CES 389C/EM 397 INTRODUCTION TO MATHEMATICAL MODELING IN SCIENCE AND ENGINEERING
Exam 1, Oct 21, 2012

1. Define the following notions and provide a non-trivial example (2+2 points each).
   - Lagrangian (in analytical mechanics),
   - Legendre transformation,
   - Green – St. Venant strain tensor
   - velocity gradient
   - second Piola-Kirchhoff stress tensor

   Please clearly distinguish between material and spatial coordinates.

2. Use Principles of Linear and Angular Momentum and formula for the velocities of points belonging to a rigid body (you need not derive them) to derive the rigid body equations of motion (15 points).

3. Derive the formulas for the velocity of a particle in a deformable body in terms of its displacement in *both* Lagrange and Euler coordinates. Illustrate the formulas with an example (15 points).

4. Derive equations of motion for a continuum in both Euler (Cauchy stress tensor) and Lagrange (Piola-Kirchhoff stress tensor) (15 points).

5. Consider the "three-quarter" homogeneous thin plate with radius R and mass m shown in Fig. 1. Compute the 3D inertia tensor at point A. Determine the direction through A for which the corresponding moment of inertia is maximal and determine its value (20 points).

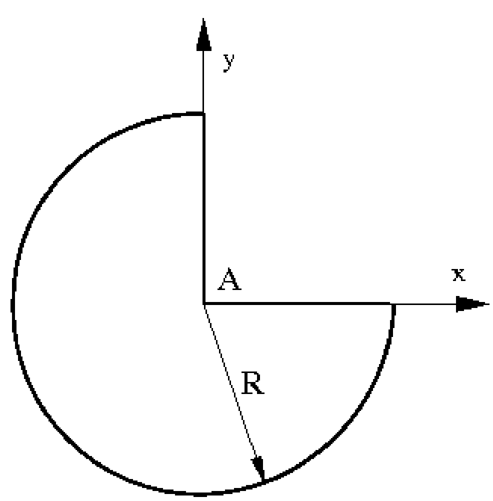

Figure 1: A semicircular plate.

We will use the standard polar coordinates to parametrize the domain

$$\begin{cases} x = r\cos\theta & 0 < r < R \\ y = r\sin\theta & \frac{\pi}{2} < \theta < 2\pi \end{cases}$$

Area:

$$A = \int_r^R \int_{\pi/2}^{2\pi} r\, drd\theta = \int_0^R r\, dr \int_{\pi/2}^{2\pi} d\theta = \frac{R^2}{2}\frac{3\pi}{2} = \frac{3}{4}\pi R^2$$

Density:

$$\rho = \frac{m}{A} = \frac{4}{3}\frac{m}{\pi R^2}$$

Moment of inertia with respect to axis x for a homogeneous thin body (variation in z neglected):

$$I_x = \int_A \rho(y^2+z^2)\, dA = \int_A \rho y^2\, dA = \rho \int_A y^2\, dA$$

and

$$\int_A y^2 dA = \int_0^R \int_{\pi/2}^{2\pi} r^2 \sin^2\theta r drd\theta = \int_0^R r^3 dr \int_{\pi/2}^{2\pi} \sin^\theta\, d\theta = \frac{R^4}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)|_{\pi/2}^{2\pi} = \frac{R^4}{4}\frac{3\pi}{4} = \frac{3\pi R^4}{16}$$

gives

$$I_x = \frac{4m}{3\pi R^2}\frac{3\pi R^4}{16} = \frac{1}{4}mR^2$$

(units OK). By symmetry,

$$I_y = I_x = \frac{1}{4}mR^2$$

For thin bodies,

$$I_z = \int_A \rho(x^2+y^2)\, dA = \int_A \rho x^2\, dA + \int_A \rho y^2\, dA = I_y + I_x = \frac{1}{mR^2}$$

Product of inertia for a homogeneous body with respect to axes x and y:

$$I_{xy} = \rho \int_A xy\, dA$$

and,

$$\begin{aligned} \int_A xy\, dA &= \int_0^R \int_{\pi/2}^{2\pi} r^2 \cos\theta \sin\theta \; rdrd\theta = \int_0^R r^3\, dr \int_{\pi/2}^{2\pi} \frac{\sin 2\theta}{2}\, d\theta \\ &= \frac{R^4}{4}\left(-\frac{\cos 2\theta}{4}\right)|_{\pi/2}^{2\pi} = \frac{R^4}{4}(-\frac{1}{4})(1+1) = -\frac{R^4}{8} \end{aligned}$$

gives

$$I_{xy} = \frac{4}{3}\frac{m}{\pi R^2}\left(-\frac{R^4}{8}\right) = -\frac{1}{6}\frac{mR^2}{\pi}$$

As products of inertia $I_{xz} = I_{yz} = 0$ ($z \approx 0$), the whole tensor of inertia at point A is:

$$\boldsymbol{I}_A = \begin{pmatrix} \frac{1}{4} & \frac{1}{6\pi} & 0 \\ \frac{1}{6\imath} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} mR^2$$

The form of the tensor proves that $\frac{1}{2}$ equals one of the eigenvalues with the z axis being the corresponding eigendirection. The eigenproblems reduces thus to two space dimensions (x, y) only.

Invariants:

$$I_1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad I_2 = \frac{1}{16} - \frac{1}{36\pi^2}$$

Charasteristic equation:

$$\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} - \frac{1}{36\pi^2} = 0$$

$$\Delta = \frac{1}{4} - 4(\frac{1}{16} - \frac{1}{36\pi^2}) = \frac{1}{9\pi^2} \qquad \sqrt{\Delta} = \frac{1}{3\pi}$$

Eigenvalues

$$\lambda_1 = \frac{1}{4} - \frac{1}{6\pi}, \qquad \lambda_2 = \frac{1}{4} + \frac{1}{6\pi}$$

represent prinicipal moments of inertial with respect to corresponding eigendirections. Both of them, however, are smaller then I_z. In other words, I_z is the largest possible moment of inertia with repect to an axis passing through point A.

6. Reproduce the Coleman–Noll argument and discuss its consequences (15 points).