CSE386L MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
Spring 22, Exam 2

1. Separation of variables. Consider the BVP defined in Fig. 1. Use separation of variables to derive the solution in a form of a series. Define the coefficients in the series but you do not need to compute them.

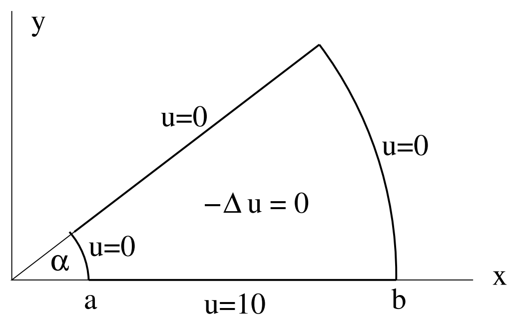

Figure 1: Laplace equation in a conical domain.

(20 points)

Solution 1: Use separation of variables: $u = R(r)\Theta(\theta)$ to arrive at:

$$-\frac{1}{r}(rR')'\Theta - \frac{1}{r^2}R\Theta'' = 0 \quad \Rightarrow \quad -\frac{r(rR')'}{R} = \frac{\Theta''}{\Theta} = \lambda \,.$$

BCs imply that we should consider the Sturm-Liouville problem in r,

$$LR := -r(rR')' = \lambda R \,.$$

The operator L is self-adjoint in the weighted space $L^1_{1/r}$ and positive definite (easy to check) so $\lambda > 0$. Assume $\lambda = k^2, k > 0$. We obtain the Cauchy-Euler equation:

$$-r^2R'' - rR' - k^2 = 0 \,.$$

Seeking $R = r^\alpha$, we obtain $\alpha = \pm ik$. This gives

$$R = r^{\pm ik} = e^{\pm ik \ln r} = \cos(k \ln r) \pm i \sin(k \ln r) \,.$$

Alternatively, we can represent the general solution as:

$$R = C \cos(k \ln r) + D \sin(k \ln r) \,.$$

The condition for the existence of non-trivial solutions with the homogeneous BCs is:

$$\begin{vmatrix} \cos(k\ln a) & \sin(k\ln a) \\ \cos(k\ln b) & \sin(k\ln b) \end{vmatrix} = \sin(k\ln\frac{b}{a}) = 0 \quad \Rightarrow \quad k = k_n = \frac{n\pi}{\ln\frac{b}{a}} \, .$$

With known separation constant, the corresponding Θ is:

$$\Theta = A_n e^{k_n\theta} + B_n e^{-k_n\theta} \, ,$$

or,

$$\Theta = A_n \cosh(k_n\theta) + B_n \sinh(k_n\theta) \, ,$$

or, looking at the BC at $\theta = \alpha$,

$$\Theta = A_n \cosh(k_n(\theta-\alpha)) + B_n \sinh(k_n(\theta-\alpha)) \, .$$

The BC dictates then:

$$\Theta = B_n \sinh(k_n(\theta-\alpha)) \, .$$

The ultimate solution is:

$$u = \sum_{n=1}^{\infty} \underbrace{\left(\cos(k_n\ln r) - \frac{\cos(k_n\ln a)}{\sin(k_n a)}\sin(k_n\ln r)\right)}_{=:R_n(r)} B_n \sinh(k_n(\theta-\alpha)) \, .$$

Constants B_n are determined from the BC for $\theta = 0$,

$$-\sum_{n=1}^{\infty} R_n(r) B_n \sinh(k_n(\alpha)) = 10 \, .$$

Multiplying both sides with $\frac{1}{r}R_m(r)$, integrating over (a,b) interval, and using the $L^2_{1/r}$-orthogonality of functions $R_n(r)$, we get

$$B_m = -\frac{\int_a^b \frac{10}{r} R_m(r)\,dr}{\int_a^b \frac{1}{r} R_m^2(r)\,dr \, \sinh(k_n\alpha)} \, .$$

Solution 2: As the Sturm-Liouville problem in r is more complicated than in θ, we may try to revrese the role of r and θ by introducing a *lift* U of Dirichlet data $u_0 = 10$ such that U satisfies the Laplace equation. Upon inspection, we propose

$$U = 10(1 - \frac{\theta}{\alpha}) \, .$$

Function U is linear in θ and independent of r, so it satisfies the Laplace equation and BCs at $\theta = 0$ and $\theta = \alpha$. We seek the ultimate solution in the form

$$u = U + v$$

where

$$v = -U \quad \Rightarrow \quad v = -U \text{ at } r = a, b \text{ and } v = 0 \text{ at } \theta = 0, \alpha\,.$$

As $-\Theta''$ is self-adjoint and positive-definite, we can look for the separation constant $\lambda = k^2$. The Sturm-Liouville problem in θ leads to

$$k = k_n = \frac{n\pi}{\alpha}, \quad \Theta = \Theta_n = \sin(k_n\theta)\,.$$

The solution of the corresponding Cauchy-Euler equation is:

$$R_n = A_n r^{k_n} + B_n r^{-k_n}\,,$$

which gives:

$$v = \sum_{n=1}^{\infty} (A_n r^{k_n} + B_n r^{-k_n}) \sin(k_n\theta)\,.$$

Enforcing BCs at $r = a$ and $r = b$, we get:

$$\sum_{n=1}^{\infty} (A_n a^{k_n} + B_n a^{-k_n}) \sin(k_n\theta) = 10(\frac{\theta}{\alpha} - 1) \quad \text{and} \quad \sum_{n=1}^{\infty} (A_n b^{k_n} + B_n b^{-k_n}) \sin(k_n\theta) = 10(\frac{\theta}{\alpha} - 1)\,.$$

Multiplying both equations with $\sin k_m\theta$, integrating over $(0, \alpha)$, utilizing the L^2-orthogonality of eigenfunctions $\sin k_n\theta$, and noticing that

$$\int_0^\alpha \sin^2(k_m\theta)\,d\theta = \int_0^\alpha \frac{1 - \cos 2k_m\theta}{2}\,d\theta = \frac{\alpha}{2}\,,$$

we get a system of two equations for constants A_n, B_n,

$$\begin{pmatrix} a^{k_n} & a^{-k_n} \\ b^{k_n} & b^{-k_n} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \frac{20}{\alpha} \int_0^\alpha (\frac{\theta}{\alpha} - 1) \sin k_n\theta\,d\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

2. Calculus of variations. Find the geodesics for a cylinder. Formulate the minimization problem, write down the corresponding variational problem, the E-L equation, and solve it. *Hint:* Look for the geodesics as a graph of function $z = z(\theta)$ or $\theta = \theta(z)$.

   (20 points)

   Solution: We can always rotate and shift the cylindrical coordinates in such a way that the two end-points have coordinates $\theta = 0, z = 0$ and $\theta = \alpha, z = b$.

   Case: $\alpha > 0$. Parametrizing a curve on the cylinder with θ, we have:

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= f(\theta)\,. \end{cases}$$

   Consequently,

$$ds = \sqrt{r^2 + (f')^2}\, d\theta\,.$$

   The minimization problem is then:

$$\min_{f(0)=0, f(\alpha)=b} \int_0^\alpha \sqrt{r^2 + (f')^2}\, d\theta\,.$$

   The corresponding variational formulation is:

$$\begin{cases} \text{Find } f(\theta), \theta \in (0,\alpha) \text{ such that:} \\ \displaystyle\int_0^\alpha \frac{f'g'}{\sqrt{r^2+(f')^2}}\, d\theta = 0 \quad \forall\, g(\theta) \,:\, g(0) = g(\alpha) = 0\,. \end{cases}$$

   The Euler-Lagrange equation is:

$$-\left(\frac{f'}{\sqrt{r^2+(f')^2}}\right)' = 0 \quad \Rightarrow \quad \frac{f'}{\sqrt{r^2+(f')^2}} = c\,.$$

   This leads to

$$f' = \frac{c^2 r^2}{1-c^2} = \text{const}\,.$$

   The geodesics is thus the straight line in (θ, z) plane, connecting the two points.

   Case: $\alpha = 0$. We must parametrize now with z,

$$\begin{cases} x &= r\cos f(z) \\ y &= r\sin f(z) \\ z &= z\,. \end{cases}$$

   This gives:

$$ds = \sqrt{r^2[f'(z)]^2 + 1}\, dz\,.$$

The minimization problem is now:

$$\min_{f(0)=f(b)=0} \int_0^b \sqrt{r^2[f'(z)]^2 + 1}\, dz\,.$$

This is essentially the same minimization problem as before. The solution of the Lagrange equation is again $f(z) = \mathrm{const}$, and the BCs imply $f(z) = 0$. The curve is again a straight line in the (θ, z) plane connecting the two points.

3. Complex variables.

   - State the Residue Theorem.
   - Use the Residue Theorem to compute the integral:

   $$\int_c \frac{dz}{z^2(z^2+3)}$$

   where c is a ccw unit circle.

   (20 points)

   Solution:

   - See the lecture notes.
   - We have a double pole at $z = 0$ inside of the curve. In order to evaluate the residue at $z = 0$, we multiply the integrand with z^2, differentiate,

   $$[(z^2+3)^{-1}]' = -(z^2+3)^{-2}2z\,,$$

   and evaluate at $z = 0$ which yields zero. The integral is simply equal zero.

4. Fourier transform. Consider the elastic beam with stiffness $EI > 0$ on an elastic foundation with stiffness $k > 0$. The beam is loaded with a concentrated force F at the origin, represented with Dirac's delta. The equation is defined on the entire real line.

$$EIu'''' + ku = F\delta\,.$$

- Fourier transform the equation.
- Solve the problem in the Fourier domain.
- Use the Residue Theorem to obtain the final solution of the problem.

(20 points)

Solution: See the lecture notes.

5. Laplace transform. Consider the parabolic initial boundary-value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha^2 \dfrac{\partial^2 u}{\partial x^2} &= 0 & x \in (0,l),\ t > 0 \\ u(0,t) &= 0 & t > 0 \\ u(l,t) &= 0 & t > 0 \\ u(x,0) &= \sin(\frac{\pi x}{l}) & x \in (0,l)\,. \end{cases}$$

(a) Use separation of variables to compute the solution (or just guess it).

(b) Laplace transform the problem in time.

(c) Solve the problem in the Laplace domain.

(d) Use the Residue Theorem to compute the final solution and compare it with the one from (a).

(20 points)

(a) Assume

$$u = \sin(\frac{\pi x}{l}) T(t)$$

to obtain:

$$T' + \alpha^2 \left(\frac{\pi}{l}\right)^2 T = 0 \quad \Rightarrow \quad T = e^{-\alpha^2 (\frac{\pi}{l})^2 t} \quad \Rightarrow \quad u = e^{-\alpha^2(\frac{\pi}{l})^2 t} \sin(\frac{\pi x}{l})\,.$$

(b) We obtain the boundary-value problem:

$$s\bar{u} - \alpha^2 \bar{u}_{,xx} = \sin(\frac{\pi x}{l}) \quad \bar{u}(0) = \bar{u}(l) = 0\,.$$

(c) The solution is the sum of a general solution to the homogeneous equation and a particular solution to the non-homogeneous equation. We obtain:

$$\bar{u} = Ae^{\frac{\sqrt{s}}{\alpha}x} + Be^{-\frac{\sqrt{s}}{\alpha}x} + \frac{1}{s + \alpha^2(\frac{\pi}{l})^2} \sin(\frac{\pi x}{l})\,,$$

with, e.g. the branch cut along the negative real axis for $\sqrt{z}$. BCs imply $A = B = 0$, so the final solution in the Laplace domain is:

$$u = \frac{1}{s + \alpha^2(\frac{\pi}{l})^2} \sin(\frac{\pi x}{l})\,.$$

(d) We need to compute the inverse Laplace transform:

$$u(x,t) = \sin(\frac{\pi x}{l}) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s + \alpha^2(\frac{\pi}{l})^2}\, ds\,.$$

Employing the standard semicircular contour to the left, we obtain:

$$u(x,t) = \sin(\frac{\pi x}{l}) \mathrm{Res}_{-\alpha^2(\frac{\pi}{l})^2} \frac{e^{st}}{s + \alpha^2(\frac{\pi}{l})^2} = e^{-\alpha^2(\frac{\pi}{l})^2 t} \sin(\frac{\pi x}{l})\,.$$