POROELASTIC FILTRATION COUPLED TO STOKES FLOW

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ABSTRACT. We report on some recent progress in the mathematical theory of fluid transport and poro-mechanics, specifically, the exchange of fluid between the Biot model of an elastic porous structure saturated with a slightly compressible viscous fluid coupled to the Stokes flow in an adjacent open channel. The coupled system is resolved by semigroup methods by developing appropriate variational formulations. These lead to either a standard weak formulation or a mixed formulation for the resolvent equation.

1. INTRODUCTION

Consider the flow of a single phase slightly compressible viscous fluid through a system composed of two regions, the first being an elastic and porous structure and the second being an adjacent open channel, possibly a macropore, an isolated cavity, or a connected fracture system. Both regions are saturated with the fluid, and we need to prescribe the stress and flow couplings on the interface between the Biot filtration flow through the deforming porous medium and the Stokes flow in the open channel. Our objective is to formulate a model of this composite hydro-mechanical system which accurately characterizes the depletion history and transient response of the fluid exchange and stress balance between the saturated elastic porous medium and the contiguous fluid-filled chamber, and to show that this model leads to a mathematically well-posed problem which is amenable to analysis and computation.

Suppose that the disjoint pair of regions Ω_1 and Ω_2 in $\mathbb{R}^3$ share the common *interface*, $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. The first region Ω_1 is the fully-saturated *elastic porous matrix* structure, and the second region Ω_2 is the fluid-filled *macro-void system* which is adjacent to Ω_1. Here we denote by $\mathbf{n}$ the unit normal vector on the boundaries, directed *out* of Ω_1 and *into* Ω_2. The derivative with respect to time will be denoted by a superscript dot,

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so $\mathbf{v}^1(x,t) = \dot{\mathbf{u}}^1(x,t)$ denotes the *velocity* corresponding to a *displacement* $\mathbf{u}^1(x,t)$ of the *porous structure* at $x \in \Omega_1$. Also, we let $\mathbf{v}^2(x,t)$ be the velocity of the *fluid* at $x \in \Omega_2$. The *pressure* of the fluid in the pores of Ω_1 is given by $p^1(x,t)$ and the pressure of the fluid in the adjacent channel system Ω_2 by $p^2(x,t)$.

1.1. **The Conservation Equations.** The laminar flow of a slightly compressible viscous fluid through the deformable porous medium Ω_1 is described by the *Biot system* [9, 10, 11]

$$c_1\dot{p}^1 - \partial_i k_{ij}\partial_j p^1 + c_0\boldsymbol{\nabla}\cdot\dot{\mathbf{u}}^1 = h_1(x,t)\,, \tag{1.1a}$$

$$\rho_1\ddot{\mathbf{u}}^1 - (\lambda_1+\mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}^1) - \mu_1\Delta\mathbf{u}^1 + c_0\boldsymbol{\nabla}p^1 = \mathbf{f}_1(x,t)\,, \tag{1.1b}$$

consisting of the *diffusion equation* for conservation of fluid mass, and the *momentum equation* for the balance of forces, respectively. The porosity of the matrix and the compressibility of the fluid or the solid material on the meso-scale are incorporated in c_1. The *conductivity* k_{ij} combines the permeability of the structure and the viscosity of the fluid to provide a measure of the *filtration velocity* or *fluid flux* $\mathbf{q} = (q_1, q_2, q_3)$, given by *Darcy's law*, $q_i = -k_{ij}\partial_j p^1$. The density of the saturated porous matrix is denoted by ρ_1, and the positive Lamé constants λ_1 and μ_1 represent the *dilation* and *shear* moduli of elasticity, respectively. The first accounts for *compression* and the second for *distortion* of the medium [21, 17]. The dilation $c_0\boldsymbol{\nabla}\cdot\mathbf{u}^1(t)$ provides the additional *pore fluid content* due to the local volume change, and the term $c_0\boldsymbol{\nabla}p^1(t)$ is the *pressure stress* of the pore fluid on the structure. The *Biot-Willis* constant c_0 is the *pressure-storage coupling* coefficient [12]. See [7, 18, 26, 19, 60, 20, 16, 53, 54, 55] for background and recent results. We shall include here the situation of consolidation problems in which the inertial effects of the matrix are negligible, hence, $\rho_1 = 0$.

The slow flow of a slightly compressible viscous fluid in the adjacent open channel Ω_2 is described by the *compressible Stokes system* [57, 51]

$$c_2(x)\dot{p}^2 + \boldsymbol{\nabla}\cdot\mathbf{v}^2 + c_2(x)\rho_2(x)\mathbf{g}(x)\cdot\mathbf{v}^2 = 0\,, \tag{1.2a}$$

$$\rho_2(x)\dot{\mathbf{v}}^2 - (\lambda_2+\mu_2)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}^2) - \mu_2\Delta\mathbf{v}^2 + \boldsymbol{\nabla}p^2 = c_2(x)\rho_2(x)\mathbf{g}(x)\,p^2. \tag{1.2b}$$

The constants λ_2 and μ_2 represent dilation and shear viscosity of the fluid, respectively. We also include the limiting case of an *incompressible* fluid (see p. 147 of [57], p. 269 of [50]) for which $c_2 = 0$ and the flow in the channel is the classical *Stokes flow*,

$$\boldsymbol{\nabla} \cdot \mathbf{v}^2 = 0\,, \qquad \rho_2(x)\dot{\mathbf{v}}^2 - \mu_2 \Delta \mathbf{v}^2 + \boldsymbol{\nabla} p^2 = \mathbf{0}\,.$$

The system is obtained by linearization about a steady situation in which ρ_2 is the density of the fluid at the reference pressure. The coefficient $c_2(\cdot)$ arises from the compressibility of the fluid, and the terms with $\mathbf{g}(\cdot)$ are the gravitational contribution to momentum and to convection.

1.2. **Interface Conditions.** The objectives below are to identify a physically consistent set of interface conditions which couple these systems together and to formulate a variational statement of the resulting problem that leads to a mathematically well-posed initial-boundary-value problem. The interface coupling conditions must recognize the conservation of mass and total momentum. Thus, they will include the continuity of the normal fluid flux and of stress. The two additional constitutive relations concern the dependence of the Darcy flux at the interface on the pressure increment and the effect of the tangential component of stress on the velocity increment at the interface. The former is the classical *Robin* boundary condition, and the latter is the slip condition of *Beavers-Joseph-Saffman*.

2. The Biot-Stokes System

We assume the mechanical behavior of the porous solid is determined by classical small-strain elasticity. In order to describe this, we denote hereafter by Σ the space of *symmetric second-order tensors*. Boldface letters will be used to indicate vectors in $\mathbb{R}^3$ and Greek letters to indicate second-order tensors in Σ. We denote by $\delta = \{\delta_{ij}\}$ the identity consisting of ones on the diagonal and zeros elsewhere. We adopt the convention that repeated indices are summed.

Standard function spaces will be used [1, 57]. Let Ω be a smoothly bounded region in $\mathbb{R}^3$, and denote its boundary by $\Gamma = \partial\Omega$. Let $H^1(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives also in $L^2(\Omega)$. The *trace* map or restriction to the boundary is the linear map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ defined by $\gamma(w) = w|_\Gamma$. Corresponding spaces of vector-valued functions will be denoted by boldface symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. We shall also use the space $\mathbf{L}^2_{\mathrm{div}}(\Omega)$ of vector functions $\mathbf{L}^2(\Omega)$ whose divergence belongs to $L^2(\Omega)$. Recall that for the functions $\mathbf{r} \in \mathbf{L}^2_{\mathrm{div}}(\Omega)$ there is a *normal trace* on the interface, and this is denoted by $\mathbf{r} \cdot \mathbf{n}$, since it takes this value on the smooth functions $\mathbf{r}$ in $\mathbf{L}^2_{\mathrm{div}}(\Omega)$. Finally, we denote by $L^2(\Omega; \Sigma)$ the indicated space of Σ-valued functions on Ω.

Let $\mathbf{n} = \{n_i\}$ be the unit normal vector on a surface. For a vector $\mathbf{w}$, we denote the normal projection $w_n = \mathbf{w} \cdot \mathbf{n}$ and the tangential component $\mathbf{w}_T = \mathbf{w} - w_n \mathbf{n}$. Likewise for the tensor τ in Σ, we have its value at $\mathbf{n}$, $\tau(\mathbf{n}) = \{\tau_{ij} n_i\} \in \mathbb{R}^3$, and its normal and tangential parts $\tau(\mathbf{n})(\mathbf{n}) = \tau_n = \tau_{ij} n_i n_j$, $\tau_T = \tau(\mathbf{n}) - \tau_n \mathbf{n}$.

2.1. **The System.** We shall write the constitutive equations together with the partial differential equations for mass and momentum balance as a system of first-order partial differential equations in each of the two regions. Recall that $\mathbf{v}^1 = \dot{\mathbf{u}}^1$ denotes the *velocity* corresponding to a *displacement* $\mathbf{u}^1$ of the *porous structure* in Ω_1, and $\mathbf{v}^2$ is the velocity of the *fluid* in Ω_2. The symmetric derivative of a vector function $\mathbf{u}(x)$ is the tensor $\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i) \in \Sigma$. The constitutive laws take the forms $\sigma^1(\mathbf{u}^1)_{ij} = \lambda_1 \delta_{ij} \varepsilon(\mathbf{u}^1)_{kk} + 2\mu_1 \varepsilon(\mathbf{u}^1)_{ij}$ in Ω_1 for the *elastic stress* corresponding to the *strain* $\varepsilon(\mathbf{u}^1)$ in the homogeneous and isotropic structure and $\sigma^2(\mathbf{v}^2)_{ij} = \lambda_2 \delta_{ij} \varepsilon(\mathbf{v}^2)_{kk} + 2\mu_2 \varepsilon(\mathbf{v}^2)_{ij}$ in Ω_2 for the *viscous stress* corresponding to the *strain rate* $\varepsilon(\mathbf{v}^2)$ of the *Newtonian fluid.* Note that $\sigma^1(\mathbf{u}^1) - c_0\, p^1\, \delta$ is the *total stress* due to elastic deformation and pore pressure p^1 within the matrix, and $\sigma^2(\mathbf{v}^2) - p^2\, \delta$ is the combined viscous and pressure stress of the fluid. Here both p^1 and p^2 are the thermodynamic pressure of the barotropic fluid in the respective regions. The *Biot-Stokes system* takes the form

$$c_1 \dot{p}^1 + \boldsymbol{\nabla} \cdot \mathbf{q} + c_0 \boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1(x,t)\,, \tag{2.3a}$$

$$\mathcal{Q}\mathbf{q} + \boldsymbol{\nabla} p^1 = 0\,, \tag{2.3b}$$

$$\rho_1 \dot{\mathbf{v}}^1 - \boldsymbol{\nabla} \cdot \sigma^1 + c_0 \boldsymbol{\nabla} p^1 = \mathbf{f}_1(x,t)\,, \tag{2.3c}$$

$$\mathcal{C}^1 \sigma^1 - \varepsilon(\mathbf{u}^1) = 0 \text{ in } \Omega_1\,, \text{ and} \tag{2.3d}$$

$$c_2(x) \dot{p}^2 + \boldsymbol{\nabla} \cdot \mathbf{v}^2 + c_2(x)\rho_2(x)\mathbf{g}(x) \cdot \mathbf{v}^2 = h_2(x,t)\,, \tag{2.3e}$$

$$\rho_2(x) \dot{\mathbf{v}}^2 - \boldsymbol{\nabla} \cdot \sigma^2 + \boldsymbol{\nabla} p^2 - c_2(x)\rho_2(x)\, p^2\, \mathbf{g}(x) = \mathbf{f}_2(x,t)\,, \tag{2.3f}$$

$$\mathcal{C}^2 \sigma^2 - \varepsilon(\mathbf{v}^2) = 0 \text{ in } \Omega_2\,. \tag{2.3g}$$

The first (2.3a) is the *storeage equation* for the fluid mass conservation in the pores of the matrix in which the *flux* $\mathbf{q}$ is the relative velocity of the fluid within the porous structure given by *Darcy's law*. This is written in the form (2.3b) of a force balance in which the flow resistance tensor $\mathcal{Q}$ is the inverse of the conductivity tensor k_{ij}. The third set of equations (2.3c) is the standard *Navier* system for the conservation of momentum of the elastic matrix structure, the constitutive relation (2.3d) is *Hookes law* for the stress-strain relationship, and the *compliance tensor* $\mathcal{C}^1$ is just the inverse of elasticity. These first four equations are equivalent to the Biot system (1.1). The last three equations are just the compressible Stokes system (1.2) for pressure $p^2(x,t)$ and velocity $\mathbf{v}^2(x,t)$ of the fluid. The equation (2.3e) accounts for the fluid mass conservation in the channel, and (2.3f) is the momentum conservation equation. The gravitational force $\mathbf{g}$ contributes to both of these. The *Newtonian fluid* is described by the constitutive relation (2.3g) in which the tensor $\mathcal{C}^2$ is the inverse to the viscosity tensor.

2.2. **Boundary and Interface Conditions.** We choose the *boundary conditions* on $\partial\Omega_1 \cup \partial\Omega_2 - \Gamma_{12}$ in a classical simple form, since they play no essential role here. On the exterior boundary of the porous medium, $\partial\Omega_1 - \Gamma_{12}$, we shall impose *drained conditions* $p_1 = 0$ on fluid pressure and the *clamped condition* $\mathbf{v}_1 = \mathbf{0}$ on velocity of the structure. On the exterior boundary of the free fluid, $\partial\Omega_2 - \Gamma_{12}$, we shall impose the *no-slip condition* $\mathbf{v}_2 = \mathbf{0}$ on fluid velocity.

In order to complete a well posed problem, additional *interface conditions* must be imposed across the interior boundary Γ_{12}. Let's begin by reviewing the interface conditions that have been used previously to couple various models of fluid and solid composites.

2.2.1. *Fluid-solid contact.* The natural transmission conditions at the interface of a free fluid and an impervious elastic solid consist of the continuity of displacement and of stress [50]. The effective flow through a rigid micro-porous and permeable matrix is described by the *Darcy law*, $q_i = -k_{ij}\partial_j p^1$, where $\mathbf{q}$ is the filtration velocity or flux of fluid driven by a pressure gradient, and k_{ij} is the *conductivity*. In fact, Darcy's law can be realized as the upscaled limit by averaging or *homogenization* of a fine-scale periodic array of a rigid solid and intertwined fluid. See [56, 2, 25]. Similar results are obtained when the solid is permitted to be *elastic*, and then various scalings of the viscosity lead to a *viscous solid* or to the *Biot model of poroelasticity* (1.1). See [4, 49, 51, 14, 59, 5, 22, 6, 58].

2.2.2. *Fluid-porous medium.* The description of a free fluid in contact with a rigid but porous solid matrix requires a means to couple the slow flow to the upscaled Darcy filtration. Since a Stokes system is used for the free fluid, we have two distinct scales of hydrodynamics, and these are represented by two completely different systems of partial differential equations. Fluid conservation is a natural requirement at the interface, and other classically assumed conditions such as continuity of pressure or vanishing tangential velocity of the viscous fluid have been investigated [23, 41], but these issues have been controversial. See the discussion on p. 157 of [51]. In fact, one can even question the *location* of the interface, since the porous medium itself is already a mixture of fluid and solid. Moreover, Beavers and Joseph [8] discovered that fluid in contact with a porous medium flows faster along the interface than a fluid in contact with a solid surface: there is a substantial *slip* of the fluid at the interface with a porous medium. They proposed that the normal derivative of the tangential component of fluid velocity $\mathbf{v}_T$ satisfy

$$\tfrac{\partial}{\partial n}\mathbf{v}_T = \tfrac{\gamma}{\sqrt{K}}(\mathbf{v}_T - \mathbf{q}_T)$$

where K is the permeability of the porous medium, and γ is the *slip rate coefficient*. This condition was developed further in [47, 29], and a substantial rigorous analysis of such interface conditions was given in [27, 28]. See [45, 42] for an excellent discussion, [48, 24, 40] for numerical work, [46] for dependence on the slip parameter, and [3] for homogenization results on related problems.

2.2.3. *Fluid-elastic porous medium.* Any model of free fluid in contact with a *deformable* and porous medium contains the upscaled filtration velocity in addition to the displacement and stress variations of the porous matrix. These must be coupled to the Stokes flow, so all of the previous issues are present in the interface conditions. See [43, 44].

We begin with the mass-conservation requirement that the normal fluid flux be continuous across the interface. For this purpose, we introduce the parameter β which represents the surface fraction of the interface on which the diffusion paths of the structure are *sealed*. The remaining fraction $1-\beta$ is the *contact surface* along Γ_{12}, where the diffusion paths of the porous medium are exposed to the fluid in the open channel, and so the motion of the structure contributes to the interfacial fluid mass flux. Thus, the solution is required to satisfy the *admissability constraint*

$$(c_0(1-\beta)\mathbf{v}^1 + \mathbf{q}) \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n} \tag{2.4a}$$

for the conservation of fluid mass across the interface. We shall assume that the Darcy flow across Γ_{12} is driven by the difference between the total normal stress of the fluid and the pressure internal to the porous medium according to

$$\sigma_n^2 - p^2 + p^1 = \alpha \mathbf{q} \cdot \mathbf{n}\,. \tag{2.4b}$$

The constant $\alpha \geq 0$ is the *fluid entry resistance.* The conservation of momentum requires that the total stress of the porous medium is balanced by the total stress of the fluid. For the normal component this means

$$\sigma_n^1 - c_0 p^1 = c_0(1-\beta)(\sigma_n^2 - p^2)\,, \tag{2.4c}$$

and for the tangential component we have

$$\sigma_T^1 = \sigma_T^2\,. \tag{2.4d}$$

Finally, this common tangential stress is assumed to be proportional to the *slip rate* according to the Beavers-Joseph-Saffman condition

$$\sigma_T^2 = \gamma\sqrt{\mathcal{Q}}(\mathbf{v}_T^2 - \mathbf{v}_T^1)\,. \tag{2.4e}$$

We shall show next that the *interface conditions* (2.4) suffice precisely to couple the Biot system (1.1) in Ω_1 to the Stokes system (1.2) in Ω_2.

2.3. **The Weak Formulation.** Our objective is to construct an appropriate *variational formulation* of the Biot-Stokes system (2.3) coupled by the interface conditions (2.4). We seek a solution in spaces

$$p^1(t) \in V_1, \quad p^2(t) \in L^2(\Omega_2), \quad \mathbf{q}(t) \in \mathbf{W}, \quad \mathbf{v}^1(t) \in \mathbf{V}_1, \quad \mathbf{v}^2(t) \in \mathbf{V}_2,$$

that are determined by boundary conditions. In order to focus on the interface conditions, we have chosen here the simplest classical examples, clamped and drained conditions on the appropriate boundaries, so the corresponding spaces are given by

$$\mathbf{V}_j = \{\mathbf{v} \in \mathbf{H}^1(\Omega_j) : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_j - \Gamma_{12}\}, \quad j = 1, 2,$$
$$V_1 = \{p \in H^1(\Omega_1) : p = 0 \text{ on } \partial\Omega_1 - \Gamma_{12}\}\,, \qquad \mathbf{W} = \mathbf{L}^2_{\mathrm{div}}(\Omega_1)\,.$$

Those functions of V_1, $\mathbf{V}_1$, or $\mathbf{V}_2$ have a well defined trace on the external boundary and on the interface Γ_{12}, and those from $\mathbf{W}$ have a normal trace, as noted above.

We want an appropriate weak form of the initial-boundary-value problem for the system (2.3)–(2.4). Multiply the momentum equations by test functions $\mathbf{w}^j \in \mathbf{V}_j$ and the Darcy law by $\mathbf{r} \in \mathbf{W}$, integrate the spatial derivatives and add to obtain

$$\begin{aligned}(2.5)\quad &\int_{\Omega_1} (\rho_1 \dot{\mathbf{v}}^1 \cdot \mathbf{w}^1 + (\sigma^1 - c_0 p^1 \delta) : \varepsilon(\mathbf{w}^1) + \mathcal{Q}\mathbf{q} \cdot \mathbf{r} - p^1 \delta : \varepsilon(\mathbf{r}))\, dx \\ &\qquad + \int_{\Omega_2} (\rho_2 \dot{\mathbf{v}}^2 \cdot \mathbf{w}^2 + (\sigma^2 - p^2\delta) : \varepsilon(\mathbf{w}^2))\, dx \\ &+ \int_{\Gamma_{12}} \left(-\sigma^1(\mathbf{n})(\mathbf{w}^1) + \sigma^2(\mathbf{n})(\mathbf{w}^2) + (c_0\mathbf{w}^1 + \mathbf{r}) \cdot \mathbf{n} p^1 - \mathbf{w}^2 \cdot \mathbf{n} p^2\right) dS \\ &\qquad\qquad = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \mathbf{f}_2 \cdot \mathbf{w}^2\, dx\,.\end{aligned}$$

For each triple of test functions satisfying the admissibility constraint (2.4a), the interface integral reduces to

$$\int_{\Gamma_{12}} \left(c_0 \beta p^1 \mathbf{n} \cdot \mathbf{w}^1 - \sigma^1(\mathbf{n})(\mathbf{w}^1) + \sigma^2(\mathbf{n})(\mathbf{w}^2) + (p^1 - p^2)\mathbf{n} \cdot \mathbf{w}^2\right) dS\,.$$

Moreover, decomposing the stress terms into their normal and tangential components, we obtain

$$\int_{\Gamma_{12}} \left((c_0\beta p^1 - \sigma_n^1) w_n^1 - \sigma_T^1 \cdot \mathbf{w}_T^1 + \sigma_T^2 \cdot \mathbf{w}_T^2 + (\sigma_n^2 + p^1 - p^2) w_n^2\right) dS\,,$$

and then the interface conditions (2.4b)– (2.4e) yield

$$\begin{aligned}&\int_{\Gamma_{12}} \left(\alpha \mathbf{q} \cdot \mathbf{n}\, (w_n^2 - c_0(1-\beta) w_n^1) + \gamma\sqrt{\mathcal{Q}}(\mathbf{v}_T^2 - \mathbf{v}_T^1)(\mathbf{w}_T^2 - \mathbf{w}_T^1)\right) dS \\ &\qquad = \int_{\Gamma_{12}} \left(\alpha(\mathbf{q} \cdot \mathbf{n})\,(\mathbf{r} \cdot \mathbf{n}) + \gamma\sqrt{\mathcal{Q}}(\mathbf{v}_T^2 - \mathbf{v}_T^1)(\mathbf{w}_T^2 - \mathbf{w}_T^1)\right) dS\,.\end{aligned}$$

Finally, multiply the fluid conservation equations by test functions $\varphi^1 \in V_1$, $\varphi^2 \in L^2(\Omega_2)$, and the constitutive equations by $\tau^1 \in L^2(\Omega_1; \Sigma)$ and $\tau^2 \in L^2(\Omega; \Sigma)$, integrate over the corresponding regions and add to the above to obtain the *variational statement*

(2.6)

$$\begin{aligned}
&\int_{\Omega_1} (\rho_1 \dot{\mathbf{v}}^1(t)\cdot\mathbf{w}^1 + (\sigma^1(t) - c_0 p^1(t)\delta) : \varepsilon(\mathbf{w}^1) + \mathcal{Q}\mathbf{q}(t)\cdot\mathbf{r} - p^1(t)\delta : \varepsilon(\mathbf{r}) \\
&+ \mathcal{C}^1\dot{\sigma}^1(t) : \tau^1 - \varepsilon(\mathbf{v}^1(t)) : \tau^1 + c_1\dot{p}^1(t)\varphi^1 + \delta : \varepsilon(\mathbf{q}(t))\varphi^1 + c_0\delta : \varepsilon(\mathbf{v}^1(t))\varphi^1)\,dx \\
&+ \int_{\Omega_2} (\rho_2\dot{\mathbf{v}}^2(t)\cdot\mathbf{w}^2 + (\sigma^2(t) - p^2(t)\delta) : \varepsilon(\mathbf{w}^2) + \mathcal{C}^2\sigma^2(t) : \tau^2 - \varepsilon(\mathbf{v}^2(t)) : \tau^2 \\
&\quad - c_2\rho_2 p^2(t)\mathbf{g}\cdot\mathbf{w}^2 + c_2\dot{p}^2(t)\varphi^2 + \delta : \varepsilon(\mathbf{v}^2(t))\varphi^2 + c_2\rho_2\mathbf{g}\cdot\mathbf{v}^2(t)\varphi^2)\,dx \\
&\quad + \int_{\Gamma_{12}} \Big(\alpha(\mathbf{q}(t)\cdot\mathbf{n})\,(\mathbf{r}\cdot\mathbf{n}) + \gamma\sqrt{\mathcal{Q}}(\mathbf{v}_T^2(t) - \mathbf{v}_T^1(t))(\mathbf{w}_T^2 - \mathbf{w}_T^1)\Big)\,dS \\
&= \int_{\Omega_1} \mathbf{f}_1(t)\cdot\mathbf{w}^1\,dx + \int_{\Omega_2} \mathbf{f}_2(t)\cdot\mathbf{w}^2\,dx + \int_{\Omega_1} h_1(t)\varphi^1\,dx + \int_{\Omega_2} h_2(t)\varphi^2\,dx\,.
\end{aligned}$$

Note that we have carefully written the operators on the *stress variables* as dual operators which contain an interior differential operator and boundary conditions, while the operator on *displacement variables* is the *local* differential operator. In summary, we define the product space

$$\begin{aligned}
\mathbb{V} = \{&(\varphi^1, \mathbf{r}, \mathbf{w}^1, \tau^1, \varphi^2, \mathbf{w}^2, \tau^2) \in \\
&V_1 \times \mathbf{W} \times \mathbf{V}_1 \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times \mathbf{V}_2 \times L^2(\Omega_2, \Sigma) : \\
&\qquad (c_0(1-\beta)\mathbf{w}^1 + \mathbf{r})\cdot\mathbf{n} = \mathbf{w}^2\cdot\mathbf{n} \text{ on } \Gamma_{12}\}\,,
\end{aligned}$$

and then the *weak formulation* of the problem is to find the vector-valued functions

$$\mathbf{v}(t) \equiv [p^1(t), \mathbf{q}(t), \mathbf{v}^1(t), \sigma^1(t), p^2(t), \mathbf{v}^2(t), \sigma^2(t)] \in \mathbb{V}, \quad t > 0,$$

such that (2.6) holds for every $[\varphi^1, \mathbf{r}, \mathbf{w}^1, \tau^1, \varphi^2, \mathbf{w}^2, \tau^2] \in \mathbb{V}$, and we have the *initial conditions*

$$\rho_1\mathbf{v}^1(0) = \rho_1\mathbf{v}_0^1,\ c_1 p^1(0) = c_1 p_0^1 \text{ in } \Omega_1\,, \tag{2.7a}$$

$$\rho_2\mathbf{v}^2(0) = \rho_2\mathbf{v}_0^2,\ c_2 p^2(0) = c_2 p_0^2 \text{ in } \Omega_2\,. \tag{2.7b}$$

Of course, $\sigma^1(0)$ is also determined from from (2.3d) and the initial displacement, $\mathbf{u}^1(0)$.

3. The Evolution Dynamics

The equations in the system are to hold in the product space

$$\mathbb{H} = L^2(\Omega_1) \times \mathbf{L}^2(\Omega_1) \times \mathbf{L}^2(\Omega_1) \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times \mathbf{L}^2(\Omega_2) \times L^2(\Omega_2, \Sigma),$$

and the solution will be sought in the space $\mathbb{V}$. Note that we have the continuous inclusions $\mathbb{V} \hookrightarrow \mathbb{H} = \mathbb{H}' \hookrightarrow \mathbb{V}'$. The (explicit) *divergence operator* $\delta : \varepsilon$ is a map of each of $\mathbf{W} \to \mathbf{L}^2(\Omega_1)$, $\mathbf{V}_1 \to \mathbf{L}^2(\Omega_1)$, and $\mathbf{V}_2 \to \mathbf{L}^2(\Omega_2)$, and then the corresponding *dual* operator $-\boldsymbol{\nabla} \cdot \delta$ mapping $\mathbf{L}^2(\Omega_1) \to \mathbf{W}'$, $\mathbf{L}^2(\Omega_1) \to \mathbf{V}_1'$, or $\mathbf{L}^2(\Omega_2) \to \mathbf{V}_2'$, respectively, consists of the gradient and a boundary condition. Note that $\mathbf{V}_1 \hookrightarrow \mathbf{W} \hookrightarrow \mathbf{L}^2(\Omega_1)$. Similar remarks hold for $\varepsilon : \mathbf{V}_j \to L^2(\Omega_j, \Sigma)$ and its dual $-\boldsymbol{\nabla}\cdot : L^2(\Omega_j, \Sigma) \to \mathbf{V}_j'$. We have two *interface operators* in the variational formulation (2.6). These are the *normal trace* $\gamma_n(\mathbf{q}) = \mathbf{q} \cdot \mathbf{n}$ and the *tangential trace* $\gamma_T(\mathbf{v}) = \mathbf{v}_T$ which define linear maps $\gamma_n : \mathbf{W} \to L^2(\Gamma_{12})$, $\gamma_T : \mathbf{V}_j \to \mathbf{L}^2(\Gamma_{12})$, for $j = 1, 2$.

3.1. The initial-value problem.

With the operators so defined, the system has the form

$$\text{(3.8a)} \qquad \mathbf{v}(t) \in \mathbb{V} : \ \frac{d}{dt}(\mathcal{A}\mathbf{v}(t)) + \mathcal{B}\mathbf{v}(t) = \mathbf{f}(t) \text{ in } \mathbb{H}',\ t > 0,$$

where the matrix of operators and variables are denoted by

$$\mathcal{A} = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix} p^1(t) \\ \mathbf{q}(t) \\ \mathbf{v}^1(t) \\ \sigma^1(t) \\ p^2(t) \\ \mathbf{v}^2(t) \\ \sigma^2(t) \end{bmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} 0 & \delta:\varepsilon & c_0\delta:\varepsilon & 0 & 0 & 0 & 0 \\ \boldsymbol{\nabla}\cdot\delta & \mathcal{Q} + \gamma_n'\alpha\gamma_n & 0 & 0 & 0 & 0 & 0 \\ c_0\boldsymbol{\nabla}\cdot\delta & 0 & \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & -\boldsymbol{\nabla}\cdot & 0 & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta:\varepsilon + c_2\rho_2\mathbf{g} & 0 \\ 0 & 0 & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & 0 & \boldsymbol{\nabla}\cdot\delta - c_2\rho_2\mathbf{g} & \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & -\boldsymbol{\nabla}\cdot \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon & \mathcal{C}^2 \end{pmatrix}.$$

The evolution equation (3.8a) is to be solved subject to the *initial condition*

$$\mathcal{A}\mathbf{v}(0) = \mathcal{A}\mathbf{v}_0\,, \tag{3.8b}$$

where $\mathcal{A}\mathbf{v}_0$ is determined from (2.7). Note that $\mathcal{A} : \mathbb{H} \to \mathbb{H}'$ is degenerate but symmetric and nonnegative, and it is easy to see that $\mathcal{B} : \mathbb{V} \to \mathbb{V}'$ is monotone. The equation (3.8a) is an example of an *implicit evolution equation* with *degenerate* operators as coefficients, sometimes known as a degenerate *Sobolev equation.* We recall that Jack Lagnese was a major contributor to the development of the theory of these abstract Sobolev equations, especially the singular perturbation theory and dependence of the solution on the operators. See [30, 39, 31, 32, 33, 34, 35, 36, 37, 38].

Since $\mathcal{A} + \mathcal{B}$ is $\mathbb{H}$-coercive in our situation, *uniqueness* for the initial value problem (3.8) is easy to establish. According to the general theory of such equations [52, 15], for *existence* of a solution it suffices to establish the *range condition* $\mathrm{Rg}(\lambda\mathcal{A} + \mathcal{B}) \supset \mathrm{Rg}(\mathcal{A})$ for $\lambda > 0$. For this, we consider the *resolvent system*

$$\mathbf{v} = [p^1(t), \mathbf{q}(t), \mathbf{v}^1(t), \sigma^1(t), p^2(t), \mathbf{v}^2(t), \sigma^2(t)] \in \mathbb{V} :$$

$$\lambda c_1 p^1 + \delta : \varepsilon(\mathbf{q}) + c_0 \delta : \varepsilon(\mathbf{v}^1) = c_1 h_1\,, \tag{3.9a}$$

$$(\mathcal{Q} + \gamma_n' \alpha \gamma_n)\mathbf{q} + \boldsymbol{\nabla} \cdot \delta p^1 = \mathbf{s}\,, \tag{3.9b}$$

$$\lambda \rho_1 \mathbf{v}^1 - \boldsymbol{\nabla} \cdot \sigma^1 + c_0 \boldsymbol{\nabla} \cdot \delta p^1 + \gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T (\mathbf{v}^1 - \mathbf{v}^2) = \rho_1 \mathbf{f}_1\,, \tag{3.9c}$$

$$\lambda \mathcal{C}^1 \sigma^1 - \varepsilon(\mathbf{v}^1) = \xi_1\,, \tag{3.9d}$$

$$\lambda c_2(x) p^2 + \delta : \varepsilon(\mathbf{v}^2) + c_2(x)\rho_2(x)\mathbf{g}(x) \cdot \mathbf{v}^2 = c_2 h_2\,, \tag{3.9e}$$

$$\begin{aligned} \lambda \rho_2(x)\mathbf{v}^2 - \boldsymbol{\nabla} \cdot \sigma^2 + \boldsymbol{\nabla} \cdot \delta p^2 - c_2(x)\rho_2(x)\, p^2\, \mathbf{g} & \\ + \gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T (\mathbf{v}^2 - \mathbf{v}^1) = \rho_2 \mathbf{f}_2\,, & \end{aligned} \tag{3.9f}$$

$$\mathcal{C}^2 \sigma^2 - \varepsilon(\mathbf{v}^2) = \xi_2\,, \tag{3.9g}$$

where the right side of this system is given as $[c_1 h_1, \mathbf{s}, \rho_1 \mathbf{f}_1, \xi_1, c_2 h_2, \rho_2 \mathbf{f}_2, \xi_2] \in \mathbb{H}'$. Note that (3.9) contains the interface conditions (2.4).

The means by which we establish the solvability of the resolvent system will depend critically on how much degeneracy occurs in the operators. For example, in the *least degenerate* case in which all the constants $c_1,\ \rho_1,\ c_2,\ \rho_2$ are strictly positive, the resolution of (3.9) is straightforward. In the mathematically more interesting and practically more relevant situations, some of these coefficients will vanish. In many of these cases, we can eliminate appropriate variables, thereby obtaining elliptic terms in the system, and then solve the reduced higher order system. We shall indicate briefly how one can establish the solvability by means of the *mixed formulation* of the resolvent system in which it is regarded as a *saddle point problem* from convex analysis [13].

3.2. **The mixed formulation.** Here we shall consider the resolvent system (3.9), but instead of writing a single operator equation in the space $\mathbb{V}'$ with 7 unknowns, we shall re-order the variables according to their role in the *physics* of the model, not in the *geometry* of the problem. Thus, we write the resolvent system on a product of two spaces so that it is realized as a *saddle point problem.* The first space $\mathbf{X}$ consists of the *displacement* variables,

$$\mathbf{X} = \{[\mathbf{q}, \mathbf{v}_1, \mathbf{v}_2] \in \mathbf{W} \times \mathbf{V}_1 \times \mathbf{V}_2 : \ (c_0(1-\beta)\mathbf{v}^1 + \mathbf{q}) \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n}\},$$

and the second space $\mathbf{Y}$ contains the *generalized stress* variables,

$$\mathbf{Y} = \{[p_1, \sigma_1, p_2, \sigma_2] \in L^2(\Omega_1) \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times L^2(\Omega_2, \Sigma)\}.$$

If we define the operators

$$A: \ \mathbf{X} \to \mathbf{X}' \quad B: \ \mathbf{X} \to \mathbf{Y}' \quad C: \ \mathbf{Y} \to \mathbf{Y}'$$

by means of the matrices

$$A = \begin{pmatrix} \mathcal{Q} + \gamma_n' \alpha \gamma_n & 0 & 0 \\ 0 & \lambda\rho_1 + \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T \\ 0 & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & \lambda\rho_2 + \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T \end{pmatrix},$$

$$B = \begin{pmatrix} \delta : \varepsilon & c_0\delta : \varepsilon & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \delta : \varepsilon + c_2\rho_2\mathbf{g}\cdot \\ 0 & 0 & -\varepsilon \end{pmatrix}, \quad C = \begin{pmatrix} \lambda c_1 & 0 & 0 & 0 \\ 0 & \lambda\mathcal{C}^1 & 0 & 0 \\ 0 & 0 & \lambda c_2 & 0 \\ 0 & 0 & 0 & \mathcal{C}^2 \end{pmatrix},$$

then the system (3.9) is obtained in the form

$$A\mathbf{x} + B'\mathbf{y} = \mathbf{f}$$
$$B\mathbf{x} + C\mathbf{y} = \mathbf{g}$$

for the unknowns $\mathbf{x} \equiv [\mathbf{q}, \mathbf{v}_1, \mathbf{v}_2] \in \mathbf{X},\ \mathbf{y} \equiv [p_1, \sigma_1, p_2, \sigma_2] \in \mathbf{Y}$. This formulation requires a *closed range condition* on the operator B, and it provides a natural and well established approach to the *numerical approximation* of such problem. In addition, the analysis of this formulation provides a means to establish the relation with the *singular limits* such as the incompressible case $c_2 = 0$ of the Stokes flow and the *quasistatic* case $\rho_1 = 0$ of consolidation processes. However, we can work directly with the original formulation (3.9) to obtain these limits and the corresponding existence results. These issues will be developed for nonlinear extensions of these models in forthcoming works.

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