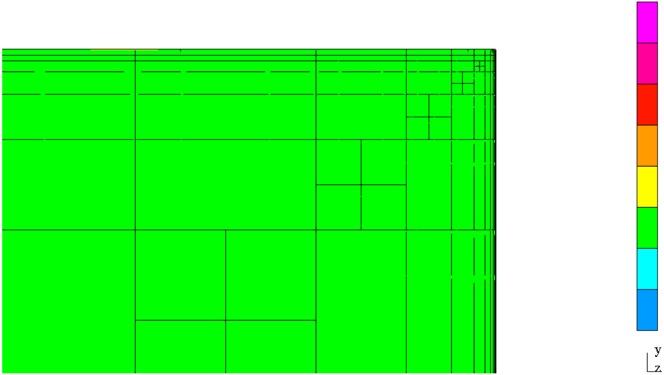
Optimal hp mesh after 45 mesh refinements. Zoom $\times 10$ on the north-east corner.

$\epsilon = 10^{-7}$



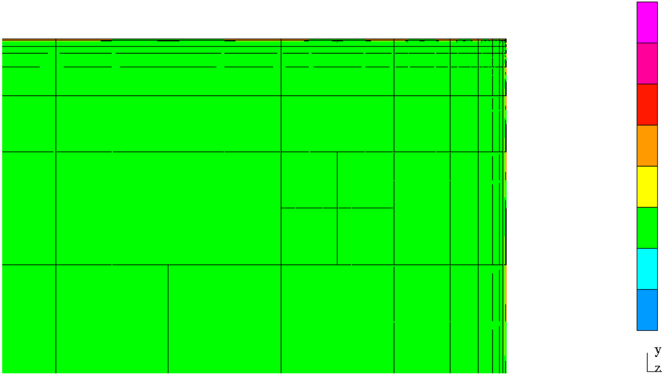
Optimal hp mesh after 45 mesh refinements. Zoom $\times 100$ on the north-east corner.

$\epsilon = 10^{-7}$

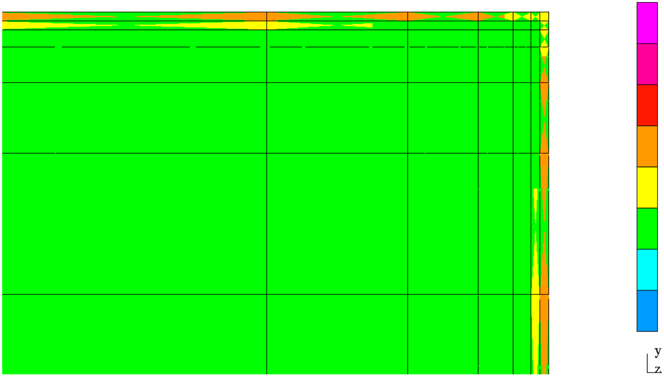


Optimal hp mesh after 45 mesh refinements. Zoom $\times 1000$ on the north-east corner.

$\epsilon = 10^{-7}$

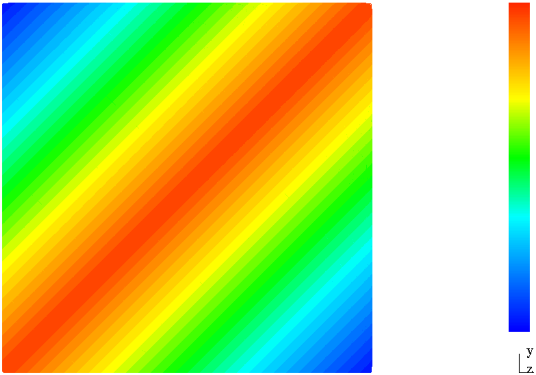


Optimal hp mesh after 45 mesh refinements. Zoom $\times 10000$ on the north-east corner.



Optimal hp mesh after 45 mesh refinements. Zoom $\times 10^5$ on the north-east corner.

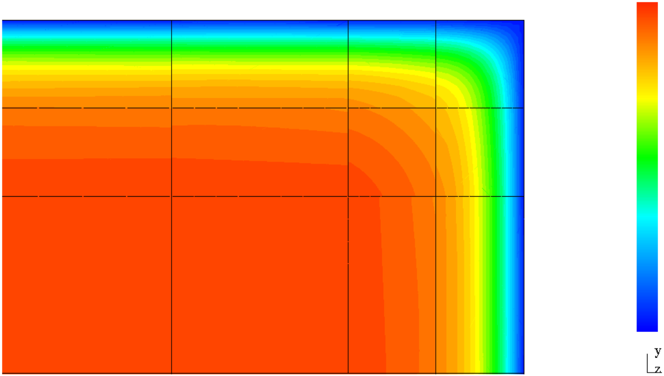
$\epsilon = 10^{-7}$



Velocity u.

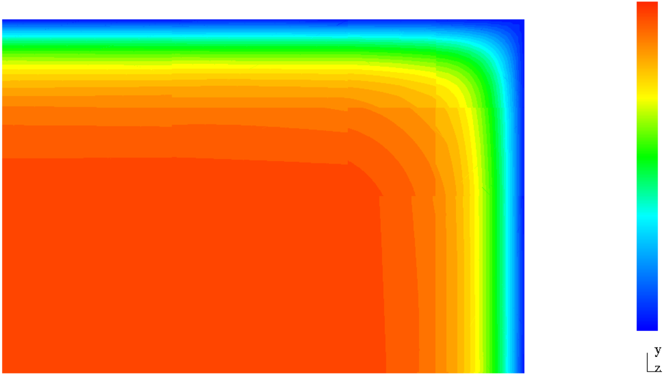
$\epsilon = 10^{-7}$

Velocity u. Zoom $\times 10^5$ on the north-east corner.



Velocity u. Zoom $\times 10^6$ on the north-east corner with the mesh.

$\epsilon = 10^{-7}$



Velocity u. Zoom $\times 10^6$ on the north-east corner w/o the mesh.
OK, is not ideal yet...

Outline

- Babuška's Theorem.
- Struggle with discrete stability.
- Optimal test functions and least squares.
- Ultraweak variational formulation and DPG Method.
- Systematic choice of test norm.

Optimal test norm

Q: Can we select the norm in the test space in such a way that the corresponding energy norm coincides with the original norm (of choice) in U ?

A: Yes! Choose:

$$\|v\|_V = \sup_{u \in U} \frac{|b(u,v)|}{\|u\|_U}$$

(under assumption that

$$V_0 = \{v \in V \ : \ b(u,v) = 0 \quad \forall u \in U\}$$

is trivial)

Linear Acoustics. Quasi-optimal test norm

Sesquilinear form

$$b(\boldsymbol{U},\boldsymbol{V}) = -(u, i\omega \boldsymbol{v} + \boldsymbol{\nabla} q)_{\Omega_h} - (p, i\omega q + \mathsf{div}\boldsymbol{v})_{\Omega_h} + < \hat{u}_n, q >_{\Gamma_h^0} + < \hat{p}, v_n >_{\Gamma_h}$$

Trial norm:

$$\|(\boldsymbol{u}, p, \hat{u}_n, \hat{p})\|_U^2 = \|\boldsymbol{u}\|_{L^2}^2 + \|p\|_{L^2}^2 + \|\hat{u}\|_{?}^2 + \|\hat{p}\|_{?}^2$$

Optimal test norm (unfortunately, non-local):

$$\|(\boldsymbol{v}, q)\|_{opt}^2 = \|i\omega\boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 + \sup_{\hat{u}_n,\hat{p}} \frac{|<\hat{u}_n,q>+<\hat{p},v_n>|}{(\|\hat{u}_n\|_{?}^2+\|\hat{p}_{?}^2)^{1/2}}$$

Quasi-optimal test norm (local):

$$\|(\boldsymbol{v}, q)\|_{opt}^2 = \|i\omega\boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 + \|\boldsymbol{v}\|^2 + \|q\|^2$$

Robust stability result

Theorem: [‡‡] Assume: Ω contractable, impedance BC
Use: the quasi-optimal norm to define the minimum energy extension norms for fluxes $\hat{u}_n$ and traces $\hat{p}$.
Then

$$\|(\boldsymbol{v},q)\|_{opt}^2 \approx \|(\boldsymbol{v},q)\|_{qopt}^2 \quad \text{(uniformly in } k \text{ and mesh)}$$

Consequently, we get the robust stability in the *desired norm*:

$$\begin{aligned}
&\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|^2 + \|p-p_h\|^2 + \|\hat{u}_n - \hat{u}_{n,h}\| + \|\hat{p}-\hat{p}_h\|^2\right)^{\frac{1}{2}} \\
&\lesssim \|(\boldsymbol{u},p,\hat{u}_n,\hat{p}) - (\boldsymbol{u}_h,p_h,\hat{u}_{n,h},\hat{p}_h)\|_E \\
&= \text{BAE of } (\boldsymbol{u},p,\hat{u}_n,\hat{p}) \text{ in energy norm} \\
&\lesssim \text{BAE of } (\boldsymbol{u},p,\hat{u}_n,\hat{p}) \text{ in desired norm}
\end{aligned}$$

[‡‡] D., J. Gopalakrishnan, I. Muga, and J. Zitelli. "Wavenumber Explicit Analysis for a DPG Method for the Multidimensional Helmholtz Equation", *ICES Report 2011-24*, submitted to *CMAME*.

No pollution in 1D case

In 1D, traces and fluxes and just numbers. Thus, the BAE of fluxes and traces is zero. We get,

$$\left(\|u-u_h\|^2+\|p-p_h\|^2+\|\hat{u}_n-\hat{u}_{n,h}\|+\|\hat{p}-\hat{p}_h\|^2\right)^{\frac{1}{2}}$$

$$\lesssim \inf_{w_h,r_h}\left(\|u-w_h\|^2+\|p-r_h\|^2\right)^{\frac{1}{2}}$$

The BAE of u, p in L^2-error is pollution free.

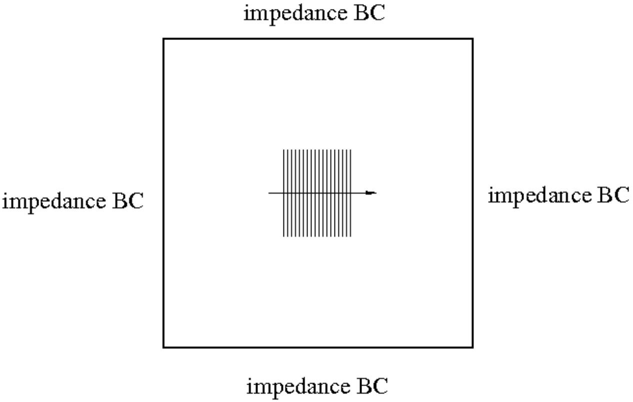
2D experiments

Discretization:

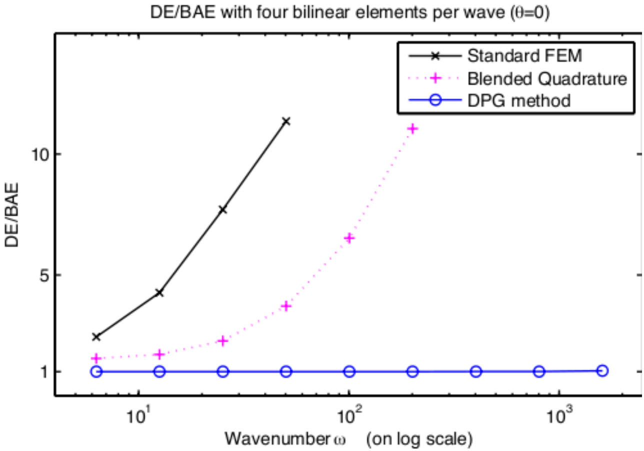
- field variables are discretized using isoparametric L^2-conforming quads of order p,
  $u_1, u_2, p \in \mathcal{P}^p \otimes \mathcal{P}^p$,
- traces are discretized using H^1-conforming elements of order $p+1$,
- fluxes are discretized using L^2-conforming elements of order $p+1$
- optimal test functions are approximated with polynomials of order $p+1+\Delta p$, i.e. $\boldsymbol{v} \in (\mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p}) \times (\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p+1})$,
  $q \in \mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p+1}$

2D experiment A

Exact solution: horizontal plane wave
Enriched space: $\Delta p = 2$.



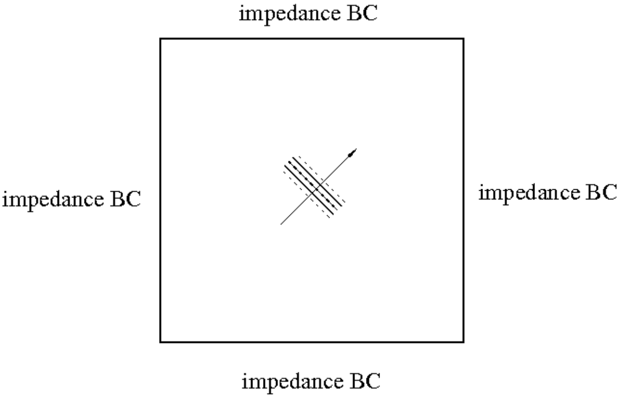
2D experiment A



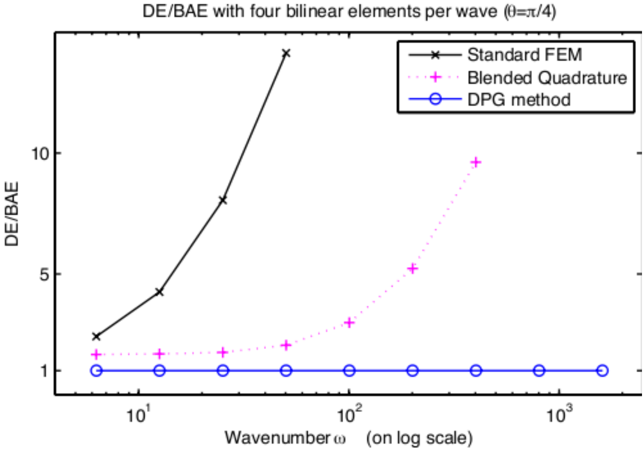
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment B

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.



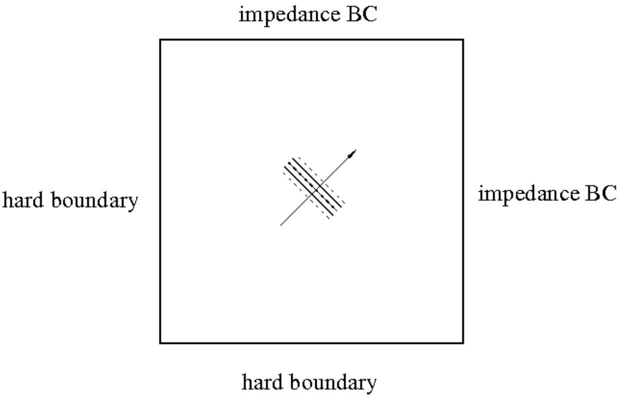
2D experiment B

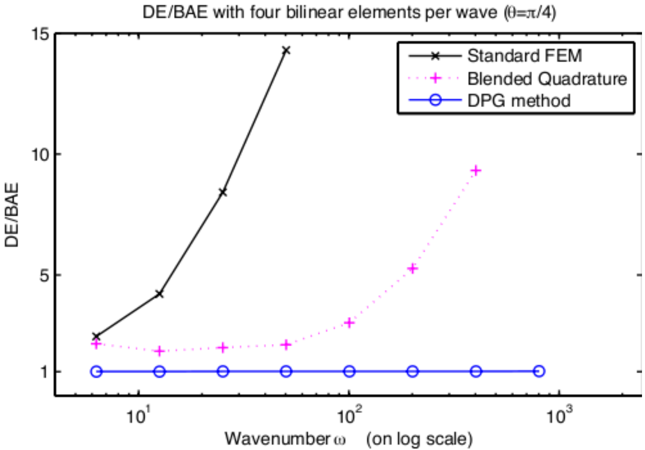


Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment C

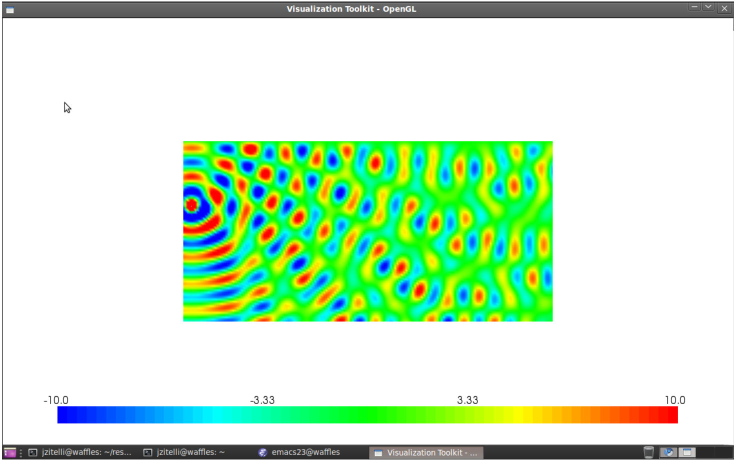
Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.





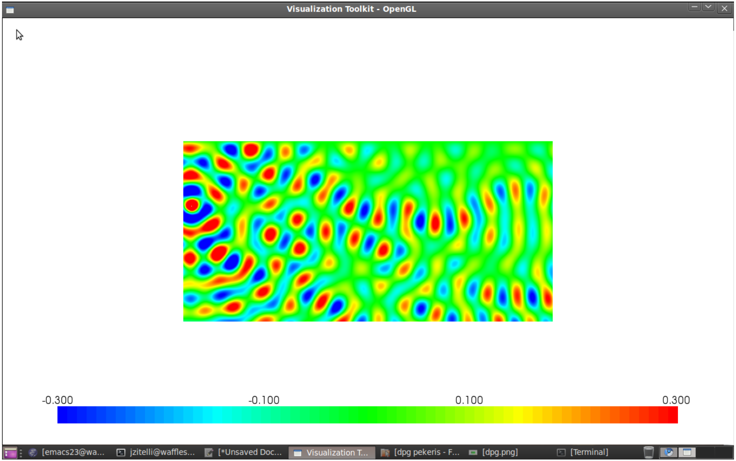
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

Pekeris problem, $k = 50$ (8 wavelengths)



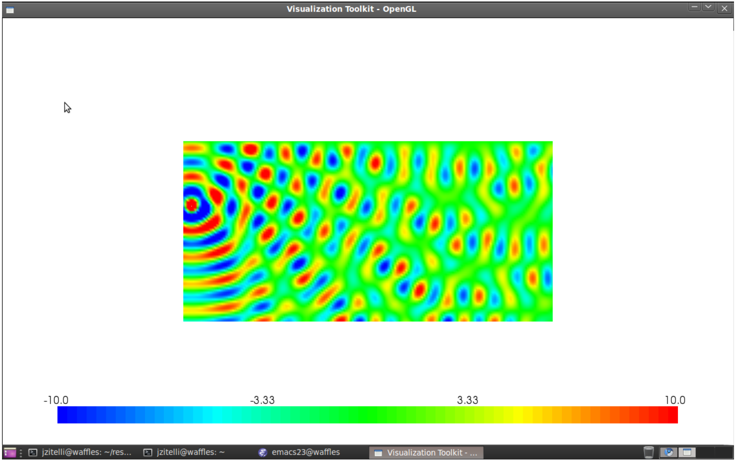
Exact solution (real part of pressure).

Pekeris problem, $k = 50$ (8 wavelengths)



Classical FEs, four biquadratic elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



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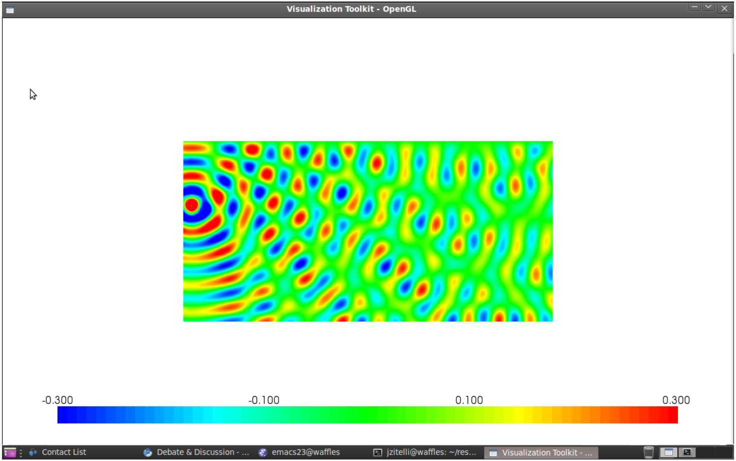
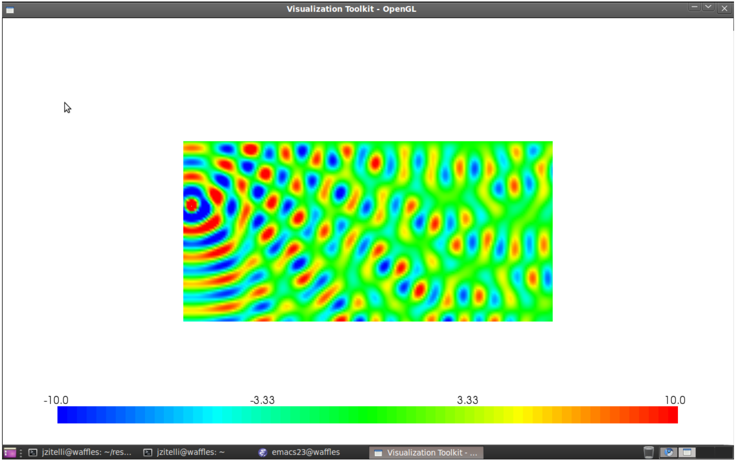


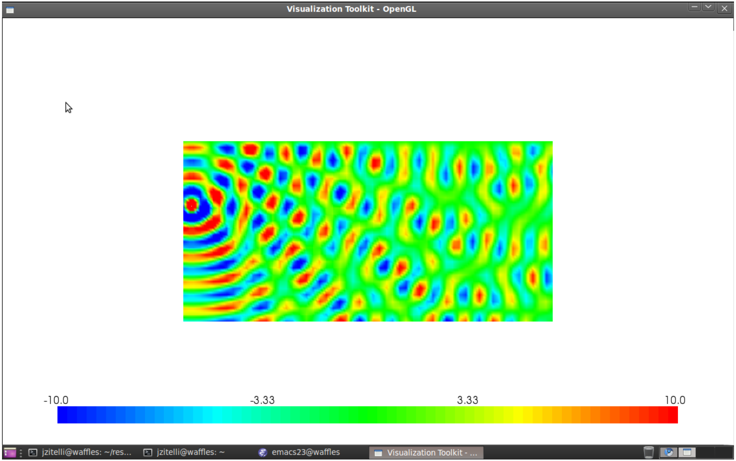
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



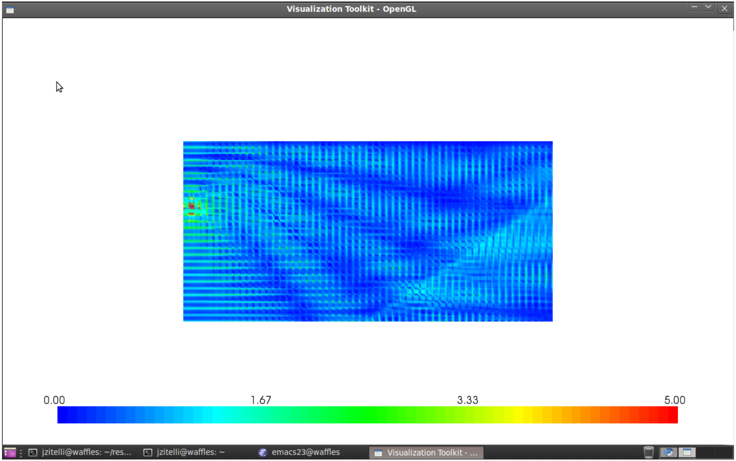
Exact solution (real part of pressure).

Pekeris problem, $k = 50$ (8 wavelengths)



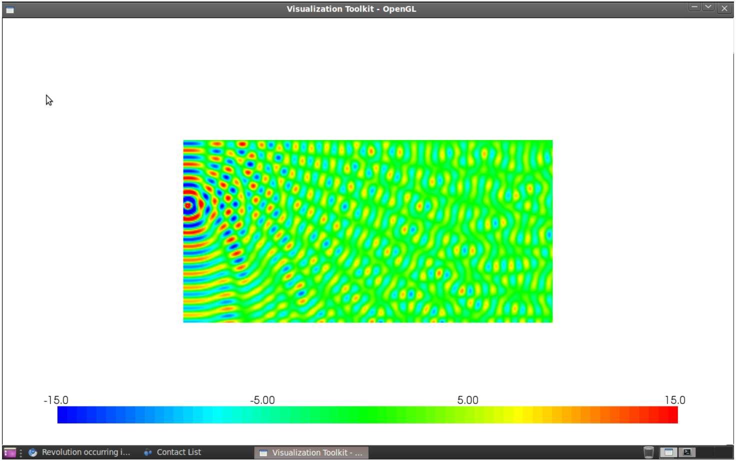
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



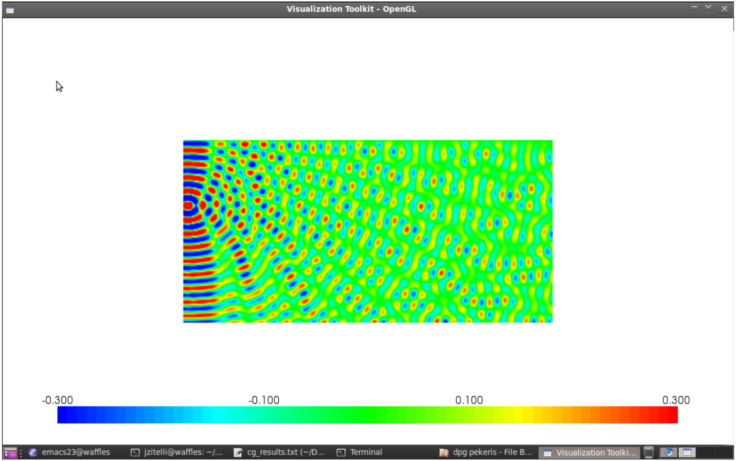
Error for the DPG method.

Pekeris problem, $k = 100$ (16 wavelenghts)



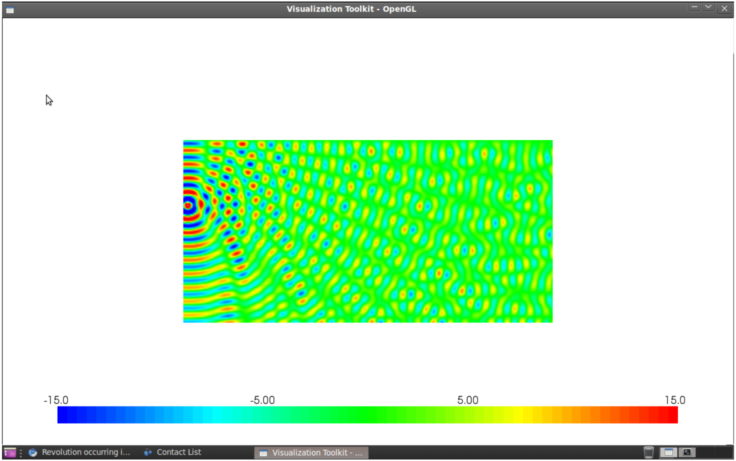
Exact solution (real part of pressure).

Pekeris problem, $k = 100$ (16 wavelenghts)



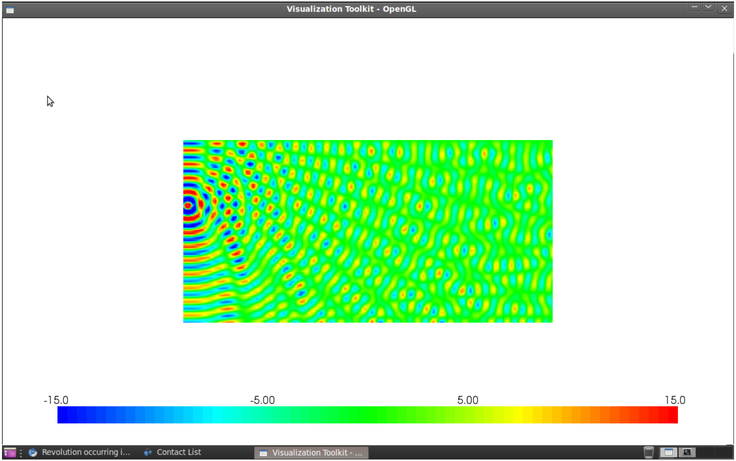
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 100$ (16 wavelenghts)



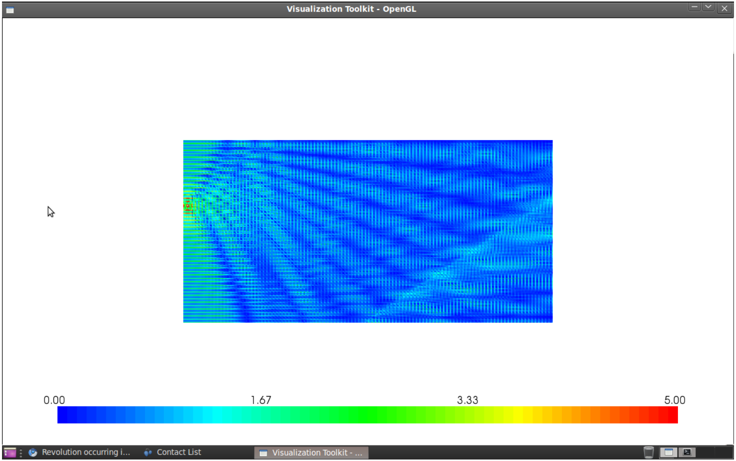
Exact solution (real part of pressure).

Pekeris problem, $k = 100$ (16 wavelenghts)



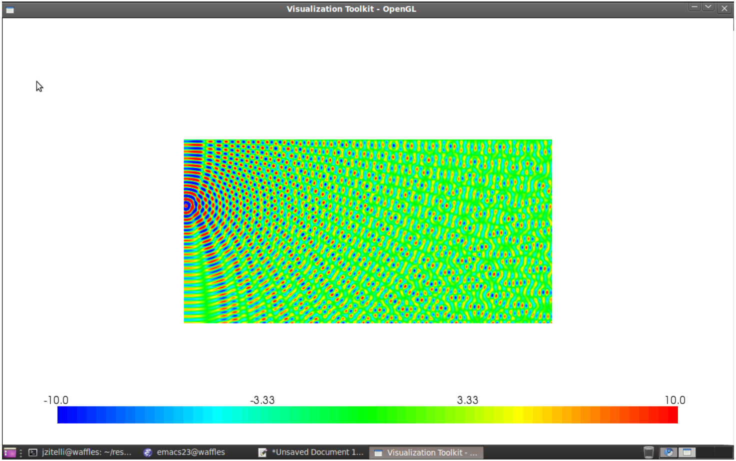
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 100$ (16 wavelenghts)



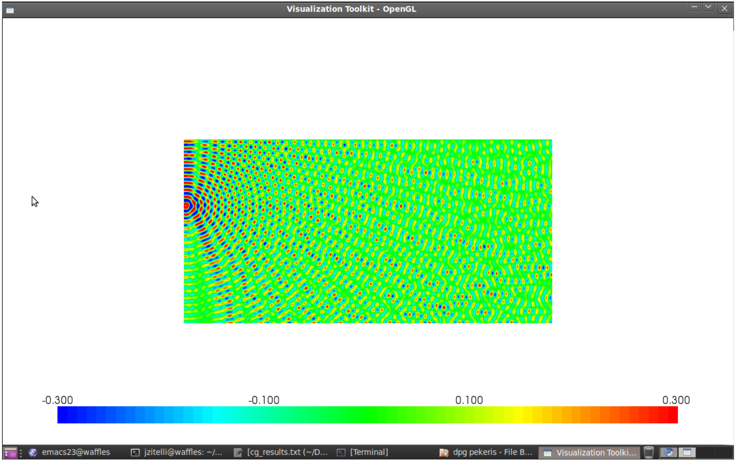
Error for the DPG method.

Pekeris problem, $k = 200$ (32 wavelenghts)



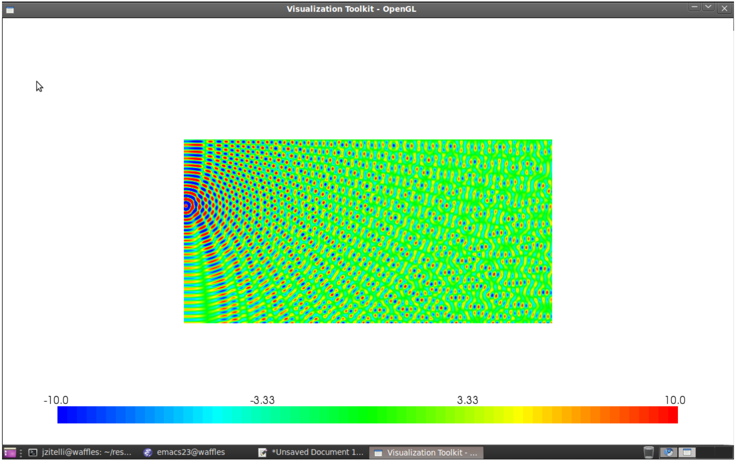
Exact solution (real part of pressure).

Pekeris problem, $k = 200$ (32 wavelenghts)



Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 200$ (32 wavelenghts)



Exact solution (real part of pressure).

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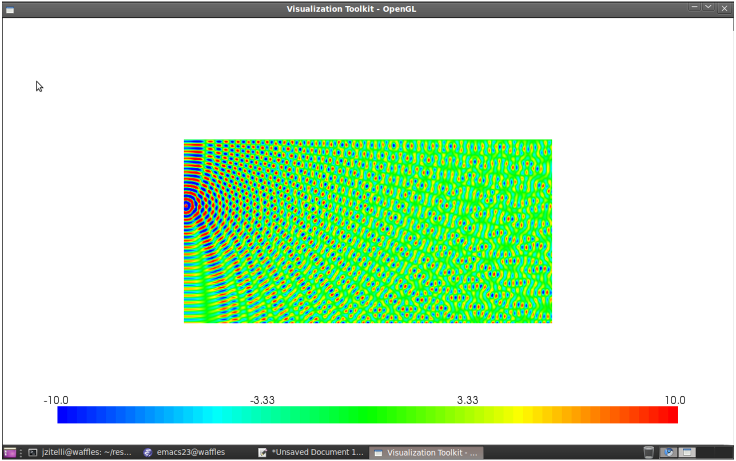

DPG method, four bilinear elements per wavelength.

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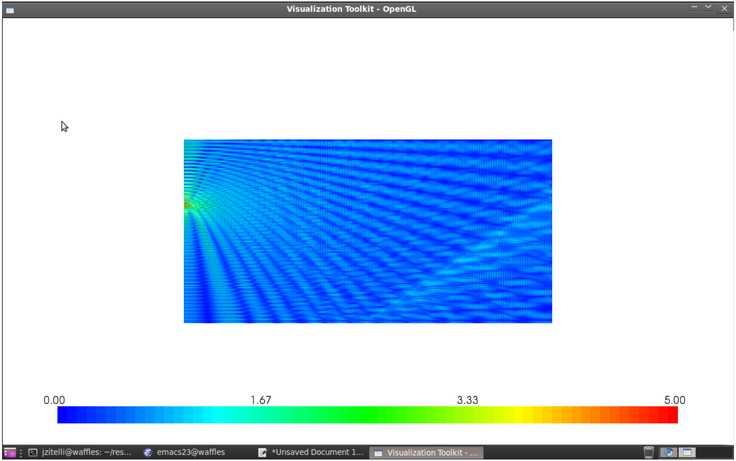


Error for the DPG method.

A Recipe:

How to Construct a Robust DPG Method

for the Confusion Problem

(and Any Other Linear Problem as Well)

Step 1: Decide what you want

We want the L^2 robustness in u:

$$\|u\| \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E$$

($a \lesssim b$ means that there exists a constant C, independent of ϵ such that $a \leq Cb$). This implies

$$\begin{aligned} \|u - u_h\| &\lesssim \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E \\ &= \underbrace{\inf_{(u_h, \boldsymbol{\sigma}_h, \hat{u}_h, \hat{q}_h)} \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E}_{\text{Best Approximation Error (BAE)}} \\ &\leq C(\epsilon) h^p \end{aligned}$$

Step 2: Select a special test function...

$$b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) \quad = (\boldsymbol{\sigma},\tfrac{1}{\epsilon}\boldsymbol{\tau}+\boldsymbol{\nabla} v)_{\Omega_h} + (u,\mathrm{div}\boldsymbol{\tau}-\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_{\Omega_h}$$

$$-<\hat{u},\tau_n>_{\Gamma_h^0} - <\hat{q},v>_{\Gamma_h}$$

Choose a test function $(v,\boldsymbol{\tau})$ such that

$$\begin{cases} v\in H_0^1(\Omega),\ \boldsymbol{\tau}\in \boldsymbol{H}(\mathrm{div},\Omega) \\ \frac{1}{\epsilon}\boldsymbol{\tau}+\boldsymbol{\nabla} v \quad = 0 \\ \mathrm{div}\boldsymbol{\tau}-\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v \quad = u \end{cases}$$

Then

$$\|u\|^2 \quad = b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) = \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V$$

$$\leq \sup_{(v,\boldsymbol{\tau})}\frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V = \|(u,\boldsymbol{\sigma},\hat{u},\hat{q})\|_E\,\|(v,\boldsymbol{\tau})\|_V$$

… and request stability of the adjoint problem

Consequently, we need to select the test norm in such a way that

$$\|(v, \boldsymbol{\tau})\|_V \lesssim \|u\|$$

This gives,

$$\|u\|^2 \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E \, \|u\|$$

Dividing by $\|u\|$, we get what we wanted.

The point: Construction of a robust DPG reduces to the classical stability analysis for the adjoint equation!

Step 3: Study the stability of the adjoint equation

Theorem (Generalization of Erickson-Johnson Theorem) (Heuer,D., 2011)

$$\left.\begin{array}{r} \|v\| \\ \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|_w, \sqrt{\epsilon}\|\boldsymbol{\nabla} v\| \\ \|\mathrm{div}\boldsymbol{\tau}\|_{w+\epsilon}, \frac{1}{\epsilon}\|\boldsymbol{\beta}\cdot\boldsymbol{\tau}\|_w, \frac{1}{\sqrt{\epsilon}}\|\boldsymbol{\tau}\| \end{array}\right\} \lesssim \|u\|$$

where $w = O(1)$ is a weight vanishing on the inflow boundary that satisfies some "mild" assumptions.

The terms on the left-hand side are our "Lego" blocks with which we can build different test norms.

Step 4: Construct test norm(s)

Quasi-optimal test norm:

$$\|(v,\boldsymbol{\tau})\|_1^2 := \|v\|^2 + \|\frac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v\|^2 + \|\mathrm{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|^2$$

Weighted norm:

$$\|(v,\boldsymbol{\tau})\|_2^2 := \epsilon\|v\|^2 + \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|_w^2 + \epsilon\|\boldsymbol{\nabla} v\|^2 + \|\boldsymbol{\tau}\|_{w+\epsilon}^2 + \|\mathrm{div}\boldsymbol{\tau}\|_{w+\epsilon}^2$$

Remark: Both choices imply also L^2-robustness in $\boldsymbol{\sigma}$, as well as in traces and fluxes measured in special energy norms.

Estimates for $\boldsymbol{\sigma}, \hat{u}, \hat{q}$

Same methodology can be used to design a test norm that will imply,

$$\|\boldsymbol{\sigma}\| \lesssim \|(\boldsymbol{\sigma}, u, \hat{u}, \hat{q})\|_E$$

In fact both quasioptimal and weighted norms imply the robust estimate for $\boldsymbol{\sigma}$. They also imply a robust estimate for traces and fluxes measured in a minimum extension norm implied by the problem,

$$(*) \qquad \|(\hat{u}, \hat{q})\|^2 := \|\frac{1}{\epsilon}\Sigma - \boldsymbol{\nabla} U\|^2 + \| - \mathrm{div}\Sigma + \boldsymbol{\beta} \cdot \boldsymbol{\nabla} U\|^2$$

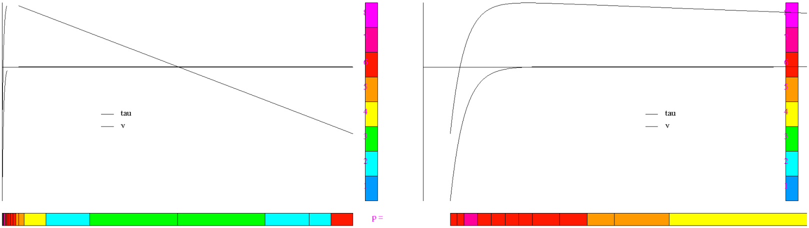
where Σ, U are extensions of $\hat{u}, \hat{q}$ from mesh skeleton to the whole domain,

$$U = \hat{u} \text{ on } \Gamma_h^0, \quad (\Sigma - \boldsymbol{\beta} U) \cdot \boldsymbol{n}_e = \hat{q} \text{ on } \Gamma_h$$

that minimize the right hand side of $(*)$.

Pros and cons for both test norms

- The quasi-optimal test norm produces strong boundary layers that need to be resolved, also in 1D,

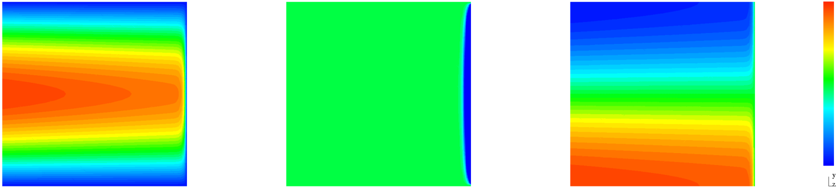


  Left: τ and v components of the optimal test function corresponding to trial function $u = 1$ and element size $h = 0.25$, along with the optimal hp subelement mesh. Right: $10 \times$ zoom on the left end of the element.
  Determining optimal test functions is expensive.
- The weighted test norm produces no boundary layers. Solving for the optimal test functions is inexpensive.
- Quasi-optimal test norm yields better estimates for the best approximation error measured in the corresponding energy norm.

2D: Model problem of Erickson and Johnson

$$\Omega = (0,1)^2, \quad \boldsymbol{\beta} = (1,0), f = 0, \qquad u_0 = \begin{cases} \sin \pi y & \text{on } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

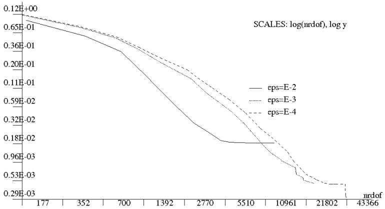
The problem can be solved analytically using separation of variables.

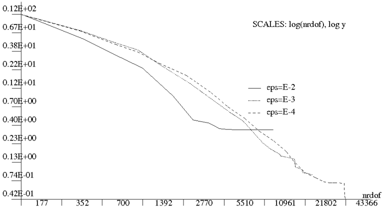


Velocity u and "stresses" σ_x, σ_y (using scale for σ_y) for $\epsilon = 0.01$.

2D: Weighted norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

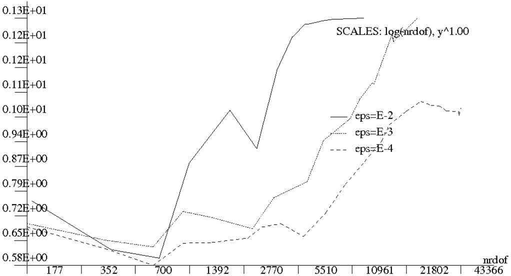
Weight: $w = x$.





Left: convergence in energy error. Right: convergence in relative L^2-error for the field variables (in percent of their L^2-norm).

2D: Weighted norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$



Ratio of L^2 and energy norms.

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- The implied discrete stability holds for hp meshes enabling hp-adaptivity.

Thank You !

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