Mixed Methods for Flow and Transport Problems on General Geometry

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Abstract

A groundwater flow or transport problem requires the solution of a second order elliptic equation with a tensor hydraulic conductivity or dispersion. These problems are posed over an aquifer domain with varying topography due to the geological layering, surface features, and the like. We present an expanded mixed finite element method that can efficiently handle these difficulties. The approximating spaces are defined on a smooth curvilinear grid, obtained by a global mapping of a simple, computational grid to the aquifer domain. Quadrature rules are introduced to transform the mixed method into a cell-centered finite difference method for the pressure or concentration. If rectangular computational elements are used, the stencil is 9 points in 2 dimensions and 19 points in 3 dimensions. Triangular computational elements give a 10 point stencil. The resulting scheme is locally mass conservative. For flow, the linear Galerkin finite element method gives first order accurate velocities, while the rectangular mixed method is second order accurate in the interior of the domain.

1 Introduction

The mixed finite element method was developed by Raviart and Thomas [13, 15, 5], and we restrict our attention to their lowest-order method. It was first used for subsurface problems by Douglas, Ewing, and Wheeler [6], though Russell and Wheeler [14] pointed out that the often used cell-centered finite difference method on rectangular grids [12] for problems with diagonal

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coefficients is actually a mixed method with appropriate quadrature rules applied to the integrals. The problem with mixed methods is that they can be difficult to implement directly, especially if the aquifer domain Ω is not rectangular.

This chapter gives a summary of some recent work on the development of numerical schemes for groundwater flow and transport problems with tensor coefficients on a geometrically general domain. The schemes are cell-centered finite difference approximations of a mixed finite element method; thus, they are easy to implement, have only one unknown per element, and are locally conservative. The elements can be either deformed rectangles or bricks, or they can be triangles. The work on rectangular elements is joint with Mary F. Wheeler and Ivan Yotov [2, 3], and the work on triangles is joint with Clint N. Dawson and Philip T. Keenan [1].

Our main requirement is that there be a smooth mapping F of a reference, computational domain $\hat{\Omega}$ onto the aquifer domain Ω in $d = 2$ or 3 dimensions. The Jacobian matrix of F is DF, $(DF)_{ij} = \partial F_i/\partial \hat{x}_j$, and the Jacobian of the mapping is $J = |\det(DF)|$. The reference domain is divided into standard elements (rectangles, bricks, or equilateral triangles) of maximal diameter $\hat{h}$ to form a regular partition $\hat{\mathcal{T}}_{\hat{h}}$ of $\hat{\Omega}$. Then F defines a smooth, curvilinear grid $\mathcal{T}_h$ on Ω with elements of maximal diameter h. (There are grid generation codes available for creating F and its Jacobian matrix.)

We consider the two main problems of subsurface flow and transport, in somewhat simplified form. In the flow problem, we solve for the pressure p and the velocity $\mathbf{u}$ satisfying

$$\nabla \cdot \mathbf{u} = f, \quad \mathbf{x} \in \Omega, \tag{1}$$

$$\mathbf{u} = -K\nabla p, \quad \mathbf{x} \in \Omega, \tag{2}$$

$$p = g, \quad \mathbf{x} \in \partial\Omega, \tag{3}$$

where K is the hydraulic conductivity tensor, f is a source term, and g gives the boundary condition. In the transport problem, we solve for the concentration c such that

$$\frac{\partial(\theta c)}{\partial t} + \nabla \cdot (\mathbf{u}c - D(\mathbf{u})\nabla c) = f_c(c),$$

where θ is the water content and D is the dispersion tensor. The dispersion part of the transport problem, $-\nabla \cdot D(\mathbf{u})\nabla c$, is of the same form as the flow problem; therefore, we concentrate on (1)–(3), a second order elliptic equation with, for simplicity of exposition, Dirichlet boundary conditions.

In the next two sections, before we begin the development of our new numerical schemes, we define the mixed method finite element spaces and review the standard mixed method.

2 The lowest order Raviart-Thomas mixed spaces

Let $L^2(\Omega)$ denote the set of square integrable functions, and let $H(\Omega;\mathrm{div})$ denote the space of vector functions that have a divergence; that is,

$$L^2(\Omega) = \left\{ w(\mathbf{x}) : \int_\Omega |w|^2 \, dx < \infty \right\},$$
$$H(\Omega;\mathrm{div}) = \{\mathbf{v}(\mathbf{x}) : \mathbf{v} \in (L^2(\Omega))^d \text{ and } \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$$

The mixed method requires a vector approximating space $V_h \subset H(\Omega;\mathrm{div})$ for the velocity $\mathbf{u}$, as well as a scalar space $W_h \subset L^2(\Omega)$ for the pressure p. In the lowest order Raviart-Thomas mixed spaces [13, 10], the pressure is approximated by a constant on each element; that is,

$$W_h = \{w \in L^2(\Omega) : w \text{ is constant on each element } E \in \mathcal{T}_h\}.$$

The nodal degrees of freedom can be considered as the function values at the centers of the elements.

The elemental space of vectors $\hat{V}(\hat{E})$ can be described on a standard reference element $\hat{E}$. If $\hat{T}$ is a rectangle or brick, then

$$\hat{V}(\hat{E}) = \{\mathbf{v} : v_i = \alpha_i x_i + \beta_i \text{ for any real numbers } \alpha_i \text{ and } \beta_i, \ i = 1, 2(, 3)\};$$

that is, the ith component of $\mathbf{v}$ is linear in the ith coordinate direction and constant in the other direction(s). If $\hat{T}$ is a triangle, then

$$\hat{V}(\hat{E}) = \{\mathbf{v} : v_i = \alpha x_i + \beta_i \text{ for any real numbers } \alpha \text{ and } \beta_i, \ i = 1, 2\}.$$

In either case, the important fact is that if ν denotes the outward unit vector (to the element), then $\mathbf{v} \cdot \nu$ is a constant; therefore, the nodal degrees of freedom can be considered as the values of $\mathbf{v} \cdot \nu$ at the centers of the element edges or faces. By matching these degrees of freedom across elements, we define $\hat{V}_{\hat{h}} \subset H(\hat{\Omega};\mathrm{div})$ on the reference domain:

$$\begin{aligned} \hat{V}_{\hat{h}} \ = \ & \{\mathbf{v} : \mathbf{v}|_{\hat{E}} \in \hat{V}(\hat{E}) \text{ for all } \hat{E} \in \hat{\mathcal{T}}_{\hat{h}} \text{ and the normal} \\ & \text{fluxes } \mathbf{v} \cdot \nu \text{ agree across element boundaries}\}. \end{aligned}$$

On general elements, there is no simple polynomial with the requisite properties. Instead, the Piola transform is used to define the space V_h [15, 5]. The construction is

$$V(E) = \{\mathbf{v} : \mathbf{v}(\mathbf{x}) = J^{-1} DF\, \hat{\mathbf{v}}(F^{-1}(\mathbf{x})) \text{ for some } \hat{\mathbf{v}} \in \hat{V}(\hat{E})\},$$

where $E = F(\hat{E})$. The Piola transform has the property that normal fluxes are preserved (in an average sense); therefore, local mass conservation is preserved under this mapping. Moreover, since $\mathbf{v} \cdot \nu$ is a constant on each edge or face, the nodal degrees of freedom are as for reference elements and V_h is defined as above.

Denote by $\{w_i\}$ a basis for W_h, and $\{\mathbf{v}_i\}$ a basis for V_h.

3 The standard mixed method

We rewrite (1)–(3) in variational form as follows. Note that (2) can be written as

$$K^{-1}\mathbf{u} = -\nabla p,$$

and then the mixed variational form of the equations is

$$\int_\Omega \nabla \cdot \mathbf{u}\, w\, dx = \int_\Omega f\, w\, dx, \quad w \in L^2(\Omega),$$
$$\int_\Omega K^{-1}\mathbf{u} \cdot \mathbf{v}\, dx = \int_\Omega p\, \nabla \cdot \mathbf{v}\, dx - \int_{\partial\Omega} g\, \mathbf{v} \cdot \nu\, ds, \quad \mathbf{v} \in H(\Omega; \mathrm{div}),$$

after integrating one term by parts.

Restricting $\mathbf{u}$ and $\mathbf{v}$ to V_h and p and w to W_h gives the standard mixed method for (1)–(3). It is a saddle point linear system; in matrix form it is

$$\begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} -G \\ -F \end{pmatrix},$$

where U represents $\mathbf{u}$ and P represents p in the bases, and

$$A_{ij} = \int_\Omega K^{-1}\mathbf{v}_i \cdot \mathbf{v}_j\, dx, \qquad B_{ij} = \int_\Omega w_i \nabla \cdot \mathbf{v}_j\, dx,$$
$$F_i = \int_\Omega f\, w_i\, dx, \qquad G_i = \int_{\partial\Omega} g\, \mathbf{v}_i \cdot \nu\, ds.$$

To reduce the size of this linear system, we can solve for the Shur complement system by eliminating U to obtain

$$(BA^{-1}B^T)P = F + BA^{-1}G.$$

This system is symmetric and positive definite and has small size. Unfortunately, although A is sparse, A^{-1} is in general full. Iterative solution will require the following steps for the application of the matrix: a matrix vector multiply BP; the solution of the system $Ay = BP$; and another matrix vector multiply $B^T y$. Thus, we need inner iterations within our overall iterative solution, which can become somewhat expensive.

Arnold and Brezzi [4] defined the hybrid form of the mixed method to obtain a sparse, symmetric, and positive definite linear system. This formulation incorporates additional unknowns approximating the pressure on the edges or faces of the elements (often referred to as the "Lagrange multipliers"). These approximations are piecewise constant; thus, the nodal degrees of freedom are the values at the centers of the edges or faces of the elements. The Shur complement system is defined in terms of these Lagrange multipliers; unfortunately, there are more unknowns than the number of elements. In 2 dimensions, if rectangles or triangles are used, there are two or three halves times as many unknowns, respectively. In 3 dimensions, brick elements give three times as many unknowns.

As mentioned in the introduction, Russell and Wheeler [14] defined an approximation to the mixed method in the case of a diagonal conductivity or permeability tensor and a rectangular grid. They used a nodal quadrature rule on the integral $\int_\Omega K^{-1}\mathbf{u}\cdot\mathbf{v}\,dx$ above, so that A becomes diagonal. This scheme is identical to the cell-centered finite difference method used in the petroleum industry [12]. The Shur complement system is then sparse, having a 5 point stencil for the pressure unknown in 2 dimensions, and 7 points in 3 dimensions. Unfortunately, if K is not diagonal, the interaction of the basis functions is problem and space dependent; thus, A cannot be diagonalized.

4 The expanded mixed method

In this and the next three sections, we generalize the approach of Russell and Wheeler [14] to define our numerical methods for the full case of tensor K and general geometry. Through a modification of the standard mixed method, we are able to simplify the interaction of the basis functions when K is not diagonal. If the elements are restricted to standard shapes, appropriate quadrature rules can be applied to approximate the Shur complement system by a sparse, symmetric, and positive definite system for the pressure unknowns. By mapping to the computational domain, geometrically general domains can be handled easily.

Let M be a tensor defined by the mapping F as

$$M = J((DF)^{-1})^T (DF)^{-1};$$

it is symmetric and positive definite. Its purpose will become clear in the next section. We introduce the additional unknown $\tilde{\mathbf{u}}$ defined by

$$M\tilde{\mathbf{u}} = -\nabla p, \tag{4}$$

$$\mathbf{u} = KM\tilde{\mathbf{u}}. \tag{5}$$

Then as before, the variational form of (1), (4)–(5) is

$$\int_\Omega \nabla \cdot \mathbf{u}\, w\, dx = \int_\Omega f\, w\, dx, \quad w \in L^2(\Omega), \tag{6}$$

$$\int_\Omega M\tilde{\mathbf{u}} \cdot \mathbf{v}\, dx = \int_\Omega p\, \nabla \cdot \mathbf{v}\, dx - \int_{\partial\Omega} g\, \mathbf{v} \cdot \nu\, ds, \quad \mathbf{v} \in H(\Omega; \mathrm{div}), \tag{7}$$

$$\int_\Omega M\mathbf{u} \cdot \tilde{\mathbf{v}}\, dx = \int_\Omega MKM\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}\, dx, \quad \tilde{\mathbf{v}} \in (L^2(\Omega))^d. \tag{8}$$

Restricting $\mathbf{u}$, $\mathbf{v}$, $\tilde{\mathbf{u}}$, and $\tilde{\mathbf{v}}$ to V_h and p and w to W_h gives the expanded mixed method. As we will see, the linear system is again a saddle point problem, but first let us transform back to the computational domain.

5 The expanded method on the reference domain

A shift back to the reference domain considerably simplifies the form of the equations (6)–(8). Let $\mathbf{x} = F(\hat{\mathbf{x}})$ for $\hat{\mathbf{x}} \in \hat{\Omega}$. Then vector and scalar functions transform according to

$$\mathbf{v}(\mathbf{x}) = \frac{1}{J} DF\, \hat{\mathbf{v}}(\hat{\mathbf{x}}), \tag{9}$$

$$w(\mathbf{x}) = \hat{w}(\hat{\mathbf{x}}). \tag{10}$$

If one notes that the Piola transform satisfies the property [15, 5] that

$$\nabla \cdot \mathbf{v} = \frac{1}{J} \hat{\nabla} \cdot \hat{\mathbf{v}},$$

then the change of variables in the integrals of (6)–(8) is straightforward. By our choice of M,

$$\int_{\hat{\Omega}} \hat{\nabla} \cdot \hat{\mathbf{u}}\, \hat{w}\, d\hat{x} = \int_{\hat{\Omega}} \hat{f}\, \hat{w}\, J\, d\hat{x}, \quad \hat{w} \in \hat{W}_{\hat{h}}, \tag{11}$$

$$\int_{\hat{\Omega}} \hat{\tilde{\mathbf{u}}} \cdot \hat{\mathbf{v}}\, d\hat{x} = \int_{\hat{\Omega}} \hat{p}\, \hat{\nabla} \cdot \hat{\mathbf{v}}\, d\hat{x} - \int_{\partial\hat{\Omega}} \hat{g}\, \hat{\mathbf{v}} \cdot \hat{\nu}\, d\hat{s}, \quad \mathbf{v} \in \hat{V}_{\hat{h}}, \tag{12}$$

$$\int_{\hat{\Omega}} \hat{\mathbf{u}} \cdot \hat{\tilde{\mathbf{v}}}\, d\hat{x} = \int_{\hat{\Omega}} \kappa \hat{\tilde{\mathbf{u}}} \cdot \hat{\tilde{\mathbf{v}}}\, d\hat{x}, \quad \hat{\tilde{\mathbf{v}}} \in \hat{V}_{\hat{h}}, \tag{13}$$

where

$$\kappa = J(DF)^{-1}K((DF)^{-1})^T.$$

This is an approximation to problem (1)–(3) on $\hat{\Omega}$, except that the conductivity tensor has been modified. The true solution is given by mapping back to Ω according to (9)–(10); one would normally map only the nodal degrees of freedom.

In matrix form, the method is

$$\begin{pmatrix} \mathcal{A} & -C & 0 \\ -C & 0 & B^T \\ 0 & B & 0 \end{pmatrix} \begin{pmatrix} \tilde{U} \\ U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ G \\ F \end{pmatrix},$$

where $\tilde{U}$, U, and P represent $\hat{\tilde{\mathbf{u}}}$, $\hat{\mathbf{u}}$, and $\hat{p}$ in the bases, respectively, B, G, and F are as before (except now defined on the computational domain), and

$$\mathcal{A}_{ij} = \int_{\hat{\Omega}} \kappa \hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j \, d\hat{x}, \qquad C_{ij} = \int_{\hat{\Omega}} \hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j \, d\hat{x}$$

(note that $\mathcal{A}$ and C are symmetric).

Again we can solve for the Shur complement system by eliminating U and $\tilde{U}$ to obtain

$$(BC^{-1}\mathcal{A}C^{-1}B^T)P = F + (BC^{-1}\mathcal{A}C^{-1})G,$$

but C^{-1} is in general full. Depending on the type of grid used, we can introduce quadrature rules to diagonalize C.

6 Approximation by rectangular cell-centered finite differences

In this section, assume that $\hat{\Omega}$ and the computational grid is rectangular in d dimensions. We denote the trapezoidal quadrature rule on the reference domain for a function $\psi(\hat{\mathbf{x}})$ by

$$Q(\psi) = \sum_{\hat{E} \in \hat{\mathcal{T}}_{\hat{h}}} \frac{1}{2^d} \sum_{i=1}^{2^d} \psi(\hat{\mathbf{x}}_i(\hat{E})) \, \mathrm{vol}(\hat{E}),$$

where the $\hat{\mathbf{x}}_i(\hat{E})$ go through the vertices of $\hat{E}$ and $\mathrm{vol}(\hat{E})$ is its area or volume. We then approximate (11)–(13) by

$$\int_{\hat{\Omega}} \hat{\nabla} \cdot \hat{\mathbf{u}}\, \hat{w}\, d\hat{x} = \int_{\hat{\Omega}} \hat{f}\, \hat{w}\, J\, d\hat{x}, \quad \hat{w} \in \hat{W}_{\hat{h}}, \tag{14}$$

$$Q(\hat{\tilde{\mathbf{u}}} \cdot \hat{\mathbf{v}}) = \int_{\hat{\Omega}} \hat{p}\, \hat{\nabla} \cdot \hat{\mathbf{v}}\, d\hat{x} - \int_{\partial\hat{\Omega}} \hat{g}\, \hat{\mathbf{v}} \cdot \hat{\nu}\, d\hat{s}, \quad \mathbf{v} \in \hat{V}_{\hat{h}}, \tag{15}$$

$$Q(\hat{\mathbf{u}} \cdot \hat{\tilde{\mathbf{v}}}) = Q(\kappa \hat{\tilde{\mathbf{u}}} \cdot \hat{\tilde{\mathbf{v}}}), \quad \hat{\tilde{\mathbf{v}}} \in \hat{V}_{\hat{h}}. \tag{16}$$

Indeed, now

$$\mathcal{A}_{ij} = Q(\kappa \hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j) \quad \text{and} \quad C_{ij} = Q(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j),$$

and so C is diagonal.

It is easy to unravel the procedure in terms of the nodal degrees of freedom of the $\hat{\mathbf{u}}$, $\hat{\tilde{\mathbf{u}}}$, and p. Consider an element $\hat{E}$ (not adjacent to the outer boundary). Equation (14) requires that the divergence of $\hat{\mathbf{u}}$ be set equal to the source term f. This involves differences of the velocities that live on the four edges or six faces of the element. Equation (16) relates the velocities to the gradients of pressure. The velocity $\hat{\mathbf{u}}$ on a given edge or face is related to the gradient $\hat{\tilde{\mathbf{u}}}$ that lives on the given edge or face and to those that live on the adjacent but perpendicular four edges or eight faces (if κ is not diagonal). Finally, (15) relates the $\hat{\tilde{\mathbf{u}}}$ living on an edge or face to the difference of the adjacent pressures. Combining this together, we get a 9 point stencil for the pressure on $\hat{E}$ if $d = 2$, and 19 points if $d = 3$.

7 Approximation by triangular cell-centered finite differences

In this section, assume that the computational grid is composed of equilateral triangles. We need to define a very special quadrature rule on equilateral triangles to diagonalize C. For a function $\psi(\hat{\mathbf{x}})$, let

$$Q(\psi) = \sum_{\hat{E} \in \hat{\mathcal{T}}_{\hat{h}}} \frac{\mathrm{area}(\hat{E})}{6} \Big[\sum_{i=1}^{3} \psi(\hat{\mathbf{x}}_i(\hat{E})) + 3\psi(\hat{\mathbf{x}}_c(\hat{E})) \Big],$$

where again $\hat{\mathbf{x}}_i(\hat{E})$ goes through the vertices of $\hat{E}$, $\hat{\mathbf{x}}_c(\hat{E})$ is the centroid of $\hat{E}$, and $\mathrm{area}(\hat{E})$ is its area. The usual formula has 3 replaced by 9 and 6

replaced by 12; this is third order accurate. Our formula is accurate only to order two; however, if

$$C_{ij} = Q(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j)$$

and each $\hat{E}$ is equilateral, then C is diagonal.

We approximate (11)–(13) by (14), (15), and

$$Q(\hat{\mathbf{u}} \cdot \hat{\tilde{\mathbf{v}}}) = \int_{\hat{\Omega}} \kappa \hat{\tilde{\mathbf{u}}} \cdot \hat{\tilde{\mathbf{v}}}\, d\hat{x}, \quad \hat{\tilde{\mathbf{v}}} \in \hat{V}_{\hat{h}}. \tag{17}$$

A 10 point stencil for the pressure results. For an element $\hat{E}$ (not adjacent to the outer boundary), (14) involves differences of the velocities that live on the three edges. Equation (17) relates the velocities to the $\hat{\tilde{\mathbf{u}}}$ that live on the nine edges of the three adjacent triangles. Finally, (15) relates the $\hat{\tilde{\mathbf{u}}}$ living on an edge to the difference of the adjacent pressures.

8 Convergence results

Because of the way (1) is approximated in mixed methods, mass is conserved cell-by-cell; that is, the test function $w \in W_h$ is a piecewise constant, and so the average divergence is set equal to the average mass source over each cell. Such is not the case in, say, the more straightforward Galerkin finite element method. Moreover, as we describe below, the standard and expanded mixed finite element methods give accurate approximations of both the velocity and pressure.

The convergence theory for mixed methods is fairly well established (see, e.g., [7, 9]). We summarize the main theoretical results (which hold under appropriate hypotheses). For a scalar function ψ, let

$$\|\psi\| = \left\{ \int_{\Omega} |\psi(\mathbf{x})|^2 \, dx \right\}^{1/2}$$

denote its $L^2(\Omega)$-norm; if ψ is a vector function, this defines its $(L^2(\Omega))^d$-norm. Let $\mathcal{C}$ denote a generic positive constant that depends on the solution, the domain, K, f, g, and the regularity of the partition $\mathcal{T}_h$, but not on the element size h.

As we refine the grid $\mathcal{T}_h$, both the standard [13, 15, 7, 5] and expanded [3] mixed methods are first order accurate in the velocities and pressures; that is, the errors satisfy

$$\|\mathbf{u}_{\text{approx}} - \mathbf{u}_{\text{true}}\| + \|p_{\text{approx}} - p_{\text{true}}\| \leq \mathcal{C}\, h.$$

This convergence rate is to be expected, since both V_h and W_h contain the piecewise constant functions.

It should be noted that at certain discrete points, one can observe computationally and prove theoretically that the pressure approximation is second order accurate. This phenomenon is referred to as superconvergence. Let the midpoint rule be used to approximate the $L^2(\Omega)$-norm:

$$|||\psi|||_M = \left\{ \sum_{E \in \mathcal{T}_h} |\psi(\mathbf{x}_c(E))|^2 \, \mathrm{vol}(E) \right\}^{1/2},$$

where $\mathbf{x}_c(E)$ denotes the centroid of E and $\mathrm{vol}(E)$ is the area or volume of E. Then in fact [13, 7, 5, 3]

$$|||p_{\mathrm{approx}} - p_{\mathrm{true}}|||_M \leq \mathcal{C}\, h^2.$$

This result is not surprising, since $\mathbf{u}$, essentially a derivative of p, is first order accurate.

A more interesting result is that superconvergence is sometimes observed for the velocity approximation. If the standard mixed method is used, and if the grid is rectangular and K is diagonal, then we can define the discrete approximation to the $(L^2(\Omega))^d$-norm for a vector ψ by

$$|||\psi|||_{TM} = \left\{ \sum_{E \in \mathcal{T}_h} \frac{1}{2d} \sum_{i=1}^{d} [|\psi_i(\mathbf{x}_i^1(E))|^2 + |\psi_i(\mathbf{x}_i^2(E))|^2] \, \mathrm{vol}(E) \right\}^{1/2},$$

where $\mathbf{x}_i^1(E)$ and $\mathbf{x}_i^2(E)$ are the centers of the two edges or faces perpendicular to the ith coordinate direction. That is, we construct the discrete norm by looking at the nodal points of V_h and applying the trapezoidal or midpoint rule as appropriate in each coordinate direction. Then the theoretical result [11, 8] is

$$|||\mathbf{u}_{\mathrm{approx}} - \mathbf{u}_{\mathrm{true}}|||_{TM} \leq \mathcal{C}\, h^2. \tag{18}$$

In fact, if homogeneous Neumann boundary conditions are used, the above result holds for the cell-centered finite difference approximation of the standard mixed method [16]; computationally speaking, other boundary conditions also exhibit some superconvergence, but perhaps not to order 2.

The cell-centered finite difference approximation of the expanded mixed method gives somewhat better velocities. On rectangular grids, but with a tensor K, we have the same superconvergence of the velocity (18) if periodic boundary conditions are used (i.e., superconvergence occurs in the interior

of the domain) [2]. We observe computationally that the velocity exhibits superconvergence for other boundary conditions, of order in the range 1.5 to 2, depending on whether Dirichlet or Neumann conditions are used, and on the eigenvalues and eigenvectors of K.

If Ω is a general domain but the image by F of a rectangular computational domain $\hat{\Omega}$, (18) holds on the computational domain (at least in the interior). Provided that F is smooth, (18) induces a superconvergent discrete norm on Ω. This induced norm is not easily described; however, the midpoint approximation of the $(L^2(\Omega))^d$-norm is also superconvergent. Therefore, the velocity approximation is superconvergent to the true velocity on rectangular or merely logically rectangular grids.

9 Conclusions

We have presented a cell-centered finite difference approximation to an expanded mixed finite element method, suitable for groundwater flow and transport problems on general geometry. It has only as many unknowns as elements. If the elements are deformed rectangles, the stencil for the presure is only 9 points, involving the given pressure and its 8 nearest neighbors. If the elements are deformed bricks, the stencil is 19 points (not 27). If the elements are triangles, the stencil is somewhat larger than the rectangular case, being 10 points.

Because these are mixed methods, mass is conserved locally cell-by-cell. The rate of convergence of the velocities is order 1 in general; however, on logically rectangular grids, superconvergence is seen in the solution. We do not require orthogonal grids, we merely require that the grid be smooth.

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