VIBRATION OF A SHAPE MEMORY ALLOY WIRE

K.-H. Hoffmann[1] and R.E. Showalter[2]

1. Introduction.

We shall prove local-in-time existence of the unique solution of an initial-boundary-value problem for the nonlinear system

$$(1.1.a)\qquad v_{tt} - \big(\mu_1 v_{xt} + T(\theta, u_x, u_{xt}) v_x - R v_{xxx}\big)_x = f_1(x,t)$$

$$(1.1.b)\qquad u_{tt} - \big(\psi_\varepsilon(\theta, u_x) + \mu_2 u_{xt}\big)_x = f_2(x,t)$$

$$(1.1.c)\qquad -\theta\big(\psi_\theta(\theta, u_x)\big)_t - k(\alpha\theta_{xt} + \theta_x)_x = \mu_1 v_{xt}^2 + \mu_2 u_{xt}^2 + R(v_{xxt})^2 + f_3(x,t)$$

in which μ_1, μ_2, α, k and R are positive numbers, $T(\theta,\varepsilon,\eta) = T_1 + \psi_\varepsilon(\theta,\varepsilon) + \mu_2\eta$, and the function $\psi(\theta,\varepsilon)$ has the special structure to be specified below. The partial differential equations (1.1) are to hold in a region $0 < x < \ell$, $0 < t < T_0$, and we require the solution to satisfy the boundary conditions

$$(1.2.a)\qquad v(0,t) = v_x(0,t) = 0\ ,\quad v(\ell,t) = v_x(\ell,t) = 0\ ,$$

$$(1.2.b)\qquad u(0,t) = u(\ell,t) = 0\ ,$$

$$(1.2.c)\qquad -k\theta_x(0,t) + k_1\big(\theta(0,t) - g_1(t)\big) = 0\ ,\quad k\theta_x(\ell,t) + k_1\big(\theta(\ell,t) - g_2(t)\big) = 0$$

for $0 < t < T_0$ and the initial conditions

$$(1.3.a)\qquad v(x,0) = v_0(x)\ ,\quad v_t(x,0) = v_1(x)\ ,$$

$$(1.3.b)\qquad u(x,0) = u_0(x)\ ,\quad u_t(x,0) = u_1(x)\ ,$$

$$(1.3.c)\qquad \theta(x,0) = \theta_0(x)$$

for $0 < x < \ell$.

In Section 2 we shall describe how such a system arises as a thermomechanical model for a vibrating wire composed of *shape memory alloy*. There $v(x,t)$ represents the vertical displacement of the wire, $u(x,t)$ is the horizontal displacement, and $\theta(x,t)$ is the absolute

---

[1] Lehrstuhl für Angewandte Mathematik, Technische UniversitMünchen, Dauchauer Str. 9a, 8000 München 2 GERMANY

[2] Department of Mathematics, The University of Texas at Austin, Austin, TX 78712 U.S.A.

The work of the second author was supported by the Deutsche Forschungsgemeinschaft, the National Science Foundation, and University of Texas Research Institute.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX temperature. The major difficulties in the resolution of this system arise from the non-convex character of the free energy function $\psi(\theta, \varepsilon)$ which defines the material. The strictly positive numbers μ_1, μ_2, α permit the estimates which make the problem tractable. These can be identified as *viscosity* coefficients in the model and they distinguish it from a highly nonlinear system consisting of a beam equation, a wave equation, and a parabolic equation. All three of these viscosity terms will be used in essential ways below.

The vibration problem considered here arises from the mathematical modeling of the dynamics of *martensitic transformations* in a *shape memory alloy*. These are diffusionless solid state phase transitions due to a deformation of the crystal lattice and which produce a substantial macroscopic stain as well as a remarkable shape memory phenomenon. Nickel-titanium alloys known as *nitinol* provide examples of such material, exhibiting an impressive shape memory effect by recovering over 8% strain through heating and being able to deform at 50% of fracture strain. We shall adopt the continuum model for such phase transitions as developed in [2], [6], [8], [9], [5], and this model is incorporated in (1.1.b) and (1.1.c). Results from numerical simulations have demonstrated that these two equations reproduce the experimentally observed phase transitions, and further numerical experiments on the full system (1.1) have rather dramatically illustrated the exchange of energy due to the phase transitions in the material induced by the strain vibration and heat generated by flexing of the wire.

We shall use the following notation for the function spaces in which we work. The continuous dual space of a Banach space $\mathbb{B}$ is denoted by $\mathbb{B}'$; $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the Banach space of (uniformly) bounded operators from $\mathbb{B}_1$ to $\mathbb{B}_2$. For each p, $1 \le p \le \infty$, we denote by $L^p(0,\ell)$ the space of p^{th} Lebesgue-integrable (equivalence classes of) real-valued functions on the interval $(0,\ell)$ for $p < \infty$ and the essentially bounded functions on $(0,\ell)$ for $p = \infty$. If $k \ge 0$ is integer, then $W^{k,p}(0,\ell)$ is the space of those functions which together with all distributional derivatives through order k belong to $L^p(0,\ell)$. In particular, $W^{0,p}(0,\ell) = L^p(0,\ell)$. For the Hilbert space case, $p = 2$, we also denote $W^{k,2}(0,\ell)$ by $H^k(0,\ell)$, and we let $H_0^k(0,\ell)$ be the subspace of those functions in $H^k(0,\ell)$ for which all derivatives through order $k-1$ vanish at the endpoints, 0 and ℓ. The dual of $H_0^k(0,\ell)$ is traditionally denoted by $H^{-1}(0,\ell)$, and we shall do so here. We shall often suppress the mention of the interval $(0,\ell)$ and write these spaces as $L^p = L^p(0,\ell)$, $W^{k,p} = W^{k,p}(0,\ell)$, $H^k = H^k(0,\ell)$, $H_0^k = H_0^k(0,\ell)$ and $H^{-1} = H^{-1}(0,\ell)$. Finally, if $\mathbb{B}$ denotes a Banach space, we have similar constructions for $\mathbb{B}$-valued functions on the interval $(0, T_0)$, and these spaces of vector-valued functions are denoted by $L^p(0,T_0;\mathbb{B})$, $W^{1,p}(0,T_0;\mathbb{B})$ and $H^k(0,T_0;\mathbb{B})$. See [1] or [7] for information on these Sobolev spaces.

Now we can present our existence results for the system (1.1), (1.2), (1.3) together with the statement of hypotheses and an indication of the roles played by these various assumptions. The formal proof will be given in Section 3.

Consider the system (1.1) for which we assume the following:

A_1 Each of the constants μ_1, μ_2, α, k and R is positive.

A_2 The function ψ belongs to $C^3(\mathbb{R}\times\mathbb{R})$ and there are constants C and $\theta_s > 0$ for which $\psi_\theta(\theta,\varepsilon) = 0$ for $\theta \le \theta_s$; we have the estimates

$$-\theta\psi_{\theta\theta}(\theta,\varepsilon) \ge 0 , \quad |\theta^2\psi_{\theta\varepsilon}(\theta,\varepsilon)| \le C|\varepsilon| ,$$
$$|\psi_\varepsilon(\theta,\varepsilon)| \le C|\varepsilon| , \quad |\psi_{\varepsilon\varepsilon}(\theta,\varepsilon)| \le C \ \text{ for } \ \theta \ge \theta_s ;$$

and

$$\Big| \int_{\theta_s}^{\infty} \theta^2 \psi_{\theta\theta\varepsilon}(\theta,\varepsilon)\, d\theta \Big| \le C|\varepsilon| \ .$$

A_3 The function T is given by

$$T(\theta,\varepsilon,\eta) = T_1 + \psi_\varepsilon(\theta,\varepsilon) + \mu_2\eta$$

where T_1 is a given constant.

A_4 The (input) functions in (1.1) satisfy

$$\begin{aligned} &f_1 \in H^1\big(0,1;H^{-1}(0,\ell)\big) \ ,\\ &f_2 \in L^2\big(0,1;L^2(0,\ell)\big) \cap H^1\big(0,1;H^{-1}(0,\ell)\big) \ ,\\ &f_3 \in L^2\big(0,1;L^2(0,\ell)\big) + L^\infty\big(0,1;L^1(0,\ell)\big) \ ,\end{aligned}$$

and f_3 is non-negative.

A_5 The data in the boundary conditions (1.2) satisfies $k_1 > 0$ and g_1, g_2 belong to $H^1(0,1)$ with $g_j'(t) \ge 0$ a.e. and $g_j(t) \ge \theta_s$ for $t \in [0,1]$, $j = 1,2$.

A_6 The initial data in (1.3) satisfies

$$\begin{aligned} &v_0 \in H_0^2(0,\ell) \ , && v_1 \in H_0^1(0,\ell) \ ,\\ &u_0 \in H^2(0,\ell) \cap H_0^1(0,\ell) \ , && u_1 \in H_0^1(0,\ell) \ ,\\ &\theta_0 \in H^2(0,\ell) \ , \quad \theta_0(x) \ge \theta_s \ \text{ for } \ x \in [0,\ell] \ . \end{aligned}$$

Theorem. *From the assumptions* A_1–A_6 *it follows that there exists a unique solution of the system* (1.1), (1.2), (1.3) *on some time interval* $(0,T_0)$, *where* $0 < T_0 \le 1$, *and this solution satisfies*

(1.4.a) $$v \in W^{1,\infty}(0,T_0;H_0^2) \cap W^{2,\infty}(0,T_0;L^2) \cap W^{2,2}(0,T_0;H_0^1)$$

(1.4.b) $$u \in H^2(0,T_0;H_0^1) \cap H^1(0,T_0;H^2)$$

(1.4.c) $$\theta \in H^1(0,T_0;H^2)$$

Remarks. 1. The positivity of the viscosity coefficients is absolutely crucial for the estimates obtained for this problem.

2. The structural assumptions differ from those of [8], [6] only for very large and very small temperature and for very large strain. Thus from the physical viewpoint we have the same model. It is noteworthy that the non-negativity of the coefficient of θ_t in (1.1.c) is assumed here, and this prevents this parabolic component of the system from becoming backward-in-time, a rather unrealistic situation that had to be reckoned with in [8].

3. Although the addition of $\mu_2 > 0$ appears essential for the estimates obtained from (1.1.b), its appearance in (1.1.a) through the *tension* $T(\theta,\varepsilon,\eta)$ requires more regularity of u than was available in previous works.

2. The Model.

We show that the nonlinear system of partial differential equations (1.1) provides a model for the vibration of a thin wire composed of a shape memory alloy. The material undergoes structural phase transitions which affect the tension and stiffness of the wire and which lead to hystereses. The internal heat created by these vibrations in turn modifies the temperature dependent stress-strain relationship for the alloy. Our objectives here include the following:

- Describe the internal dynamics of the material with the general theory of structural phase transitions due to Landau [5], specifically with the Landau-Devonshire potential for the Helmholtz free energy. This thermodynamic potential is a (non-convex) function of the temperature θ and the order parameter ε (= strain) by which the phases are recognized. The model equations are obtained from foundations of thermodynamics-mechanics and are *not* ad-hoc constructions to match observed phenomena.

- Formulate the model to permit external time-dependent sources of heat, stress or displacement in horizontal or vertical directions. Such inputs will provide various strategies to control or stabilize the structure and to activate the phase transitions.

- Restrict the methods to those which might apply to higher dimensional situations in which a membrane or solid has been reinforced with fibers of the shape memory alloy. Thus the results and techniques may be directly applicable to these more general situations.

The state of the system is given by the thermodynamic temperature, $\theta(x,t)$, the horizontal displacement, $u(x,t)$, and the vertical displacement, $v(x,t)$, where position is denoted by $0 < x < \ell$ and time by $t > 0$. The vertical stress at a point on the wire is given by the classical linear combination

$$Tv_x - (Rv_{xx})_x + \mu_1 v_{xt}$$

where T is the tension in the wire, $R > 0$ is rigidity, and $\mu_1 > 0$ is the apparent viscosity due to vertical strain rate. Since we assume $|v_x| \ll 1$, the vertical component of tension is approximated by

$$T\frac{v_x}{\sqrt{1+v_x^2}} \cong Tv_x \ .$$

The rigidity R results in a difference of the bending moments over an interval $[x_1, x_2]$ which gives in the limit the upward force

$$\frac{Rv_{xx}(x_1) - Rv_{xx}(x_2)}{x_2 - x_1} \cong (-Rv_{xx})_x$$

distributed over the interval. The segment $[x_1, x_2]$ has its length changed due to vertical displacement by the amount

$$\int_{x_1}^{x_2} \left(\sqrt{1+v_x^2} - 1\right) dx \cong \int_{x_1}^{x_2} \frac{1}{2}(v_x)^2\, dx$$

which represents stored energy, and the internal friction generates a force $\mu_1 v_{xt}$ due to this length-rate-of-change. (This term provides a source of internally generated heat; the rate-of-change of bending moment is another such source.) Combining the above effects to balance the linear momentum, we obtain

$$(\rho v_t)_t = \big(T v_x - (R v_{xx})_x + \mu_1 v_{xt}\big)_x + \rho f_1 \ . \tag{2.1.a}$$

This equation is linear in v, but T and R are material properties which depend on the temperature and strain,

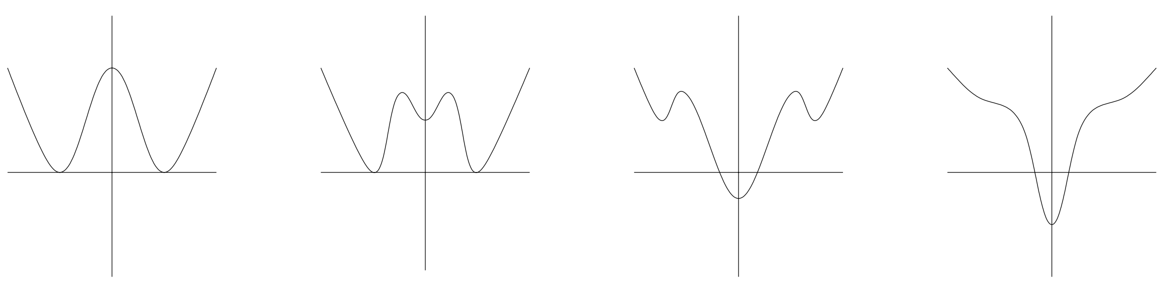
$$T = T(\theta, \varepsilon, \varepsilon_t) \quad , \quad R = R(\theta, \varepsilon) \ ,$$

in a highly nonlinear fashion to be described below.

The macroscopic behavior of a shape memory alloy is determined by the crystal lattice structure at the microscopic level. A single lattice particle has either of two equilibrium configurations, a highly symmetric austenitic form or the two twining martenistic forms obtained as sheared structures. The internal dynamics of the alloy are determined by a thermodynamic potential $\psi(\theta, \varepsilon)$ which characterizes the material. Equilibrium states of the system at a given temperature θ are determined by the minimum of $\psi(\theta, \cdot)$. In order to describe the shape-memory alloy, this potential should have the following properties:

- at low temperatures, $\theta < \theta_1$, $\psi(\theta, \cdot)$ has two symmetric minima,
- at somewhat higher temperature, $\theta_1 < \theta < \theta_2$, ψ has two symmetric and one central minima,
- and at yet higher temperature, $\theta_2 < \theta$, only the central minimum remains.

A typical progression of such curves is shown below.



$\theta < \theta_1$ $\qquad\qquad$ $\theta_1 < \theta < \theta_2$ $\qquad\qquad$ $\theta_2 < \theta$

The simplest form for ψ which fulfills the preceding required conditions is the *Landau-Devonshire potential*

$$\psi(\theta, \varepsilon) = \psi_0(\theta) + \psi_1(\theta)\varepsilon^2 + \psi_2(\theta, \varepsilon) \tag{2.2}$$

in which $\psi_0(\theta) = a_0(\theta - \theta \ln \theta)$ represents pure heat conduction, $\psi_1(\theta)\varepsilon^2 = a_1(\theta - \theta_1)\varepsilon^2$ gives rise to the shape memory, and $\psi_2(\theta, \varepsilon) = -a_2\varepsilon^4 + a_3\varepsilon^6$ accounts for a nonlinear elasticity. Such a form is the simplest in structure which is capable of reproducing all of the required shape-memory effects. Here we follow [9] and permit the last term to be temperature dependent and alter the structural assumptions on (2) from those of [8], [6] only for very large or very small temperatures and for large strain. Thus in the region in which shape-memory effects are to be found, we have the very same model from the physical viewpoint.

The *free energy* functional (2.2) corresponds to the autonomous situation with no load. The response of the autonomous system to a change of the order parameter ε is the *quasiconservative* component of *stress*

$$\sigma \equiv \rho \frac{\partial \psi}{\partial \varepsilon} \ , \tag{2.3}$$

and the *dissipative* part of the stress is given by $\rho\mu_2\varepsilon_t$, where ρ is the density and μ_2 is the viscosity arising from an internal dissipation proportional to the rate-of-change of strain. The strain is assumed linearly related to the horizontal displacement $u(x,t)$ by

$$\varepsilon = u_x \ ,$$

so we consider only the case $|\varepsilon| \ll 1$. Thus the balance of (linear) horizontal momentum requires that

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial u}{\partial t}\right) = \sigma_x + (\rho\mu_2\varepsilon_t)_x + \rho f_2 \ , \tag{2.1.b}$$

where f_2 is the distributed horizontal load.

For the balance of mass we have

$$\frac{d\rho}{dt} + \rho\varepsilon_t = 0 \ .$$

If, in addition, we assume ρ is independent of temperature, $\rho = \rho(\varepsilon)$, then we obtain

$$\rho = \rho_0 \exp(-\varepsilon) \ ,$$

and for small ε we may assume $\rho = \rho_0$ is constant. Hereafter we shall set $\rho = 1$.

The response of the system to a change of temperature is given by the *entropy*, $-\psi_\theta$, and the corresponding *internal energy* of the system is

$$e = \psi - \theta\psi_\theta \ . \tag{2.4}$$

The conservation of internal energy leads to

$$\frac{\partial e}{\partial t} = -q_x + (\sigma + \mu_2\varepsilon_t)\varepsilon_t + f_3 + \mu_1(v_{xt})^2 + R(v_{xxt})^2 \ , \tag{2.5}$$

where q is the *heat flux*, f_3 is a distributed heat source, $\sigma\varepsilon_t$ arises from internal heating due to rate of increase of $\psi(\theta,\varepsilon)$ at constant temperature, and $\mu_2\varepsilon_t^2$ accounts for the heat generated due to the internal motion.

For the heat flux we assume the constitutive equation

$$q = -k(\theta + \alpha\theta_t)_x \tag{2.6}$$

in which k is conductivity and $\alpha > 0$ corresponds to the characteristic delay time of a short thermal memory. Such constitutive laws arise in the work of Gurtin and Chen [4]; see also [3]. In such a model there is a conductive temperature, φ, possibly different from the thermodynamic temperature, θ, and the conservation of energy is given by the pair of equations

$$c\theta_t + \frac{1}{\alpha}(\theta - \varphi) = 0 \quad , \quad \frac{1}{\alpha}(\varphi - \theta) - (k\varphi_x)_x = f$$

in which $c\theta$ is the stored energy, $q \equiv -k\varphi_x$ is the heat flux, and $\frac{1}{\alpha}(\theta - \varphi)$ is the rate of exchange of heat energy between these two temperature fields. Thus the thermodynamic temperature θ follows the conductive temperature φ according to its recent history,

$$\theta(t) = \frac{1}{c\alpha}\int_{-\infty}^{t} e^{-\frac{t-s}{c\alpha}}\varphi(s)\,ds \ .$$

By eliminating φ one obtains the effective flux in terms of thermodynamic temperature as postulated above in (2.6). Also we note this system is equivalent to the pseudoparabolic fissured medium equation [11],

$$c\theta_t - \alpha k\theta_{txx} - k\theta_{xx} = f \ .$$

An application of the chain rule to (2.4) shows

$$e_t = \psi_\theta\theta_t + \psi_\varepsilon\varepsilon_t - \theta\psi_{\theta t} - \theta_t\psi_\theta \ ,$$

from which we cancel the first and last terms to obtain

$$e_t = -\theta(\psi_\theta)_t + \psi_\varepsilon\varepsilon_t \ .$$

The conservation equation (2.5) then can be written as

$$-\theta(\psi_\theta)_t = -q_x + \mu_2\varepsilon_t^2 + \mu_1(v_{xt})^2 + R(v_{xxt})^2 + f_3 \ . \tag{2.1.c}$$

To complete the system it remains only to specify $T(\theta,\varepsilon)$ and $R(\theta,\varepsilon)$. Since we consider only small vibrations with $|v_x| \ll 1$, the tension in the wire will be approximated by the horizontal forces

$$T(\theta,\varepsilon,\varepsilon_t) = T_1 + \psi_\varepsilon(\theta,\varepsilon) + \mu_2 u_{xt} \tag{2.1.d}$$

obtained as the sum of the applied load T_1 and the local stress $\sigma + \mu_2\varepsilon_t$ due to local expansion (2.3) and to expansion rate. Finally, for the existence-uniqueness theory to be developed here we shall assume the rigidity is constant, $R(\theta,\varepsilon) = R$.

3. The Proofs.

Our plan is to resolve the system (1.1), (1.2), (1.3) by obtaining the solution as the fixed point of a mapping which is a *contraction* on appropriate spaces for a sufficiently small time interval, $[0, T_0]$. The strategy is to iterate between the single viscoelastic beam equation (1.1.a) and the system of thermomechanical state equations (1.1.b), (1.1.c) in order to exploit the linear structure of the former and the known estimates from [8], [9] for the latter.

We consider (1.1.a) as an abstract wave equation in the form

$$\ddot{v}(t) + \mathcal{B}\dot{v}(t) + \mathcal{A}(t)v(t) = f(t) \tag{3.1}$$

in a triple of Hilbert spaces $V \hookrightarrow W \hookrightarrow H$. Here each space is dense and continuously imbedded in the following one, and we identity $H = H'$ by the Riesz map, so we also have $H \hookrightarrow W' \hookrightarrow V'$ by restriction. The operators in (3.1) are given with

$$\mathcal{B} \in \mathcal{L}(W, W') \quad , \quad \mathcal{A}(t) \in \mathcal{L}(V, V') \ , \qquad 0 \le t \le T \ ,$$

so $\mathcal{B}$ corresponds in examples to a partial differential operator of order less than that of $\mathcal{A}$, and the equation (3.1) holds in V'.

Lemma 1. *Assume $\mathcal{A}(t)$ is a regular* [10] *family of symmetric operators with,*

$$\begin{aligned} \mathcal{A}(t)v(v) &\ge c\|v\|_V^2 \ , && v \in V \ , \\ \|\mathcal{A}'(t)\|_{\mathcal{L}(V,V')} &\le g(t) \ , && a.e. \quad t \in [0,1] \ , \\ \mathcal{B}w(w) &\ge \varepsilon\|w\|_W^2 \ , && w \in W \end{aligned}$$

with $\varepsilon > 0$, $c > 0$ and $g \in L^1(0,1)$. Then for each triple

$$f \in L^2(0,1;W') \ , \quad v_0 \in V \ , \quad v_1 \in H$$

there exists a unique solution v of (3.1) and $v(0) = v_0$, $\dot{v}(0) = v$, satisfying

$$v \in L^\infty(0,1;V) \ , \quad \dot{v} \in L^\infty(0,1;H) \cap L^2(0,1;W) \ .$$

Proof. The essential estimate for this result is obtained by applying (3.1) to $2v(t)$ to get

$$\frac{d}{dt}\{|\dot{v}(t)|_H^2 + \mathcal{A}v(t)(v(t))\} + 2\mathcal{B}\dot{v}(t)(\dot{v}(t)) = 2f(t)\dot{v}(t) + \mathcal{A}'(t)v(t)(v(t))$$

from which we obtain the inequality

$$\sigma'(t) + \varepsilon\|\dot{v}(t)\|_W^2 \le \frac{1}{\varepsilon}\|f(t)\|_{W'}^2 + \frac{1}{c}g(t)\sigma(t)$$

in which $\sigma(t) = |\dot{v}(t)|_H^2 + \mathcal{A}(t)v(t)(v(t))$. This Gronwall inequality yields the estimate which leads by standard techniques to the desired result [7].

Lemma 2. *Assume in addition that*

$$\|\mathcal{A}'(t)\|_{\mathcal{L}(V,W')} \le G(t) \ , \quad a.e. \quad t \in [0,1]$$

with $G \in L^2(0,1)$. *Then if*

$$f \in H^1(0,1;W') \ , \quad v_1 \in V \ , \quad \mathcal{B}v_1 + \mathcal{A}(0)v_0 - f(0) \in H$$

it follows that the solution v *satisfies*

$$\dot v \in L^\infty(0,1;V) \ , \quad \ddot v \in L^\infty(0,1;H) \cap L^2(0,1;W) \ .$$

Proof. Proceeding formally, differentiate (3.1) to see that $w \equiv \dot v$ satisfies

$$\ddot w + \mathcal{B}\dot w + \mathcal{A}(t)w = \dot f - \mathcal{A}'(t)v \ .$$

Just as in Lemma 1 we obtain

$$\sigma'(t) + \varepsilon\|\dot w(t)\|_W^2 \le \frac{1}{c} g(t)\sigma(t) + \frac{1}{\varepsilon}\left(\|\dot f\|_{W'}^2 + G(t)\|v\|_{L^\infty(V)}\right)^2 \ ,$$

and this leads by the Gronwall inequality to the desired estimates on $\dot v(t)$. This provides the basis for the desired result by standard arguments as before.

Consider two solutions, v_1, v_2 of (3.1) corresponding to $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$ in the situation of Lemma 2. Then their difference $v \equiv v_1 - v_2$ and its derivative $w \equiv \dot v$ satisfy

$$\ddot v + \mathcal{B}\dot v + \mathcal{A}_1(t)v = \big(\mathcal{A}_2(t) - \mathcal{A}_1(t)\big)v_2 \ ,$$

$$\ddot w + \mathcal{B}\dot w + \mathcal{A}_1 w = \big(\mathcal{A}_2'(t) - \mathcal{A}_1'(t)\big)v_2 + \big(\mathcal{A}_2(t) - \mathcal{A}_1(t)\big)\dot v_2 - \mathcal{A}_1'(t)(v) \ ,$$

so we obtain estimates as above which show the following.

Lemma 3. *In the situation of Lemma 2, the mapping* $\mathcal{A} \mapsto w : H^1(0,1;\mathcal{L}(V,W')) \to L^\infty(0,1;V)$ *is Lipschitz.*

Remark. Suppose that $\mathcal{B}$ is symmetric. By making the change-of-variable $v \mapsto e^{-\lambda t}v$, $f \mapsto e^{-\lambda t}f$ in (3.1), it follows that the preceding results all hold if we replace $\mathcal{B}$ by $\mathcal{B} + 2\lambda I$ and $\mathcal{A}$ by $\mathcal{A} + \lambda\mathcal{B} + \lambda^2 I$ in the hypotheses.

Next we turn to the pair of equations (1.1.b), (1.1.c) with their corresponding boundary and initial data in (1.2) and (1.3). The local existence of solutions to this pair of equations was proved in [8] by the construction of a very special Galerkin approximation, and the uniqueness was established in [6]. Alternatively, one can consider a *linearization* of this pair in the form

(3.2.a) $\qquad u_{tt} - \mu_2 u_{xxt} = \psi_\varepsilon(\bar\theta, \bar u_x)_x + f_2(x,t) \ ,$

(3.2.b) $\qquad -\bar\theta\psi_{\theta\theta}(\bar\theta, \bar u_x)\theta_t - k(\alpha\theta_{xt} + \theta_x)_x = \mu_2 \bar u_{xt}^2 + \bar\theta\psi_{\theta\varepsilon}(\bar\theta, \bar u_x)\bar u_{xt} + f_3(x,t)$

where $\bar\theta(x,t), \bar u(x,t)$ are given. It can be shown by standard but tedious arguments that the mapping $\bar u, \bar\theta \mapsto u, \theta$ is a contraction on the spaces determined by the a-priori estimates of [8], but only for a short time interval, $[0, T_0]$. Moreover, in the situation of our assumptions A_1–A_6, global-in-time a-priori estimates were obtained in [9] for the solution of (3.2) with $\bar\theta = \theta$, $\bar u = u$, (see (3.3) below) and these yield from above the existence of a unique *global* solution of this pair. We summarize the essential results from [9] with some modifications necessary for our purposes here.

Lemma 4. *Assume* A_1, A_2, A_4–A_6. *Then there exists a unique solution*

$$u \in H^2(0,1;H_0^1) \cap H^1(0,1;H^2)\ , \quad \theta \in H^1(0,1;H^2)$$

of the system

$$u_{tt} - \mu_2 u_{xxt} = \psi_\varepsilon(\theta, u_x)_x + f_2(x,t) \tag{3.3.a}$$

$$-\theta\big(\psi_\theta(\theta,u_x)\big)_t - k(\alpha\theta_{xt} + \theta_x)_x = \mu_2 u_{xt}^2 + f_3(x,t) \tag{3.3.b}$$

satisfying (1.2) *and* (1.3).

Proof. According to the main result of [9] we obtain a solution $u \in H^1(0,1;H_0^1 \cap H^2) \cap C^1(0,1;H^1)$. The additional estimates needed to get $u \in H^2(0,1;H_0^1)$ then follow as in Lemma 2 above. The assumption in [9] was that $f_3 \in L^2(0,1;L^2)$. However, by modifying the procedure there to treat f_3 similar to the way the term $\mu_2 u_{xt}^2$ was handled there, it follows that we may obtain the same estimates if $f_3 \in L^\infty(0,1;L^1)$.

Corollary. *There is a function* $K : \mathbb{R}_+ \to \mathbb{R}_+$ *and a number* T_0, $0 < T_0 \le 1$, *such that the solution* u, θ *of* (3.3) *with initial and boundary data from* (1.2) *and* (1.3) *satisfies*

$$\|u\|_X + \|\theta\|_Y \le K(\|f_3\|_{Z_0})$$

and the mapping $f_3 \mapsto u, \theta$ *of* $Z_0 \equiv L^\infty(0,T_0;L^1) + L^2(0,T_0;L^2)$ *into* $X \times Y$ *with*

$$X \equiv H^2(0,T_0;H_0^1) \cap H^1(0,T_0;H^2) \quad , \quad Y \equiv H^1(0,T_0;H^2)$$

is a strict contraction.

Finally we set up the mapping whose fixed-point will yield the desired solution of (1.1), (1.2), (1.3). Define the space $Z \equiv W^{1,\infty}(0,T_0;V)$ with $0 < T_0 \le 1$. Note that in the situation of Lemma 2, the solution of (3.1) belongs to Z. Now choose $V = H_0^2(0,\ell)$ and consider the mapping of $Z = W^{1,\infty}(0,T_0;H_0^2)$ into $L^\infty(0,T_0;L^1)$ given by

$$w \mapsto \mu_2 w_{xt}^2 + R w_{xxt}^2 + f_3\ .$$

This mapping is Lipschitz continuous on bounded sets. Thus, for each $w \in Z$ we can by Lemma 4 uniquely solve (3.3) with "f_3" replaced by "$\mu_2 w_{xt}^2 + R w_{xxt}^2 + f_3$" as suggested by (1.1.c), and the mapping

$$w \mapsto u, \theta : Z \to X \times Y$$

is Lipschitz on bounded sets. Moreover, it is a contraction for $T_0 > 0$ chosen sufficiently small, and we denote this mapping by $\mathcal{S} : Z \to X \times Y$, i.e., $\mathcal{S}(w) = [u, \theta]$.

Consider the mapping $\mathcal{T} : X \times Y \to H^1(0,T_0;L^2)$ suggested by the tension-coefficient in (1.1.a): $\mathcal{T}(u,\theta) = a$, where

$$a(x,t) \equiv T(\theta, u_x, u_{xt}) = T_1 + \psi_\varepsilon(\theta, u_x) + \mu_2 u_{xt}\ . \tag{3.4}$$

It is straightforward to check that this mapping is Lipschitz on bounded sets.

In order to recover the equation (1.1.a) with boundary conditions (1.2.a) from the abstract evolution equation (3.1), we choose $W \equiv H_0^1(0,\ell)$ and define

$$\text{(3.5.a)}\quad \mathcal{B}v(\psi) = \mu_1 \int_0^\ell v_x \psi_x \, dx \ , \qquad v, \psi \in W$$

$$\text{(3.5.b)}\quad \mathcal{A}(t)v(\psi) = \int_0^\ell \left\{ a(x,t) v_x \psi_x + R v_{xx} \psi_{xx} \right\} dx \ , \qquad v, \psi \in V \ , \ 0 \le t \le T_0 \ ,$$

where the coefficient $a(x,t)$ is chosen according to (3.4) in $H^1(0,T_0;L^2)$. Then the inclusion $\mathcal{B} \in \mathcal{L}(W,W')$ is clear and the W-coerciveness follows by Poincaré's inequality. The corresponding results are not quite so easy for the operators $\mathcal{A}(t)$.

Lemma 5. *There is a $\lambda > 0$ for which $\mathcal{A}(t) + \lambda\mathcal{B}$ is V-coercive, uniformly in t.*

Proof. The difficulty is that $a(x,t)$ as given by (3.4) is not uniformly lower-bounded; the pivotal term is $u_{xt} \in H^1(0,T_0;L^2) \cap L^2(0,T_0;H^1)$. From (3.5.b) we have

$$\begin{aligned} \mathcal{A}(t)v(v) &\ge R\|v_{xx}\|_{L^2}^2 - \|a(\cdot,t)\|_{L^2}\|v_x^2\|_{L^2} \\ &= R\|v_{xx}\|_{L^2}^2 - \|a(\cdot,t)\|_{L^2}\|v_x\|_{L^4}^2 \end{aligned}$$

and we know that $\|a(\cdot,t\| \le a_\infty < \infty$. The space H_0^2 is compactly imbedded in $W_0^{1,4}$, which in turn is continuously imbedded in H_0^1. Thus, there is a constant $C(R,a_\infty)$ for which

$$a_\infty \|v_x\|_{L^4}^2 \le \frac{R}{2}\|v_{xx}\|_{L^2}^2 + C(R,a_\infty)\|v_x\|_{L^2}^2 \ , \quad v \in V \ ,$$

so we obtain

$$\mathcal{A}(t)v(v) + C(R,a_\infty)\mathcal{B}v(v) \ge \frac{R}{2}\|v_{xx}\|_{L^2}^2 \ , \quad v \in V \ ,$$

as desired.

Note that $C(R,a_\infty)$ depends on the norm of $a(\cdot,\cdot)$ in $L^\infty(0,T_0;L^2)$, and this is bounded for $a(\cdot,\cdot)$ chosen by (3.4) with u,θ bounded in $X \times Y$. In order to apply Lemma 2, we compute

$$\mathcal{A}'(t)v(\psi) = \int_0^\ell a_t(x,t) v_x \psi_x \, dx$$

and note that

$$a_t = \psi_{\varepsilon\theta}\theta_t + \psi_{\varepsilon\varepsilon}\varepsilon_t + \mu_2 u_{xxt} \in L^2(0,T_0;L^2) \ .$$

Again, the last term is pivotal, and the indicated norm of a_t is bounded by that of u,θ in $X \times Y$. Using the estimate

$$\|v_x\|_{L^\infty} \le \|v\|_{H_0^2} \ , \qquad v \in V \ ,$$

we obtain $\|\mathcal{A}'(t)\|_{\mathcal{L}(V,W')} \le \|a_t(\cdot,t)\|_{L^2}$ and hence Lemma 2 applies. Thus, we have shown that the function $\mathcal{W} : H^1(0,T_0;L^2) \to Z$ defined by $W(a) = v$, where v is the solution of (3.1) with operators chosen by (3.5), maps bounded sets into bounded sets. It follows from Lemma 3 that it is Lipschitz on bounded sets, and one can check that it is a contraction for T_0 sufficiently small. In summary the composition $\mathcal{W} \circ \mathcal{T} \circ \mathcal{S} : Z \to Z$ has a unique fixed point for T_0 sufficiently small, and this is the desired solution of (1.1), (1.2), (1.3).

References

1. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
2. H.W. Alt, K.-H. Hoffmann, M. Niezgodka, J. Sprekels, *A numerical study of structural phase transitions in shape memory alloys*, Preprint No. 90, Institut für Mathematik, Universität Augsburg, 1985.
3. R.W. Carroll and R.E. Showalter, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, 1976.
4. P. Chen and M. Gurtin, *On a theory of heat conduction involving two temperatures*, Zeit. Angew. Math. Phys. **19** (1968), 614–627.
5. F. Falk, *Landau theory and martensitic transformations*, Proceedings of the International Conference on Martensitic Transformations (L. Delaey and M. Chandrasekaran, eds.), Les Editions de Physique, Les Ulis, 1984.
6. K.-H. Hoffman, Zheng Songmu, *Uniqueness for nonlinear coupled equations arising from alloy mechanism*, Tech. Report No. 14 (1986), Center for Applied Mathematics, Purdue University, West Lafayette.
7. J.L. Lions and E. Magenes,, *Problemes aux Limite non Homogenes et Applicationes*, Dunod, Paris, 1968.
8. M. Niezgodka, J. Sprekels, *Existence of solutions for a mathematical model of structural phase transistions in shape memory alloys*, Institut für Mathematik, Universität Augsburg, Preprint, No.89, 1985.
9. M. Niezgodka, Z. Songmu, J. Sprekels, *Global solutions to a model of structural phase transistions in shape memory allows*, Institut für Mathematik, Universität Augsburg, Preprint No.105, 1986.
10. R.E. Showalter, *Degenerate evolution equations*, Indiana Univ. Math. J. **23** (1974), 655–677.
11. R.E. Showalter, *The fissured medium equation*, Physical Mathematics and Nonlinear Partial Differential Equations (J.H. Lightbourne, III and S.M. Rankin, III, eds.), Marcel Dekker, New York, 1985.

K.-H. Hoffmann
Lehrstuhl für Angewandte Mathematik
Technische Universität München
Dauchauer Str. 9a
8000 München 2 GERMANY

R.E. Showalter
Department of Mathematics
The University of Texas at Austin
Austin, TX 78712
U.S.A.
e-mail: *show@math.utexas.edu*