CSE386C: METHODS OF APPLIED MATHEMATICS
Fall 2019, Exam 2

1. Define the following notions and provide a non-trivial example (2+2 points each):
   - closed operator,
   - weak topology in a normed vector space,
   - reflexive space,
   - orthogonal complement of a subspace of a normed vector space,
   - compact operator.

   See the book.

2. State and prove *three* out of the four theorems (10 points each):
   - Closed Graph Theorem (Thm. 5.10.1)
   - Completness of qoutient Banach space (Lemma 5.17.1)
   - Characterization of injective operators with closed range (Thm. 5.17.1)
   - Properties of the transpose of a continuous operator (Prop. 5.16.1)

   See the book.

3. Let V be a Banach space, and $P : V \to V$ a linear, continuous projection, i.e. $P^2 = P$. Prove that the range of P is closed. (10 points)

4. Let U annd V be normed spaces. Prove that the follwoing conditions are equivalent to each other.
   (i) $T : U \to V$ is compact.
   (ii) $T(B(0, 1))$ is precompact in V.
   (10 points)

   Assume T is linear and maps unit ball in U into a precompact set in V. Let C be an arbitrary bounded set in U,

$$\|\boldsymbol{u}\|_U \leq M, \quad \forall \boldsymbol{u} \in C$$

Set $M^{-1}C$ is then a subset of unit ball $B = B(0,1)$ and, consequently, $M^{-1}T(C)$ is a subset of $T(B)$. Thus,

$$\overline{M^{-1}T(C)} \subset \overline{T(B)}$$

as a closed subset of a compact set, is compact as well. Finally, since multiplication by a non-zero constant is a homeomorphism, $\overline{T(C)}$ is compact as well.

5. Consider the subset

$$c_0 := \{x = \{x_n\} \in l^\infty \, : \, x_n \to 0\}$$

Prove that

- c_0 is a closed subspace of l^∞.
- Its topological dual coincides with space l^1,

$$c_0' = l^1 \, .$$

- Its topological bidual coincides with the (whole) space l^∞ (you may recall the appropriate representation theorem).

Conclude that space c_0 is not reflexive. Consider now the sequence

$$e_n = (0, \ldots, \underbrace{1}_{(n)}, \ldots) \in l^1 \, .$$

Show that

$$e_n \overset{*}{\rightharpoonup} 0 \qquad \text{but} \qquad e_n \not\rightharpoonup \quad 0$$

(20 points)

- A linear combination of seqeunces converging to zero converges to zero as well so c_0 does have the structure of a subspace. To show the closedness in l^∞, consider a seqeunce $c_0 \ni \{x_n^m\} \to \{x_n\}$. We need to show that x_n converges to zero as well. Take an arbitrary $\epsilon > 0$. From the definition of convergence in l^∞, there exists an M such that, for every $m \geq M$ $\sup_n |x_n^m - x_n| < \epsilon/2$. In particular, $|x_n^M - x_n| < \epsilon/2 \, \forall n$. Now, from the convergence of x_n^M to zero, there exists an N such that $|x_n^M| < \epsilon/2$ for $n \geq N$. Consequently, for $n \geq N$, $|x_n| < |x_n - x_n^M| + |x_n^M| < \epsilon$.
- Define

$$x_N = \sum_{i=1}^{N} x_i e_i = (x_1, \ldots, x_N, \ldots)$$

It follows from the definition of c_0-space that

$$\|x - x_N\| = \sup_{i>N} |x_i| \to 0$$

Let $f \in c_0'$ and set $\phi_i = f(e_i)$. Then

$$\sum_{i=1}^{\infty} \phi_i x_i := \lim_{N\to\infty} \sum_{i=1}^{N} \phi_i x_i = \lim_{N\to\infty} f(x_N) = f(x)$$

Consequently,

$$|f(x)| \le \|\phi\|_{\ell^1} \, \|x\|$$

In order to show that the bound equals the supremum, it is sufficient to take a sequence of vectors

$$x_N = (\operatorname{sgn}\phi_1, \ldots, \operatorname{sgn}\phi_N, 0, \ldots) \in c_0$$

Then

$$f(x_N) = \sum_{i=1}^{N} |\phi_i| \to \sum_{i=1}^{\infty} |\phi_i|$$

- This follows from $\ell_1' = \ell_\infty$.

We have

$$< e_N, x >_{\ell_1 \times c_0} = x_N \to 0, \quad \forall x \in c_0$$

but

$$< \phi, e_N >_{\ell_\infty \times \ell_1} = 1 \nrightarrow 0$$

for $\phi = (1, 1, \ldots) \in \ell_\infty$.

6. Let q be a linear functional on $\mathcal{D}(K)$ where K is a compact subset of $\mathbb{R}^n$. Prove that q is continuous iff there exist constants $C > 0$ and k such that

$$|q(\phi)| \le C \sup_{x\in K} \sup_{|\alpha|\le k} |D^\alpha \phi(x)| \qquad \phi \in \mathcal{D}(K)\,.$$

(10 points)

Proof is a direct consequence of the fact that topology in $\mathcal{D}(K)$ is defined by the sequence of seminorms

$$p_k(\phi) = \sup_{x\in K} \sup_{|\alpha|\le k} |D^\alpha \phi(x)|, \quad k = 0, 1, 2, 3 \ldots$$

and Exercise 5.2.6 in the book.