UNIFORM CONVERGENCE AND SUPERCONVERGENCE OF MIXED FINITE ELEMENT METHODS ON ANISOTROPICALLY REFINED GRIDS

JICHUN LI * AND MARY F. WHEELER †

Abstract. The lowest order Raviart-Thomas rectangular element is considered for solving the singular perturbation problem $-\text{div}(a\nabla p) + bp = f$, where the diagonal tensor $a = (\varepsilon^2, 1)$ or $a = (\varepsilon^2, \varepsilon^2)$. Global uniform convergence rates of $O(N^{-1})$ for both p and $a^{1/2}\nabla p$ in the L^2-norm are obtained in both cases, where N is the number of intervals in both directions. The pointwise interior (away from the boundary layers) convergence rates of $O(N^{-1})$ for p are also proved. Superconvergence (i.e., $O(N^{-2})$) at special points and $O(N^{-2})$ global L^2 estimate for both p and $a^{1/2} \bigtriangledown p$ are obtained by a local postprocessing. Numerical results support our theoretical analysis. Moreover numerical experiments show that an anisotropic mesh gives more accurate results than the standard global uniform mesh.

Key words. Mixed finite element method, singular perturbation, anisotropic refined meshes, superconvergence, uniform convergence

AMS subject classifications. 65N30, 35J60, 58G18

1. Introduction. Singular perturbation problems (SPP) arise in many application areas, such as in chemical kinetics, fluid dynamics and system control etc. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain. Such sharp transitions require very fine meshes globally (which is very inefficient) or locally to resolve the boundary layers. The challenging SPP [23, 25, 27] serve frequently as test models for new algorithms, e.g., in multigrid methods [12, Ch.10], domain decomposition methods (many papers in the proceedings of Domain Decomposition Methods) and adaptive methods [32, 7, 33, 31]. More details can be found in the above mentioned papers and references therein.

Though many specially designed algorithms have been developed for solving SPP over the past three decades (see [23, 22, 28, 14, 1, 15, 30] and references therein), many unsolved problems remain as described in the survey by Roos [27]. Recently, anisotropically refined meshes [23, 22, 28] were proved to be uniformly convergent, convergence independent of perturbation parameters, for standard finite element methods [27, 16, 17, 18, 19]. However, such highly nonuniform anisotropic meshes complicate the error analysis which frequently assumes quasi-uniformity.

Although there is an extensive literature on mixed finite element methods (MFEM) for second order elliptic problems [6], to the best of our knowledge, no uniformly convergent results have been obtained for SPP. For example, the standard MFEM for solving (3.1)-(3.2) on the lowest order Raviart-Thomas [26] RT_0 give the error estimate (5.1), which is not uniformly convergent, and to ensure the global convergence, the mesh size h must be in the order of $o(\varepsilon)$. This is very impractical, since ε can be as small as 10^{-10}. Because of the low regularity of the SPP only the (RT_0) rectangular elements will be considered, the techniques used here can be applied to higher order elements.

* Texas Institute for Computational and Applied Mathematics (TICAM), The University of Texas at Austin, Austin, TX 78712 (jli@ticam.utexas.edu).

† TICAM and Department of Aerospace Engineering & Engineering Mechanics, and Department of Petroleum and Geosystems Engineering, The University of Texas at Austin, Austin, TX 78712 (mfw@ticam.utexas.edu).

In this paper we consider mixed finite element methods based on the anisotropically refined meshes. For simplicity, we exam only two dimensional problems. However results can be directly extended to three dimension. Here we focus on the following reaction-diffusion model:

$$-\text{div}(a(x,y)\nabla p) + b(x,y)\ p = f(x,y) \quad \text{in} \quad \Omega \equiv (0,1)^2, \tag{1.1}$$

$$p = -g(x,y) \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where the diagonal tensor $a = (\varepsilon^2, \varepsilon^2)$ or $a = (\varepsilon^2, 1)$, and

$$b(x,y) > \beta^2 > 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

Here $0 < \varepsilon \ll 1$ is a small positive parameter. Extensions to other models are discussed in our forthcoming paper.

In this paper, we first derive our theoretical results assuming exact quadrature. Here global uniform convergence rates of $O(N^{-1})$ for both p and $a^{1/2}\nabla p$ in the L^2-norm are obtained, where N is the number of intervals in both directions. The pointwise interior (away from the boundary layers) convergence rates of $O(N^{-1})$ for p are also proved. Superconvergence (i.e., $O(N^{-2})$) at Gaussian points and $O(N^{-2})$ global L^2 estimate for both p and $a^{1/2} \bigtriangledown p$ are obtained by a local postprocessing. Modifications in the theoretical analyses can be extended to treat cell-centered finite differences, RT_0 with numerical quadrature [29, 24, 35, 3, 2]. Numerical results in this case which support our theory are presented and show that the anisotropic mesh gives more accurate results than the standard uniform mesh.

The organization of this paper is as follows. In §2, a general MFEM with exact quadrature is presented for (1.1). For completeness, general RT projections and the corresponding approximation properties are provided. Then refined approximations are proved for RT_0. Section 3 devotes to the anisotropic case ($a = (\varepsilon^2, 1)$). The isotropic case ($a = (\varepsilon^2, \varepsilon^2)$) is discussed is §4. Numerical results confirming our theoretical analysis are provided in §5.

Throughout the paper, C (or C_i) will denote a generic positive constant, which is independent of the mesh size and the perturbation parameter ε. We use the notation $||\cdot||_{k,\tau}$ for the Sobolev space $H^k(\tau)$ norm on τ, and v_{ξ^k} for the derivative of v with respect to (w.r.t.) the variable ξ.

2. The mixed formulation and the Raviart-Thomas projections. Let $\mathbf{u} = -a\nabla p, \alpha = a^{-1}$, then the weak formulation suitable for mixed methods of (1.1)-(1.2) is given by selecting a pair of $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{W}$ such that

$$(\alpha\mathbf{u}, \mathbf{v}) - (\text{div}\mathbf{v}, p) = < g, \mathbf{v}\cdot\nu >, \quad \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$(\text{div}\mathbf{u}, w) + (bp, w) = (f, w), \quad w \in \mathbf{W}, \tag{2.2}$$

where $\mathbf{V} = H(\text{div}, \Omega) = \{\mathbf{u} \in [L^2(\Omega)]^2 : \text{div}\mathbf{u} \in L^2(\Omega)\}, \mathbf{W} = L^2(\Omega), \nu$ is the outward normal to $\partial\Omega$ and the inner product in $[L^2(\Omega)]^2$ (or $L^2(\Omega)$) is indicated by $(\cdot,\cdot)$ and in $L^2(\partial\Omega)$ by $< \cdot,\cdot >$.

The MFEM for approximating the solution of (1.1)-(1.2) is: Finding $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{W}_h$ such that

$$(\alpha\mathbf{u}_h, \mathbf{v}) - (\text{div}\mathbf{v}, p_h) = < g, \mathbf{v}\cdot\nu >, \quad \mathbf{v} \in \mathbf{V}_h, \tag{2.3}$$

$$(\text{div}\mathbf{u}_h, w) + (bp_h, w) = (f, w), \quad w \in \mathbf{W}_h, \tag{2.4}$$

where $\mathbf{V}_h$ and $\mathbf{W}_h$ are the k-th ($k \geq 0$) order Raviart-Thomas space (RT_k) on any rectangular partition T_h of Ω, i.e.,

$$\mathbf{V}_h = \{\mathbf{v} \in H(\text{div}, \Omega) : \mathbf{v}|_R \in Q_{k+1,k}(R) \times Q_{k,k+1}(R), \quad R \in T_h\}, \tag{2.5}$$

and

$$\mathbf{W}_h = \{w \in L^2(\Omega) : w \in Q_{k,k}(R), \quad R \in T_h\}. \tag{2.6}$$

Here we denote $Q_{i,j}(R)$ by the space of polynomials of degree less than or equal to i in the first variable and to j in the second one restricted to R. The special partitions T_h will be specified later for different cases.

The existence and uniqueness of a solution of (2.3)-(2.4) can be established easily. Taking $\mathbf{v} = \mathbf{u}_h$ and $p = p_h$ in (2.3)-(2.4) and $f = g = 0$, then adding (2.3) and (2.4) together, we have

$$(\alpha \mathbf{u}_h, \mathbf{u}_h) + (b p_h, p_h) = 0, \tag{2.7}$$

from which and the assumptions of the coefficients we see that $\mathbf{u}_h = p_h = 0$.

The local Raviart-Thomas projection

$$\Pi_h : [H^1(R)]^2 \to \mathbf{V}_h(R), \quad \forall \ R \in T_h, \tag{2.8}$$

satisfies the following properties [8, 9, 34]

$$(\text{div}(\mathbf{v} - \Pi_h \mathbf{v}), w) = 0, \quad \forall w \ \in \mathbf{W}_h, \tag{2.9}$$

$$||\mathbf{v} - \Pi_h \mathbf{v}||_{0,R} \leq C h^r ||\mathbf{v}||_{r,R}, \quad 1 \leq r \leq k+1, \tag{2.10}$$

$$\text{div}\Pi_h = P_h \text{div}, \tag{2.11}$$

where P_h is the local L^2-projection: $L^2(R) \to \mathbf{W}_h(R)$. Furthermore, we have [8]:

$$(\text{div}\mathbf{v}_h, p - P_h p) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h(R), \tag{2.12}$$

$$||p - P_h p||_{0,R} \leq C h^r ||p||_{r,R}, \quad 0 \leq r \leq k+1. \tag{2.13}$$

It follows immediately from (2.1)-(2.4) that

$$(\alpha(\mathbf{u} - \mathbf{u}_h), \mathbf{v}) - (\text{div}\mathbf{v}, p - p_h) = 0, \quad \mathbf{v} \in \mathbf{V}_h, \tag{2.14}$$

$$(\text{div}(\mathbf{u} - \mathbf{u}_h), w) + (b(p - p_h), w) = 0, \quad w \in \mathbf{W}_h, \tag{2.15}$$

or

$$(\alpha(\Pi_h \mathbf{u} - \mathbf{u}_h), \mathbf{v}) - (\text{div}\mathbf{v}, P_h p - p_h) = (\alpha(\Pi_h \mathbf{u} - \mathbf{u}), \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \tag{2.16}$$

$$(\text{div}(\Pi_h \mathbf{u} - \mathbf{u}_h), w) + (b(P_h p - p_h), w) = (b(P_h p - p), w), \quad w \in \mathbf{W}_h, \tag{2.17}$$

where we used the properties of (2.9) and (2.12). Let $\mathbf{v} = \mu \equiv \Pi_h \mathbf{u} - \mathbf{u}_h$, $w = \tau \equiv P_h p - p_h$ in (2.16)(2.17), and adding (2.16) and (2.17) gives us:

$$(\alpha\mu, \mu) + (b\tau, \tau) = (\alpha(\Pi_h \mathbf{u} - \mathbf{u}), \mu) + (b(P_h p - p), \tau). \tag{2.18}$$

In the following, we will present some refined results for the lowest order Raviart-Thomas space RT_0.

LEMMA 2.1. *On any rectangular element R with width and length denoted as h_x and h_y respectively, the local L^2-projection P_h to $Q_{0,0}(R)$ satisfies that*

$$(i)\quad ||q - P_h q||_{0,R} \le C(h_x||q_x||_{0,R} + h_y||q_y||_{0,R}), \quad \forall q \in H^1(R),$$
$$(ii)\quad ||q - P_h q||_{\infty,R} \le C(h_x||q_x||_{\infty,R} + h_y||q_y||_{\infty,R}), \quad \forall q \in C^1(\overline{R}).$$

Proof. (i) The proof can be obtained following the idea of [4, Lemma 3.3]. Let $\hat{R} = [0,1]^2$, and $\hat{P}_{\hat{R}}$ be the L^2 projection to $Q_{0,0}(\hat{R})$, we have

$$||q - P_h q||_{0,R} = (\int_{\hat{R}} |\hat{q} - \hat{P}_{\hat{R}}\hat{q}|^2 h_x h_y d\hat{x}d\hat{y})^{1/2} = (h_x h_y)^{1/2}||\hat{q} - \hat{P}_{\hat{R}}\hat{q}||_{0,\hat{R}} \tag{2.19}$$
$$\le C \cdot (h_x h_y)^{1/2} \cdot \inf_{\hat{c}\in\mathcal{R}} ||\hat{q} + \hat{c}||_{1,\hat{R}} \le C \cdot (h_x h_y)^{1/2}|\hat{q}|_{1,\hat{R}} \tag{2.20}$$
$$= C \cdot (h_x h_y)^{1/2}(\int_{\hat{R}} (|\frac{\partial \hat{q}}{\partial \hat{x}}|^2 + |\frac{\partial \hat{q}}{\partial \hat{y}}|^2) d\hat{x}d\hat{y})^{1/2} \tag{2.21}$$
$$\le C \cdot (h_x h_y)^{1/2}(\int_R (|\frac{\partial q}{\partial x} \cdot h_x|^2 + |\frac{\partial q}{\partial y} \cdot h_y|^2) \frac{1}{h_x h_y} dxdy)^{1/2} \tag{2.22}$$
$$\le C \cdot (h_x||q_x||_{0,R} + h_y||q_y||_{0,R}), \tag{2.23}$$

which finishes the proof of (i).

(ii) Let (x_c, y_c) be the center of R, we have

$$||q - P_h q||_{\infty,R} = ||q(x,y) - q(x_c,y_c) - \frac{1}{R}\int_R [q(x,y) - q(x_c,y_c)]dxdy||_{\infty,R} \tag{2.24}$$
$$\le 2||q(x,y) - q(x_c,y_c)||_{\infty,R} \tag{2.25}$$
$$\le C \cdot (h_x||q_x||_{\infty,R} + h_y||q_y||_{\infty,R}), \tag{2.26}$$

from which we concludes the proof of (ii). ◻

LEMMA 2.2. *Let R be any rectangular element with width and length denoted as $2h_x$ and $2h_y$ respectively, and $\mathbf{q} \equiv (q^1, q^2) \in [H^2(R)]^2, \mathbf{v} \equiv (v^1, v^2) \in Q_{1,0} \times Q_{0,1}(R)$, then the local RT_0 projection Π_h has the following properties:*

$$(i)\quad |\int_R (q^1 - \Pi_h q^1)v^1| \le Ch_x^2||q^1_{x^2}||_{0,R} \cdot ||v^1||_{0,R},$$
$$|\int_R (q^2 - \Pi_h q^2)v^2| \le Ch_y^2||q^2_{y^2}||_{0,R} \cdot ||v^2||_{0,R},$$
$$(ii)\quad ||q^k - \Pi_h q^k||_{r,R} \le C(h_x||q^k_x||_{r,R} + h_y||q^k_y||_{r,R}), \quad k = 1,2, r = 0,\infty.$$

Proof. The proof will use the integral technique we developed in [20, 36]. More details can be found in [21].

(i) For simplicity, consider a rectangular element $R = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$. Let $E(x) = \frac{1}{2}[(x - x_c)^2 - h_x^2]$ and $\chi = q^1 - \Pi_h q^1$. By Taylor expansion, we have

$$\int_R \chi v^1 = \int_R \chi[v^1(x_c,y) + (x - x_c)v^1_x(x_c,y)] \tag{2.27}$$
$$= \int_R \chi[E_{x^2}v^1(x_c,y) + \frac{1}{6}(E^2)_{x^3}v^1_x(x_c,y)], \tag{2.28}$$

where we used the facts that $E_{x^2} = 1$ and $x - x_c = \frac{1}{6}(E^2)_{x^3}$.

Let $l_2 = \{x = x_c + h_x\} \times [y_c - h_y, y_c + h_y], l_1 = \{x = x_c - h_x\} \times [y_c - h_y, y_c + h_y]$. By the property of Π_h [34, (2.1)] and the fact that $E(x) = 0$ on l_1, l_2, we have

$$\int_R \chi E_{x^2} v^1(x_c, y) = (\int_{l_2} - \int_{l_1})\chi E_x v^1(x_c, y)dy - \int_R \chi_x E_x v^1(x_c, y) \tag{2.29}$$

$$= -\int_R \chi_x E_x v^1(x_c, y) = \int_R \chi_{x^2} E v^1(x_c, y) \tag{2.30}$$

$$= \int_R q^1_{x^2} E(x) \cdot [v^1(x, y) + (x_c - x)v^1_x(x, y)] \tag{2.31}$$

$$\leq Ch_x^2 ||q^1_{x^2}||_{0,R}||v^1||_{0,R}, \tag{2.32}$$

where in the last step, we used the fact that $E(x) = O(h_x^2)$ and the standard inverse estimate [5].

Repeating the same arguments as above, we obtain

$$\int_R \chi \cdot \frac{1}{6}(E^2)_{x^3} v^1_x(x_c, y) \tag{2.33}$$

$$= (\int_{l_2} - \int_{l_1})\chi \cdot \frac{1}{6}(E^2)_{x^2} v^1_x(x_c, y)dy - \int_R \chi_x \frac{1}{6}(E^2)_{x^2} v^1_x(x_c, y) \tag{2.34}$$

$$= \int_R \chi_{x^2} \frac{1}{6}(E^2)_x v^1_x(x_c, y) \tag{2.35}$$

$$= \int_R q^1_{x^2} \frac{1}{6}(E^2)_x v^1_x(x, y) \leq Ch_x^2 ||q^1_{x^2}||_{0,R}||v^1||_{0,R}. \tag{2.36}$$

Hence, we have

$$|\int_R (q^1 - \Pi_h q^1)v^1| \leq Ch_x^2 ||q^1_{x^2}||_{0,R}||v^1||_{0,R}. \tag{2.37}$$

Similarly, by taking the Taylor expansion in y and using $F(y) = \frac{1}{2}[(y - y_c)^2 - h_y^2]$ instead of $E(x)$, we have

$$|\int_R (q^2 - \Pi_h q^2)v^2| \leq Ch_y^2 ||q^2_{y^2}||_{0,R}||v^2||_{0,R}, \tag{2.38}$$

which together with (2.37) completes the proof of (i).

(ii) For $r = 0, \infty$, by Lemma 2.1 of [34], we have

$$\begin{aligned}
||q^1 - \Pi_h q^1||_{r,R} &= ||P_h[q^1 + (\xi - x_c)q^1_\xi] + P_h[q^1_\xi](x - x_c) - q^1||_{r,R} \\
&\leq ||P_h q^1 - q^1||_{r,R} + ||P_h[(\xi - x_c)q^1_\xi]||_{r,R} + ||P_h[q^1_\xi](x - x_c)||_{r,R} \\
&\leq C(h_x ||q^1_x||_{r,R} + h_y ||q^1_y||_{r,R}),
\end{aligned}$$

where we used the results of Lemma 2.1 and the fact that $||P_h v||_{r,R} \leq C||v||_{r,R}$.

The proof for the second component q^2 is all the same by noting that

$$||q^2 - \Pi_h q^2||_{r,R} = ||P_h[q^2 + (\eta - y_c)q^2_\eta] + P_h[q^2_\eta](y - y_c) - q^2||_{r,R}. \tag{2.39}$$

□

3. The anisotropic case. For simplicity, let $a = (\varepsilon^2, 1)$ and $g = 0$, i.e., we consider the following problem:

$$L_\varepsilon p \equiv -(\varepsilon^2 p_{x^2} + p_{y^2}) + b(x,y)p = f, \quad \text{in} \quad \Omega \equiv (0,1)^2, \tag{3.1}$$

$$p = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

Hence we have boundary layers at the sides $x = 0$ and $x = 1$ of Ω.

3.1. The anisotropic mesh and the derivative estimates. In this section, we will present some estimates for the smooth solution p of (3.1)-(3.2), where the corresponding compatibility conditions are assumed [13].

Considering the behaviour of the analytic solution p of (3.1)-(3.2), we divide Ω into three matching subdomains $\Omega_i, 1 \le i \le 3$, i.e., $\Omega = \cup_{i=1}^3 \Omega_i$, where

$$\Omega_1 \equiv (0, \sigma_x) \times (0,1), \Omega_2 \equiv (\sigma_x, 1-\sigma_x) \times (0,1), \Omega_3 \equiv (1-\sigma_x, 1) \times (0,1).$$

Here $\sigma_x = 2\beta^{-1}\varepsilon|\ln\varepsilon|$. Each subdomain Ω_i is then divided quasiuniformly [11, p.28] into N_{i_x} and N_{i_y} subintervals in the x- and y-directions, respectively. Hence we obtain a highly nonuniform rectangular mesh, which is a *a prior* anisotropically refined mesh (see Figure 3.1). To avoid lengthy notations, we assume $N_{i_x} \simeq N_{i_y} \simeq N$. Here $N_1 \simeq N_2$ means that $C_1 N_1 \le N_2 \le C_2 N_1$, where C_1 and C_2 are positive constants, N is the total number of partitions in each direction. Since we are considering the singularly perturbed case, without loss of generality, we assume that $\sigma_x \le 1/3$.

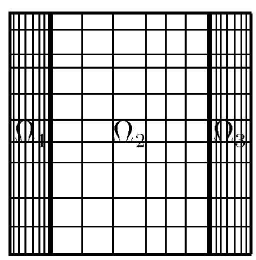

FIG. 3.1. *An example of the anisotropic mesh for the problem (3.1)-(3.2)*

LEMMA 3.1. *For the solution p of (3.1)-(3.2) and $0 \le k \le 4$, we have*

(i) $|p_{x^k}(x,y)| \le C(1 + \varepsilon^{-k}\exp(-\beta x/\varepsilon) + \varepsilon^{-k}\exp(-\beta(1-x)/\varepsilon))$, *on* $\overline{\Omega} \equiv \Omega \cup \partial\Omega$,

(ii) $|p_{y^k}(x,y)| \le C$, *on* $\overline{\Omega}$.

Proof. The results were proved for k=0,1,2 [16, Lemmas 2.1-2.5]. The proof for k=3 and 4 is provided in Lemmas 3.2-3.7. □

Remark 1. From Lemma 3.1, we see that the solution p of (3.1)-(3.2) has sharp boundary layers at sides x=0 and x=1.

LEMMA 3.2. *[16, Theorem 2.1] For any functions $w(x,y) \in C^2(\Omega) \cap C^0(\overline{\Omega})$, if $w \ge 0$ on $\partial\Omega$ and $L_\varepsilon w \ge 0$ on Ω, then $w \ge 0$ on $\overline{\Omega}$.*

By using the boundary condition (3.2) in (3.1), we have

$$p_{x^2}|_{x=0} = -\varepsilon^{-2} f(0,y), \quad p_{x^2}|_{x=1} = -\varepsilon^{-2} f(1,y), \tag{3.3}$$

$$p_{x^2}|_{y=0} = p_{x^2}|_{y=1} = 0, \tag{3.4}$$

or written in one simple form as

$$p_{x^2}(x,y) = \tilde{g}(x,y,\varepsilon) \quad \text{on} \quad \partial\Omega,$$

where $\tilde{g}(x,y,\varepsilon) = -\varepsilon^{-2}\frac{1-\exp(-\beta(1-x)/\varepsilon)}{1-\exp(-\beta/\varepsilon)}f(0,y) - \varepsilon^{-2}\frac{1-\exp(-\beta x/\varepsilon)}{1-\exp(-\beta/\varepsilon)}f(1,y)$. Here the compatibility conditions [13] of $f(m_i) = 0$ were used, where $m_i, 1 \leq i \leq 4$, are the corners of Ω.

Denote $\tilde{p}(x,y) = p_{x^2}(x,y) - \tilde{g}(x,y,\varepsilon)$, differentiating (3.1) twice w.r.t. x gives us:

$$L_\varepsilon \tilde{p} \equiv -(\varepsilon^2 \tilde{p}_{x^2} + \tilde{p}_{y^2}) + b\tilde{p} = \tilde{F} \quad \text{in} \quad \Omega, \tag{3.5}$$

$$\tilde{p} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{3.6}$$

where $\tilde{F} = f_{x^2} - 2b_x p_x - b_{x^2} p + \varepsilon^2 \tilde{g}_{x^2} + \tilde{g}_{y^2} - b\tilde{g}$.

Lemma 3.3. *For the solution $\tilde{p}$ of (3.5)-(3.6), we have*

$$|\tilde{p}_x(x,y)| \leq C\varepsilon^{-3}, \quad on \quad \partial\Omega. \tag{3.7}$$

Proof. Using the barrier function

$$\phi = C\varepsilon^{-2}(1-\exp(-\beta x/\varepsilon))(1-\exp(-\beta(1-x)/\varepsilon)),$$

and after some simple calculations, we have

$$\begin{aligned} L_\varepsilon(\phi \pm \tilde{p}) &= \frac{C\beta^2}{\varepsilon^2}[\exp(-\beta x/\varepsilon) + \exp(-\beta(1-x)/\varepsilon)] \\ &\quad + \frac{Cb}{\varepsilon^2}(1-\exp(-\beta x/\varepsilon))(1-\exp(-\beta(1-x)/\varepsilon)) \pm \tilde{F} \\ &\geq \frac{C\beta^2}{\varepsilon^2}(1+\exp(-\beta/\varepsilon)) \pm \tilde{F} \\ &\geq 0, \quad \text{for sufficiently large C,} \end{aligned}$$

where we used the fact that $b > \beta^2$ and $|\tilde{F}| \leq C_1\varepsilon^{-2}$.

Hence, by Lemma 3.2 and the fact that $(\phi \pm \tilde{p})|_{\partial\Omega} \geq 0$, we obtain

$$|\tilde{p}| \leq \phi = C\varepsilon^{-2}(1-\exp(-\beta x/\varepsilon))(1-\exp(-\beta(1-x)/\varepsilon)), \quad \text{on} \quad \overline{\Omega}, \tag{3.8}$$

from which we have

$$|\tilde{p}_x(0,y)| = |\lim_{x\to 0+} \frac{\tilde{p}(x,y) - \tilde{p}(0,y)}{x}| \leq \lim_{x\to 0+} |\frac{\tilde{p}(x,y)-\tilde{p}(0,y)}{x}| \tag{3.9}$$

$$\leq \lim_{x\to 0+} \frac{C\varepsilon^{-2}(1-\exp(-\beta x/\varepsilon))(1-\exp(-\beta(1-x)/\varepsilon))}{x} \leq C\beta\varepsilon^{-3}\ . \tag{3.10}$$

In the same way, we can obtain

$$|\tilde{p}_x(1,y)| \leq \lim_{x\to 1^-} |\frac{\tilde{p}(1,y)-\tilde{p}(x,y)}{1-x}| \leq C\beta\varepsilon^{-3}. \tag{3.11}$$

The boundary condition (3.6) implies that $\tilde{p}_x(x,0) = 0 = \tilde{p}_x(x,1)$, which along with the above inequalities (3.10)(3.11) completes our proof. □

Lemma 3.4. *For the solution p of (3.1)-(3.2), we have*

$$|p_{x^3}(x,y)| \leq C(1 + \varepsilon^{-3}\exp(-\beta x/\varepsilon) + \varepsilon^{-3}\exp(-\beta(1-x)/\varepsilon)), \quad on \quad \overline{\Omega}. \tag{3.12}$$

Proof. Consider the barrier function

$$\phi = C(1+\varepsilon^{-3}\exp(-\beta x/\varepsilon)+\varepsilon^{-3}\exp(-\beta(1-x)/\varepsilon)).$$

By simple calculations, we have

$$\begin{aligned} L_\varepsilon(\phi \pm \tilde{p}_x) &= bC + C(b-\beta^2)\varepsilon^{-3}(\exp(-\beta x/\varepsilon)+\exp(-\beta(1-x)/\varepsilon)) \pm (\tilde{F}_x - b_x\tilde{p}) \\ &\geq 0, \quad \text{for sufficiently large C,} \end{aligned}$$

where the fact $b > \beta^2$ and the estimates $p_{x^k}, 0 \leq k \leq 2$, were used.

By Lemma 3.3, it is easy to see that $(\phi \pm \tilde{p}_x)|_{\partial\Omega} \geq 0$, which together with Lemma 3.2 gives us

$$|\tilde{p}_x(x,y)| \leq \phi = C(1+\varepsilon^{-3}\exp(-\beta x/\varepsilon)+\varepsilon^{-3}\exp(-\beta(1-x)/\varepsilon)) \quad \text{on} \quad \overline{\Omega}.$$

Then by the definition of $\tilde{p}$, we have

$$|p_{x^3}(x,y)| = |\tilde{p}_x(x,y) + \tilde{g}_x(x,y,\varepsilon)| \tag{3.13}$$

$$\leq C(1+\varepsilon^{-3}\exp(-\beta x/\varepsilon)+\varepsilon^{-3}\exp(-\beta(1-x)/\varepsilon)), \tag{3.14}$$

which completes our proof, where we used the fact that $1/(1-\exp(-\beta/\varepsilon)) \leq 1/(1-\exp(-\beta))$ for $0 < \varepsilon \leq 1$. □

LEMMA 3.5. *For the solution p of (3.1)-(3.2), we have*

$$|p_{x^4}(x,y)| \leq C(1+\varepsilon^{-4}\exp(-\beta x/\varepsilon)+\varepsilon^{-4}\exp(-\beta(1-x)/\varepsilon)), \quad on \quad \overline{\Omega}. \tag{3.15}$$

Proof. Let $\phi = C(1+\varepsilon^{-4}\exp(-\beta x/\varepsilon)+\varepsilon^{-4}\exp(-\beta(1-x)/\varepsilon))$. By simple calculations, we have

$$\begin{aligned} L_\varepsilon(\phi \pm \tilde{p}_{x^2}) = bC &+ C(b-\beta^2)\varepsilon^{-4}(\exp(-\beta x/\varepsilon)+\exp(-\beta(1-x)/\varepsilon)) \\ &\pm(\tilde{F}_{x^2} - b_{x^2}\tilde{p} - 2b_x\tilde{p}_x). \end{aligned}$$

Using the estimates of $p_{x^i}, 0 \leq i \leq 3$, and g, we have

$$\begin{aligned} |\tilde{F}_{x^2} - b_{x^2}\tilde{p} - 2b_x\tilde{p}_x| &= |f_{x^4} - (bp)_{x^4} + bp_{x^4} - L_\varepsilon(\tilde{g}_{x^2})| \\ &\leq C_4(1+\varepsilon^{-4}\exp(-\beta x/\varepsilon)+\varepsilon^{-4}\exp(-\beta(1-x)/\varepsilon)). \end{aligned}$$

Hence for sufficiently large C, we have

$$L_\varepsilon(\phi \pm \tilde{p}_{x^2}) \geq 0, \quad \text{on} \quad \overline{\Omega}. \tag{3.16}$$

From (3.5) and (3.6), we know that

$$\begin{aligned} \tilde{p}_{x^2}|_{y=0,1} &= 0, \\ \tilde{p}_{x^2}|_{x=0,1} &= -\varepsilon^{-2}\tilde{F}|_{x=0,1} \leq C_5\varepsilon^{-4}. \end{aligned}$$

Hence we have $(\phi \pm \tilde{p}_{x^2})|_{\partial\Omega} \geq 0$ for sufficiently large C, which along with (3.16) and Lemma 3.2 we obtain

$$|\tilde{p}_{x^2}(x,y)| \leq \phi = C(1+\varepsilon^{-4}\exp(-\beta x/\varepsilon)+\varepsilon^{-4}\exp(-\beta(1-x)/\varepsilon)).$$

Hence by the definition of $\tilde{p}$, we have

$$\begin{aligned}|p_{x^4}(x,y)| &\le |\tilde{p}_{x^2}(x,y)| + |\tilde{g}_{x^2}(x,y,\varepsilon)| \\ &\le C(1+\varepsilon^{-4}\exp(-\beta x/\varepsilon)+\varepsilon^{-4}\exp(-\beta(1-x)/\varepsilon)),\end{aligned}$$

which completes the proof. □

Repeating the above arguments, we can obtain:

$$L_\varepsilon \tilde{p}^2 \equiv -(\varepsilon^2 \tilde{p}^2_{x^2} + \tilde{p}^2_{y^2}) + b\tilde{p}^2 = \tilde{F}^2 \quad \text{in} \quad \Omega, \tag{3.17}$$

$$\tilde{p}^2 = 0 \quad \text{on} \quad \partial\Omega\ , \tag{3.18}$$

where $\tilde{p}^2(x,y) = p_{y^2} - \tilde{g}^2(x,y,\varepsilon), \tilde{g}^2(x,y,\varepsilon) = -[(1-y)f(x,0)+yf(x,1)]$ and

$$\tilde{F}^2 = f_{y^2} - 2b_y p_y - b_{y^2}p + \varepsilon^2 \tilde{g}^2_{x^2} + \tilde{g}^2_{y^2} - b\tilde{g}^2.$$

LEMMA 3.6. *For the solution p of (3.1)-(3.2), we have*

$$|p_{y^3}(x,y)| \le C \quad on \quad \overline{\Omega}.$$

Proof. Consider the barrier function $\phi = Cy(1-y)$, we have

$$L_\varepsilon(\phi \pm \tilde{p}^2) = 2C \pm \tilde{F}^2 \ge 0, \quad \text{for sufficiently large C},$$

which along with $(\phi \pm \tilde{p}^2)|_{\partial\Omega} \ge 0$ gives us

$$|\tilde{p}^2(x,y)| \le Cy(1-y), \quad \text{on} \quad \overline{\Omega}.$$

Hence, we have

$$|\tilde{p}^2_y(x,0)| \le \lim_{y\to 0+} |\frac{\tilde{p}^2(x,y)-\tilde{p}^2(x,0)}{y}| \le \lim_{y\to 0+} C(1-y) = C,$$

and similarly,

$$|\tilde{p}^2_y(x,1)| \le \lim_{y\to 1^-} |\frac{\tilde{p}^2(x,1)-\tilde{p}^2(x,y)}{1-y}| \le \lim_{y\to 1^-} Cy = C.$$

Boundary condition (3.18) gives that $\tilde{p}^2_y(x,y)|_{x=0,1} = 0$, which along the above two inequalities gives us:

$$|\tilde{p}^2_y(x,y)| \le C, \quad \text{on} \quad \overline{\Omega}. \tag{3.19}$$

Then consider the barrier function $\phi = C$, we have

$$L_\varepsilon(\phi \pm \tilde{p}^2_y(x,y)) = bC \pm (\tilde{F}^2_y - b_y\tilde{p}^2) \tag{3.20}$$

$$\ge 0, \quad \text{for sufficiently large C}, \tag{3.21}$$

which together with (3.19) and Lemma 3.2 shows that

$$|\tilde{p}^2_y(x,y)| \le \phi = C, \quad \text{on} \quad \overline{\Omega}. \tag{3.22}$$

Therefore,

$$|p_{y^3}(x,y)| = |\tilde{p}^2_y(x,y) + \tilde{g}^2_y(x,y)| \le C,$$

which completes our proof. ∎

LEMMA 3.7. *For the solution p of (3.1)-(3.2), we have*

$$|p_{y^4}(x,y)| \le C \quad on \ \ \overline{\Omega}.$$

Proof. Let $\phi = C$. By simple calculations, we have

$$L_\varepsilon(\phi \pm \tilde{p}_{y^2}) = bC \pm (\tilde{F}^2_{y^2} - b_{y^2}\tilde{p}^2 - 2b_y\tilde{p}^2_y).$$

Note that

$$|\tilde{F}^2_{y^2} - b_{y^2}\tilde{p}^2 - 2b_y\tilde{p}^2_y| = |f_{y^4} - (bp)_{y^4} + bp_{y^4}| \le C_6.$$

Hence for sufficiently large C, we have

$$L_\varepsilon(\phi \pm \tilde{p}_{y^2}) \ge 0, \quad \text{on} \ \ \overline{\Omega}. \tag{3.23}$$

From (3.17) and (3.18), we have

$$\tilde{p}^2_{y^2}|_{x=0,1} = 0,$$
$$\tilde{p}^2_{y^2}|_{y=0,1} = -\tilde{F}^2 \le C_7.$$

Hence for sufficiently large C, we have $(\phi \pm \tilde{p}^2_{y^2})|_{\partial\Omega} \ge 0$, which along with (3.23) and Lemma 3.2 completes the proof. ∎

In the following we derive the estimates on the mixed derivatives.

LEMMA 3.8. *For the solution p of (3.1)-(3.2), we have*

$$(i) \quad ||p_{xy}||_{0,\Omega_i} \le C(\varepsilon^{-3/4}|\ln^{1/4}\varepsilon| + 1), \quad i = 1,3, \tag{3.24}$$

$$(ii) \quad ||p_{xy}||_{0,\Omega_2} \le C. \tag{3.25}$$

Proof. (i) Integrating by steps and using the facts that $p_y|_{x=0} = 0 = p_{x^2}|_{y=0,1} = p_x|_{y=0,1}$, we have

$$||p_{xy}||^2_{0,\Omega_1} = \int_{\Omega_1} p_{xy}\cdot p_{xy}\,dxdy \tag{3.26}$$

$$= \int_0^1 (p_y \cdot p_{xy})|_{x=0}^{x=\sigma_x} dy - \int_{\Omega_1} p_y \cdot p_{x^2y}\,dxdy \tag{3.27}$$

$$= \int_0^1 (p_y\cdot p_{xy})(\sigma_x,y)dy - \int_0^{\sigma_x} (p_y\cdot p_{x^2})|_{y=0}^{y=1}dx + \int_{\Omega_1} p_{y^2}p_{x^2}\,dxdy \tag{3.28}$$

$$= (p_yp_x)(\sigma_x,1) - (p_yp_x)(\sigma_x,0) - \int_0^1 (p_{y^2}p_x)(\sigma_x,y)dy + \int_{\Omega_1} p_{y^2}p_{x^2}\,dxdy \tag{3.29}$$

$$= -\int_0^1 (p_{y^2}p_x)(\sigma_x,y)dy + \int_{\Omega_1} p_{y^2}p_{x^2}\,dxdy. \tag{3.30}$$

Then by Lemma 3.1 and the fact that $\gamma\varepsilon^{-1}\exp(-\gamma/\varepsilon) \le 1$ for any $\gamma \ge 0$, we have

$$\begin{aligned} ||p_{xy}||^2_{0,\Omega_1} &\le C(1 + \varepsilon^{-1}\exp(-\beta\sigma_x/\varepsilon) + \varepsilon^{-1}\exp^{-\beta(1-\sigma_x)/\varepsilon}) + ||p_{y^2}||_{0,\Omega_1}||p_{x^2}||_{0,\Omega_1} \\ &\le C + C||p_{x^2}||_{\infty,\Omega_1}\cdot(\text{meas}(\Omega_1))^{1/2} \\ &\le C + C\varepsilon^{-2}\cdot|(\varepsilon\ln\varepsilon)^{1/2}|. \end{aligned}$$

Hence,

$$||p_{xy}||_{0,\Omega_1} \leq C(\varepsilon^{-3/4}|\ln^{1/4}\varepsilon| + 1).$$

By symmetry, we have

$$||p_{xy}||_{0,\Omega_3} \leq C(\varepsilon^{-3/4}|\ln^{1/4}\varepsilon| + 1).$$

(ii) By repeating the above arguments, we have

$$||p_{xy}||^2_{0,\Omega_2} = \int_{\Omega_2} p_{xy} \cdot p_{xy}\, dxdy \tag{3.31}$$

$$= \int_0^1 (p_y p_{xy})|_{x=\sigma_x}^{x=1-\sigma_x} dy - \int_{\Omega_2} p_y p_{x^2y}\, dxdy \tag{3.32}$$

$$= -\int_0^1 (p_{y^2} p_x)|_{x=\sigma_x}^{x=1-\sigma_x} dy + \int_{\Omega_2} p_{y^2} p_{x^2} dxdy \tag{3.33}$$

$$\leq C, \tag{3.34}$$

where we used Lemma 3.1 and the facts that $p_x|_{y=0,1} = 0 = p_{x^2}|_{y=0,1}$. ☐

LEMMA 3.9. *For the solution p of (3.1)-(3.2), we have*

$$(i) \quad ||p_{x^2y}||_{0,\Omega_i} \leq C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|, \quad i = 1, 3, \tag{3.35}$$

$$(ii) \quad ||p_{x^2y}||_{0,\Omega_2} \leq C. \tag{3.36}$$

Proof. Differentiating (3.1) twice w.r.t. y and using Lemma 3.1, we have

$$|p_{x^2y^2}| \leq \varepsilon^{-2}(|f_{y^2}| + |(bp)_{y^2}| + |p_{y^4}|) \leq C\varepsilon^{-2}, \quad \text{on } \overline{\Omega}.$$

Differentiating (3.1) twice w.r.t. x and using Lemma 3.1, we have

$$|p_{x^2y^2}| \leq \varepsilon^2|p_{x^4}| + |f_{x^2}| + |(bp)_{x^2}| \tag{3.37}$$

$$\leq C(1 + \varepsilon^{-2}\exp(-\beta x/\varepsilon) + \varepsilon^{-2}\exp(-\beta(1-x)/\varepsilon)). \tag{3.38}$$

Note that $\sigma_x \leq x \leq 1 - \sigma_x$ on $\overline{\Omega}_2$, we have

$$|p_{x^2y^2}| \leq C(1 + \varepsilon^{-2}\exp(-\beta\sigma_x/\varepsilon)) \leq C, \text{ on } \overline{\Omega}_2.$$

By the same arguments, we have

$$||p_{x^2}||_{\infty,\Omega_i} \leq C\varepsilon^{-2}, \quad i = 1, 3, \tag{3.39}$$

$$||p_{x^2}||_{\infty,\Omega_i} \leq C, \quad i = 2. \tag{3.40}$$

Integrating by steps and using the fact that $p_{x^2}|_{y=0,1} = 0$, we have

$$\int_{\Omega_i} p_{x^2y} \cdot p_{x^2y}\, dxdy = -\int_{\Omega_i} p_{x^2y^2} \cdot p_{x^2} dxdy \tag{3.41}$$

$$\leq ||p_{x^2y^2}||_{\infty,\Omega_i}||p_{x^2}||_{\infty,\Omega_i}\text{meas}(\Omega_i). \tag{3.42}$$

Therefore, using the fact that $\text{meas}(\Omega_i) = O(\varepsilon|\ln\varepsilon|), i = 1, 3$, we have

$$||p_{x^2y}||^2_{0,\Omega_i} \leq C\varepsilon^{-2} \cdot \varepsilon^{-2} \cdot \varepsilon|\ln\varepsilon| \leq C\varepsilon^{-3}|\ln\varepsilon|,$$

or, $||p_{x^2y}||_{0,\Omega_i} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|$.

While for i=2, meas$(\Omega_i) \le 1$, we have

$$||p_{x^2y}||^2_{0,\Omega_2} \le C,$$

from which completes our proof. □

LEMMA 3.10. *For the solution p of (3.1)-(3.2), we have*

$$(i) \quad ||p_{xy^2}||_{0,\Omega_i} \le C\varepsilon^{-1/2}|\ln^{1/2}\varepsilon|, \quad i=1,3, \tag{3.43}$$

$$(ii) \quad ||p_{xy^2}||_{0,\Omega_2} \le C. \tag{3.44}$$

Proof. Differentiating (3.1) w.r.t. x and using Lemma 3.1, we obtain

$$|p_{xy^2}(x,y)| \le |f_x| + |(bp)_x| + \varepsilon^2|p_{x^3}| \tag{3.45}$$

$$\le C(1 + \varepsilon^{-1}\exp(-\beta x/\varepsilon) + \varepsilon^{-1}\exp(-\beta(1-x)/\varepsilon)). \tag{3.46}$$

Using the fact that $\sigma_x \le x \le 1 - \sigma_x$ on $\overline{\Omega}_2$, and the definition of σ_x, we have

$$||p_{xy^2}||_{\infty,\Omega_2} \le C.$$

While $||p_{xy^2}||_{\infty,\Omega_i} \le C\varepsilon^{-1}, \quad i=1,3.$

Hence we obtain

$$||p_{xy^2}||_{0,\Omega_i} \le ||p_{xy^2}||_{\infty,\Omega_i}(\text{meas}(\Omega_i))^{1/2} \tag{3.47}$$

$$\le C\varepsilon^{-1}\cdot|\varepsilon\ln\varepsilon|^{1/2} = C\varepsilon^{-1/2}|\ln^{1/2}\varepsilon|, \quad i=1,3 \tag{3.48}$$

and $||p_{xy^2}||_{0,\Omega_2} \le C$. □

3.2. The error estimates in the anisotropic case. In the following, for any $\mathbf{v} \in \mathbf{V}_h$ or $\mathbf{V}$, we denote its two components as $\mathbf{v} = (v^1, v^2)$.

Note that in this case, $\alpha = (\varepsilon^{-2}, 1)$, hence $\mathbf{u} = -a\nabla p = -(\varepsilon^2 p_x, p_y)$. The equation (2.18) becomes

$$\begin{aligned} &\varepsilon^{-2}(\mu^1,\mu^1) + (\mu^2,\mu^2) + (b\tau,\tau) \\ &= \varepsilon^{-2}(\Pi_h u^1 - u^1, \mu^1) + (\Pi_h u^2 - u^2, \mu^2) + (b(P_h p - p), \tau). \end{aligned} \tag{3.49}$$

LEMMA 3.11. *Let $\mathbf{u} = -(\varepsilon^2 p_x, p_y)$ and $\mathbf{u}_h = -(\varepsilon^2 p_x^h, p_y^h)$ be the solutions of (3.1)-(3.2) and (2.3)-(2.4) respectively, then*

$$\varepsilon||\Pi_h p_x - p_x^h||_{0,\Omega} + ||\Pi_h p_y - p_y^h||_{0,\Omega} + ||P_h p - p_h||_{0,\Omega} \le CN^{-2}. \tag{3.50}$$

Proof. For any i=1,2,3, by Lemma 2.2, we have

$$M_{1,\Omega_i} \equiv \varepsilon^{-2}\int_{\Omega_i}(\Pi_h u^1 - u^1)\mu^1 \le C\varepsilon^{-2}h_x^2||u^1_{x^2}||_{0,\Omega_i}||\mu^1||_{0,\Omega_i} \tag{3.51}$$

$$= Ch_x^2||p_{x^3}||_{0,\Omega_i}||\mu^1||_{0,\Omega_i} \tag{3.52}$$

$$\le Ch_x^2||p_{x^3}||_{\infty,\Omega_i}(\text{meas}(\Omega_i))^{1/2}||\mu^1||_{0,\Omega_i}, \tag{3.53}$$

$$M_{2,\Omega_i} \equiv \int_{\Omega_i}(\Pi_h u^2 - u^2)\mu^2 \le Ch_y^2||u^2_{y^2}||_{0,\Omega_i}||\mu^2||_{0,\Omega_i} \tag{3.54}$$

$$= Ch_y^2||p_{y^3}||_{0,\Omega_i}||\mu^2||_{0,\Omega_i}. \tag{3.55}$$

By Lemma 2.1 and the property of the projection P_h, we have

$$M_{3,\Omega_i} \equiv \int_{\Omega_i} b(P_h p - p)\tau = \int_{\Omega_i} (b - P_h b)(P_h p - p)\tau \tag{3.56}$$

$$\leq CN^{-1}(h_x||p_x||_{0,\Omega_i} + h_h||p_y||_{0,\Omega_i})||\tau||_{0,\Omega_i} \tag{3.57}$$

$$\leq CN^{-1}[h_x||p_x||_{\infty,\Omega_i}(\text{meas}(\Omega_i))^{1/2} + h_y||p_y||_{0,\Omega_i}]||\tau||_{0,\Omega_i}. \tag{3.58}$$

By the construction of T_h, we know that $h_x = O(\varepsilon|\ln\varepsilon|/N)$ on Ω_1 and Ω_3, $h_x = O(N^{-1})$ on Ω_2, and $h_y = O(N^{-1})$ on $\Omega_i, i = 1, 2, 3$.

For i=1,3, by (3.53) and Lemma 3.1, we have

$$M_{1,\Omega_i} \leq C(\varepsilon|\ln\varepsilon|/N)^2 \cdot \varepsilon^{-3} \cdot (\varepsilon|\ln\varepsilon|)^{1/2} \cdot ||\mu^1||_{0,\Omega_i} \tag{3.59}$$

$$= CN^{-2}\varepsilon^{1/2}|\ln^{5/2}\varepsilon| \cdot \varepsilon^{-1}||\mu^1||_{0,\Omega_i}, \tag{3.60}$$

while for i=2,

$$M_{1,\Omega_i} \leq CN^{-2} \cdot \varepsilon^{-1}||\mu^1||_{0,\Omega_i}. \tag{3.61}$$

For i=1,3, from (3.58), we obtain

$$M_{3,\Omega_i} \leq CN^{-1}[\frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-1} \cdot (\varepsilon|\ln\varepsilon|)^{1/2} + N^{-1}]||\tau||_{0,\Omega_i} \tag{3.62}$$

$$\leq CN^{-2}[\varepsilon^{1/2}|\ln^{3/2}\varepsilon| + 1]||\tau||_{0,\Omega_i}. \tag{3.63}$$

While for i=2, we have

$$M_{3,\Omega_i} \leq CN^{-2}||\tau||_{0,\Omega_i}. \tag{3.64}$$

For i=1,2,3, by (3.55) and Lemma 3.1, we have

$$M_{2,\Omega_i} \leq CN^{-2}||\mu^2||_{0,\Omega_i}. \tag{3.65}$$

Note that

$$\varepsilon^{1/2}|\ln^{3/2}\varepsilon| < 1.2, \quad \varepsilon^{1/2}|\ln^{5/2}\varepsilon| < 4.6, \quad \text{for } \forall\ \varepsilon \in (0, 1],$$

which along with the above inequalities, (3.49), Cauchy-Schwarz inequality completes the proof. □

Theorem 3.12. *Under the assumptions of Lemma 3.11, we have*

$$\varepsilon||p_x - p_x^h||_{0,\Omega} + ||p_y - p_y^h||_{0,\Omega} + ||p - p_h||_{0,\Omega} \leq CN^{-1}. \tag{3.66}$$

Furthermore, we have the following interior error estimates:

$$||p - p_h||_{\infty,\Omega_2} \leq CN^{-1}. \tag{3.67}$$

Proof. For i=1,2,3, by Lemma 2.2, we have

$$M_{1,\Omega_i} \equiv \varepsilon||p_x - \Pi_h p_x||_{0,\Omega_i} \leq C\varepsilon(h_x||p_{x^2}||_{0,\Omega_i} + h_y||p_{xy}||_{0,\Omega_i}) \tag{3.68}$$

$$\leq C\varepsilon(h_x||p_{x^2}||_{\infty,\Omega_i} \cdot (\text{meas}(\Omega_i))^{1/2} + h_y||p_{xy}||_{0,\Omega_i}), \tag{3.69}$$

$$M_{2,\Omega_i} \equiv ||p_y - \Pi_h p_y||_{0,\Omega_i} \leq C(h_x||p_{xy}||_{0,\Omega_i} + h_y||p_{y^2}||_{0,\Omega_i}), \tag{3.70}$$

$$M_{3,\Omega_i} \equiv ||p - P_h p||_{0,\Omega_i} \leq C(h_x||p_x||_{0,\Omega_i} + h_y||p_y||_{0,\Omega_i}) \tag{3.71}$$

$$\leq C(h_x||p_x||_{\infty,\Omega_i} \cdot (\text{meas}(\Omega_i))^{1/2} + h_y||p_y||_{0,\Omega_i}). \tag{3.72}$$

Especially, when i=1,3, by Lemmas 3.1 and 3.8, we obtain

$$M_{1,\Omega_i} \le C\varepsilon(\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-2}\cdot|\varepsilon\ln\varepsilon|^{1/2}+N^{-1}\cdot\varepsilon^{-3/4}\cdot|\ln^{1/4}\varepsilon|+N^{-1}) \tag{3.73}$$

$$\le CN^{-1}(\varepsilon^{1/2}|\ln^{3/2}\varepsilon|+\varepsilon^{1/4}|\ln^{1/4}\varepsilon|+1), \tag{3.74}$$

$$M_{2,\Omega_i} \le C(\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-3/4}|\ln^{1/4}\varepsilon|+N^{-1}) \tag{3.75}$$

$$\le CN^{-1}(\varepsilon^{1/4}|\ln^{5/4}\varepsilon|+1), \tag{3.76}$$

$$M_{3,\Omega_i} \le C(\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-1}\cdot|\varepsilon\ln\varepsilon|^{1/2}+N^{-1}) \tag{3.77}$$

$$= CN^{-1}(\varepsilon^{1/2}|\ln^{3/2}\varepsilon|+1). \tag{3.78}$$

Note that for any $\varepsilon \in (0,1]$, we have

$$\varepsilon^{1/4}|\ln^{5/4}\varepsilon| < 2.2, \quad \varepsilon^{1/2}|\ln^{3/2}\varepsilon| < 1.2,$$

which ensures that $M_{k,\Omega_i} \le CN^{-1}, k=1,2,3, i=1,3$, where C is independent of ε.

While for i=2, it is easy to see that

$$\varepsilon||p_x-\Pi_h p_x||_{0,\Omega_2} \le C\varepsilon N^{-1}, \quad ||p_y-\Pi_h p_y||_{0,\Omega_2} \le CN^{-1}, \quad ||p-P_h p||_{0,\Omega_2} \le CN^{-1}$$

which combining the above inequalities concludes the proof of the first part.

On the other hand, by Lemma 3.11 and the standard inverse estimate, we have

$$||P_h p - p_h||_{\infty,\Omega_2} \le CN^{-1}. \tag{3.79}$$

Then by Lemmas 2.1 and 3.1, we have

$$||P_h p - p||_{\infty,\Omega_2} \le C(h_x||p_x||_{\infty,\Omega_2}+h_y||p_y||_{\infty,\Omega_2}) \le CN^{-1},$$

which along with (3.79) and the triangular inequality completes the proof of the second part. □

Remark 2. If $||p_{xy}||_{\infty,\Omega_2} \le C$, then it is easy to obtain

$$\varepsilon||p_x-p_x^h||_{\infty,\Omega_2}+||p_y-p_y^h||_{\infty,\Omega_2} \le CN^{-1}.$$

This estimate is observed in our numerical experiments (see Tables 5.5-5.8 in §5). However, the rigorous proof of $||p_{xy}||_{\infty,\Omega_2} \le C$ is open.

3.3. Postprocessing and superconvergence at Gaussian points. In this section we apply Lemma 3.11 to obtain a higher order approximation of $\mathbf{u}$ by a simple local postprocessing procedure of [9] for $\mathbf{u}_h$. In the following we will use the operators I_h and K_h defined in [9, §4,5]. For completeness, we repeat the definitions of [9] here.

Let $P=(x_1,y_1)$ be an interior mesh node of the partition T_h and let $x_0<x_1<x_2, y_0<y_1<y_2$ be such that $(x_i,y_j), i=0,1,2, j=0,1,2, (i,j)\neq(1,1)$ are the nodes neighbor to P.

Denote $R_P=\cup_{P\in R, R\in T_h} R$. Given $\mathbf{r}=(r_1,r_2)$ such that point values of its normal components are well defined, let

$$K_{h,P}=(K^1_{h,P}(r_1), K^2_{h,P}(r_2)) \in Q_{2,1}(R_P)\times Q_{1,2}(R_P)$$

be such that, $K^1_{h,P}(r_1)$ interpolates r_1 at the midpoints of the sides

$$\{x_i\}\times[y_0,y_1] \quad \text{and} \quad \{x_i\}\times[y_1,y_2], \quad i=0,1,2$$

and $K_{h,P}^2(r_2)$ interpolates r_2 at the midpoints of the sides

$$[x_0,x_1]\times\{y_j\} \quad \text{and} \quad [x_1,x_2]\times\{y_j\}, \quad j=0,1,2.$$

We define

$$K_h(\mathbf{r}) = \sum_{P\in NT} K_{h,P}(\mathbf{r})(P)\psi_P,$$

here NT denotes the set of nodes of T_h, and $\{\psi_P\}_{P\in NT}$ denotes the standard basis of $M_h = \{\psi \in H^1(\Omega) : \psi \in Q_{1,1}(R), \forall R \in T_h\}$. When P is a boundary node or inter-subdomain node we choose an interior node $\tilde{P}$ neighbor to P and define $K_{h,P}(\mathbf{r})(P) = K_{h,\tilde{P}}(\mathbf{r})(P)$.

The operator $I_h : [C^0(R)]^2 \to Q_{1,0}(R)\times Q_{0,1}(R)$ is defined as the interpolation of the normal components at the midpoints of the sides, i.e., for every side l of an element $R\in T_h$, $I_h\mathbf{r}\cdot n_l(m_l) = \mathbf{r}\cdot n_l(m_l)$ where m_l is the midpoint of l.

By carrying out the same procedures as [9], we have the following refined results:

Lemma 3.13. *If* $\mathbf{r}\in[H^2(R_p)]^2$, *then*

$$||\mathbf{r} - K_h(I_h\mathbf{r})||_{0,R_p} \le C(h_x^2||\mathbf{r}_{x^2}||_{0,R_p} + h_xh_y||\mathbf{r}_{xy}||_{0,R_p} + h_y^2||\mathbf{r}_{y^2}||_{0,R_p}). \tag{3.80}$$

Proof. By the Bramble-Hilbert lemma, the proof is all the same as [9, Lemma 4.1] except that [9, (4.3)] is changed into

$$||\mathbf{r} - K_h(\mathbf{r})||_{0,R_p} \le C(h_x^2||\mathbf{r}_{x^2}||_{0,R_p} + h_xh_y||\mathbf{r}_{xy}||_{0,R_p} + h_y^2||\mathbf{r}_{y^2}||_{0,R_p}) \tag{3.81}$$

and [9, (4.5)] is changed into

$$||\mathbf{r} - \mathbf{r}^I||_{0,R} \le C(h_x^2||\mathbf{r}_{x^2}||_{0,R} + h_xh_y||\mathbf{r}_{xy}||_{0,R} + h_y^2||\mathbf{r}_{y^2}||_{0,R}), \tag{3.82}$$

where $\mathbf{r}^I = \sum_{P\in NT}\mathbf{r}(P)\psi_P$ is the bilinear interpolation of $\mathbf{r}$. $\square$

Lemma 3.14. *[9, Lemma 4.2] For any*

$$\mathbf{r}\in Q_h^0 = \{\mathbf{r}\in H(div,\Omega) : \mathbf{r}\in Q_{1,0}(R)\times Q_{0,1}(R), \forall R\in T_h\},$$

we have

$$||K_h(\mathbf{r})||_{0,\Omega} \le C||\mathbf{r}||_{0,\Omega}. \tag{3.83}$$

Lemma 3.15. *[9, Lemma 4.3] If* $\mathbf{r}\in[H^2(R)]^2$ *then*

$$||I_h\mathbf{r} - \Pi_h\mathbf{r}||_{0,R} \le C(h_x^2||\mathbf{r}_{x^2}||_{0,R} + h_xh_y||\mathbf{r}_{xy}||_{0,R} + h_y^2||\mathbf{r}_{y^2}||_{0,R}). \tag{3.84}$$

Theorem 3.16. *Under the assumptions of Lemma 3.11, we have*

$$\varepsilon||p_x - K_h(p_x^h)||_{0,\Omega} + ||p_y - K_h(p_y^h)||_{0,\Omega} + ||p - K_h(p_h)||_{0,\Omega} \le CN^{-2}. \tag{3.85}$$

Proof. By triangular inequality, for each subdomain Ω_i, $i=1,2,3$, we have

$$\begin{aligned}&\varepsilon||p_x - K_h(p_x^h)||_{0,\Omega_i}\\ &\le \varepsilon(||p_x - K_h(I_hp_x)||_{0,\Omega_i} + ||K_h(I_hp_x - \Pi_hp_x)||_{0,\Omega_i} + ||K_h(\Pi_hp_x - p_x^h)||_{0,\Omega_i}).\end{aligned}$$

Using Lemmas 3.9-3.10, 3.13-3.15 and the same techniques used in Theorem 3.12, it is easy to see that all the terms on the right hand side will be bounded by N^{-2}.

By the same arguments, $||p_y - K_h(p_y^h)||_{0,\Omega}$ and $||p - K_h(p_h)||_{0,\Omega}$ can be bounded by N^{-2}, which completes our proof. ☐

The superconvergence at Gaussian points has been obtained by Nakata *et al.* [24], Weiser and Wheeler [35], Ewing *et al.* [10] and Duran [9] for non-singular perturbation problems. For any rectangle $R \equiv [x_{k-1}, x_k] \times [y_{l-1}, y_l]$, and $q \in \mathbf{W}$ and $\mathbf{v} = (v^1, v^2) \in \mathbf{V}$. Let

$$|||q|||_0 \equiv [\sum_{R\in T_h} \sum_{j=1}^{m+1} \sum_{i=1}^{m+1} \frac{1}{4} A_i A_j (x_k - x_{k-1})(y_l - y_{l-1}) q^2(g_i, g_j)]^{1/2}, \tag{3.86}$$

$$|||v^1|||_1 \equiv [\sum_{R\in T_h} \sum_{j=1}^{m+1} \frac{1}{2} A_j (y_l - y_{l-1}) \int_{x_{k-1}}^{x_k} (v^1(x, g_j))^2 dx]^{1/2}, \tag{3.87}$$

$$|||v^1|||_2 \equiv [\sum_{R\in T_h} \sum_{j=1}^{m+1} \frac{1}{2} A_j (x_k - x_{k-1}) \int_{y_{l-1}}^{y_l} (v^2(g_j, y))^2 dy]^{1/2}, \tag{3.88}$$

where $A_j > 0$ and $g_j, j = 1, \cdots, m+1$, are the coefficients of the Gauss quadrature rule and m+1 Gauss points on $[-1, 1]$, respectively. For the lowest order of Raviart-Thomas space RT_0, m is zero in (3.86)-(3.88).

Note that for $\mathbf{v} \in \mathbf{V}_h$, the expressions (3.87) and (3.88) are equal to the L^2-norms of v^1 and v^2 respectively [10]. Also for $q \in \mathbf{W}_h$, (3.86) is equal to the L^2-norm [10, Remark 2.3].

THEOREM 3.17. *Under the assumptions of Lemma 3.11, we have*

$$\varepsilon |||p_x - p_x^h|||_1 + |||p_y - p_y^h|||_2 + |||p - p_h|||_0 \le C N^{-2}. \tag{3.89}$$

Proof. The proof follows the idea of [10, Theorem 3.1]. Note that

$$\varepsilon |||p_x - p_x^h|||_1 + |||p_y - p_y^h|||_2 + |||p - p_h|||_0 \tag{3.90}$$

$$\le \varepsilon |||p_x - \Pi_h p_x|||_1 + \varepsilon |||\Pi_h p_x - p_x^h|||_1 + |||p_y - \Pi_h p_y|||_2 \tag{3.91}$$

$$+ |||\Pi_h p_y - p_y^h|||_2 + |||p - p^I|||_0 + |||p^I - P_h p|||_0 + |||P_h p - p_h|||_0 \tag{3.92}$$

$$\le \varepsilon |||p_x - \Pi_h p_x|||_1 + \varepsilon ||\Pi_h p_x - p_x^h||_{0,\Omega} + |||p_y - \Pi_h p_y|||_2 \tag{3.93}$$

$$+ ||\Pi_h p_y - p_y^h||_{0,\Omega} + |||p - p^I|||_0 + ||p^I - P_h p||_{0,\Omega} + ||P_h p - p_h||_{0,\Omega} \tag{3.94}$$

$$\le C N^{-2} + \varepsilon |||p_x - \Pi_h p_x|||_1 + |||p_y - \Pi_h p_y|||_2 + ||p^I - P_h p||_{0,\Omega}, \tag{3.95}$$

where we used the fact that $p = p^I$ at the Gauss points (g_i, g_j) and Lemma 3.11. Here the interpolant p^I is a polynomial in $Q_{0,0}(R)$, which assumes the value of p at the lowest order Gaussian point.

By the same arguments as [9, p.296], we have

$$\varepsilon |||p_x - \Pi_h p_x|||_1 \tag{3.96}$$

$$\le C\varepsilon \sum_{i=0}^{2} (h_x^2 ||p_{x^3}||_{0,\Omega_i} + h_x h_y ||p_{x^2 y}||_{0,\Omega_i} + h_y^2 ||p_{xy^2}||_{0,\Omega_i}), \tag{3.97}$$

$$|||p_y - \Pi_h p_y|||_2 \tag{3.98}$$

$$\le C \sum_{i=0}^{2} (h_x^2 ||p_{yx^2}||_{0,\Omega_i} + h_x h_y ||p_{y^2 x}||_{0,\Omega_i} + h_y^2 ||p_{y^3}||_{0,\Omega_i}). \tag{3.99}$$

A refined result of [10, Lemma 4.3] gives

$$\|p^I - P_h p\|_{0,\Omega} \tag{3.100}$$

$$\leq C \sum_{i=0}^{2} (h_x^2 \|p_{x^2}\|_{0,\Omega_i} + h_x h_y \|p_{xy}\|_{0,\Omega_i} + h_y^2 \|p_{y^2}\|_{0,\Omega_i}), \tag{3.101}$$

which corresponds to the constant coefficient case [10, (4.6)].

By repeating the same techniques used in Theorem 3.12 and using Lemmas 3.1, 3.9 and 3.10, we have

$$\varepsilon |||p_x - \Pi_h p_x|||_1 + |||p_y - \Pi_h p_y|||_2 + \|p^I - P_h p\|_{0,\Omega} \leq C N^{-2}, \tag{3.102}$$

which along with (3.95) completes the proof. □

4. The isotropic case. For simplicity, let $a = (\varepsilon^2, \varepsilon^2)$ and $g = 0$, i.e., we consider

$$L_\varepsilon p \equiv -\varepsilon^2 (p_{x^2} + p_{y^2}) + b(x,y) p = f, \quad \text{in} \quad \Omega \equiv (0,1)^2, \tag{4.1}$$

$$p = 0 \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

Hence we have boundary layers at four sides of Ω.

4.1. Another anisotropic mesh and the derivative estimates. Since the solution p of (4.1)-(4.2) has boundary layers on four boundaries of Ω, we divide Ω into nine matching subdomains Ω_i, $1 \leq i \leq 9$, i.e., $\Omega = \cup_{i=1}^{9} \Omega_i$, where

$$\begin{aligned}
&\Omega_1 \equiv (0,\sigma_x) \times (0,\sigma_y), \quad \Omega_2 \equiv (\sigma_x, 1-\sigma_x) \times (0,\sigma_y), \quad \Omega_3 \equiv (1-\sigma_x, 1) \times (0,\sigma_y), \\
&\Omega_4 \equiv (0,\sigma_x) \times (\sigma_y, 1-\sigma_y), \quad \Omega_5 \equiv (\sigma_x, 1-\sigma_x) \times (\sigma_y, 1-\sigma_y), \\
&\Omega_6 \equiv (1-\sigma_x, 1) \times (\sigma_y, 1-\sigma_y), \quad \Omega_7 \equiv (0,\sigma_x) \times (1-\sigma_y, 1), \\
&\Omega_8 \equiv (\sigma_x, 1-\sigma_x) \times (1-\sigma_y, 1), \quad \Omega_9 \equiv (1-\sigma_x, 1) \times (1-\sigma_y, 1).
\end{aligned}$$

Here $\sigma_x = \sigma_y = 2\beta^{-1}\varepsilon |\ln \varepsilon|$. Then each subdomain Ω_i is divided quasiuniformly into N_{ix} and N_{iy} subintervals in the x- and y-directions, respectively. Also we assume $N_{i_x} \simeq N_{i_y} \simeq N$.

Here we will provide some estimates for the problem (4.1)-(4.2).

LEMMA 4.1. *For the solution p of (4.1)-(4.2) and* $0 \leq k \leq 4$, *we have*

$$\begin{aligned}
&(i) \quad |p_{x^k}(x,y)| \leq C(1 + \varepsilon^{-k} \exp(-\beta x/\varepsilon) + \varepsilon^{-k} \exp(-\beta(1-x)/\varepsilon)), \quad \text{on} \ \overline{\Omega}, \\
&(ii) \quad |p_{y^k}(x,y)| \leq C(1 + \varepsilon^{-k} \exp(-\beta y/\varepsilon) + \varepsilon^{-k} \exp(-\beta(1-y)/\varepsilon)), \quad \text{on} \ \overline{\Omega}.
\end{aligned}$$

Proof. The results was proved for k=0,1,2 [16, Lemmas 2.1-2.5]. The proof for k=3,4 follows the same way as for the anisotropic case.

Denote $\tilde{p}(x,y) = p_{x^2}(x,y) - \tilde{g}(x,y,\varepsilon)$, differentiating (4.1) twice with respect to x gives us:

$$L_\varepsilon \tilde{p} \equiv -\varepsilon^2 (\tilde{p}_{x^2} + \tilde{p}_{y^2}) + b\tilde{p} = \tilde{F} \quad \text{in} \quad \Omega, \tag{4.3}$$

$$\tilde{p} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{4.4}$$

where $\tilde{F} = f_{x^2} - 2b_x u_x - b_{x^2} u + \varepsilon^2 (\tilde{g}_{x^2} + \tilde{g}_{y^2}) - b\tilde{g}$ and

$$\tilde{g}(x,y,\varepsilon) = -\varepsilon^{-2} \frac{1 - \exp(-\beta(1-x)/\varepsilon)}{1 - \exp(-\beta/\varepsilon)} f(0,y) - \varepsilon^{-2} \frac{1 - \exp(-\beta x/\varepsilon)}{1 - \exp(-\beta/\varepsilon)} f(1,y).$$

Note that this L_ε also satifies the maximum principle [17]. The rest proofs are similar to the anisotropic case. □

LEMMA 4.2. *For the solution p of (4.1)-(4.2), we have*

$$\begin{aligned} (i) &\quad ||p_{xy}||_{0,\Omega_i} \le C\varepsilon^{-3/4}|\ln^{1/4}\varepsilon|, \quad i=2,4,6,8,\\ (ii) &\quad ||p_{xy}||_{0,\Omega_2} \le C, \quad i=5,\\ (iii) &\quad ||p_{xy}||_{0,\Omega_i} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|, \quad i=1,3,7,9. \end{aligned}$$

Proof. (i) For i=2, integration by steps and using the boundary condition $p=0$, we have

$$\begin{aligned} &\int_{\Omega_2} p_{xy}\cdot p_{xy}dxdy = \int_0^{\sigma_y}(p_y\cdot p_{xy})|_{x=\sigma_x}^{x=1-\sigma_x}dy - \int_{\Omega_2} p_y\cdot p_{x^2y}dxdy\\ &= (p_yp_x)|_{(1-\sigma_x,0)}^{(1-\sigma_x,\sigma_y)} - (p_yp_x)|_{(\sigma_x,0)}^{(\sigma_x,\sigma_y)} - \int_0^{\sigma_y}(p_{y^2}\cdot p_x)|_{x=\sigma_x}^{x=1-\sigma_x}dy\\ &\quad - \int_{\sigma_x}^{1-\sigma_x}(p_y\cdot p_{x^2})|_{y=0}^{y=\sigma_y}dx + \int_{\Omega_2} p_{y^2}\cdot p_{x^2}dxdy\\ &= (p_yp_x)|_{(\sigma_x,\sigma_y)}^{(1-\sigma_x,\sigma_y)} - \int_0^{\sigma_y}(p_{y^2}\cdot p_x)|_{x=\sigma_x}^{x=1-\sigma_x}dy\\ &\quad - \int_{\sigma_x}^{1-\sigma_x}(p_y\cdot p_{x^2})(x,\sigma_y)dx + \int_{\Omega_2} p_{y^2}\cdot p_{x^2}dxdy\\ &\le C\varepsilon^{-2}\cdot(\varepsilon|\ln\varepsilon|)^{1/2} = C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|. \end{aligned}$$

Hence we have $||p_{xy}||_{0,\Omega_2} \le C\varepsilon^{-3/4}|\ln^{1/4}\varepsilon|$. Same estimates hold true for $i = 4,6,8$.

(ii) For i=5, by simple integration and using the boundary condition, we have

$$\int_{\Omega_5} p_{xy}\cdot p_{xy}dxdy = (p_yp_x)|_{\text{vertices of }\Omega_5} - \int_{\sigma_y}^{1-\sigma_y}(p_{y^2}p_x)|_{x=\sigma_x}^{x=1-\sigma_x}dy \tag{4.5}$$

$$- \int_{\sigma_x}^{1-\sigma_x}(p_yp_{x^2})|_{y=\sigma_y}^{y=1-\sigma_y}dx - \int_{\Omega_5} p_{y^2}p_{x^2}dxdy \le C, \tag{4.6}$$

which shows that $||p_{xy}||_{0,\Omega_5} \le C$.

(iii) By the same techniques as above, it is not difficult to obtain

$$\begin{aligned} \int_{\Omega_1} p_{xy}\cdot p_{xy}dxdy &= (p_yp_x)|(\sigma_x,\sigma_y) - \int_0^{\sigma_y}(p_{y^2}p_x)(\sigma_x,y)dy\\ &\quad - \int_0^{\sigma_x}(p_yp_{x^2})(x,\sigma_y)dx + \int_{\Omega_1} p_{y^2}p_{x^2}dxdy\\ &\le C\varepsilon^{-3}|\ln\varepsilon|, \end{aligned}$$

from which we obtain

$$||p_{xy}||_{0,\Omega_1} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|.$$

Same estimates hold true for i=3,7,9. □

LEMMA 4.3. *For the solution p of (4.1)-(4.2), we have*

$$(i)\quad ||p_{x^2y}||_{0,\Omega_i} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|, \quad i=2,4,6,8,$$
$$(ii)\quad ||p_{x^2y}||_{0,\Omega_i} \le C\varepsilon^{-1}, \quad i=5,$$
$$(iii)\quad ||p_{x^2y}||_{0,\Omega_i} \le C\varepsilon^{-2}|\ln\varepsilon|, \quad i=1,3,7,9.$$

Proof. (i) By differentiating (4.1) w.r.t. y, we have

$$\begin{aligned}|p_{x^2y}| &\le \varepsilon^{-2}(|f_y|+|(bp)_y|)+|p_{y^3}|\\ &\le C\varepsilon^{-2}(1+\varepsilon^{-1}\exp(-\beta y/\varepsilon)+\varepsilon^{-1}\exp(-\beta(1-y)/\varepsilon)),\end{aligned}$$

from which we have

$$||p_{x^2y}||_{\infty,\Omega_i} \le C\varepsilon^{-2}, \quad i=4,5,6,$$
$$||p_{x^2y}||_{\infty,\Omega_i} \le C\varepsilon^{-3}, \quad i=1,2,3,7,8,9.$$

By differentiating (4.1) twice w.r.t. y, we have

$$\begin{aligned}|p_{x^2y^2}| &\le \varepsilon^{-2}(|f_{y^2}|+|(bp)_{y^2}|)+|p_{y^4}|\\ &\le C\varepsilon^{-2}(1+\varepsilon^{-2}\exp(-\beta y/\varepsilon)+\varepsilon^{-2}\exp(-\beta(1-y)/\varepsilon)).\end{aligned}$$

Similarly by differentiating (4.1) twice w.r.t. x, we have

$$|p_{x^2y^2}| \le C\varepsilon^{-2}(1+\varepsilon^{-2}\exp(-\beta x/\varepsilon)+\varepsilon^{-2}\exp(-\beta(1-x)/\varepsilon)).$$

Hence we obatin

$$||p_{x^2y^2}||_{\infty,\Omega_i} \le C\varepsilon^{-2}, \quad i=2,5,8,4,6,$$
$$||p_{x^2y^2}||_{\infty,\Omega_i} \le C\varepsilon^{-4}, \quad i=1,3,7,9.$$

By the same arguments and definitions of σ_x and σ_y, it is not difficult to obtain

$$||p_{x^2}||_{\infty,\Omega_i}+\varepsilon||p_{x^3}||_{\infty,\Omega_i} \le C, \quad i=2,5,8,$$
$$||p_{x^2}||_{\infty,\Omega_i}+\varepsilon||p_{x^3}||_{\infty,\Omega_i} \le C\varepsilon^{-2}, \quad i=1,4,7,3,6,9$$
$$||p_{y^2}||_{\infty,\Omega_i}+\varepsilon||p_{y^3}||_{\infty,\Omega_i} \le C, \quad i=4,5,6$$
$$||p_{y^2}||_{\infty,\Omega_i}+\varepsilon||p_{y^3}||_{\infty,\Omega_i} \le C\varepsilon^{-2}, \quad i=1,2,3,7,8,9.$$

Integrating by steps and using the fact that $p_{x^2}|_{y=0}=0$, we have

$$\begin{aligned}|\int_{\Omega_2} p_{x^2y}\cdot p_{x^2y}\,dxdy| &= |\int_{\sigma_x}^{1-\sigma_x}(p_{x^2y}\cdot p_{x^2})(x,\sigma_y)dx - \int_{\Omega_2} p_{x^2y^2}\cdot p_{x^2}\,dxdy|\\ &\le C\varepsilon^{-3}\cdot C + C\varepsilon^{-2}\cdot C\cdot|\varepsilon\ln\varepsilon| \le C\varepsilon^{-3}|\ln\varepsilon|.\end{aligned}$$

Hence we obtain

$$||p_{x^2y}||_{0,\Omega_2} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|.$$

In the same way, we can obtain

$$||p_{x^2y}||_{0,\Omega_8} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|.$$

On the other hand, we have

$$||p_{x^2y}||_{0,\Omega_4} \le ||p_{x^2y}||_{\infty,\Omega_4} \cdot (\text{meas}(\Omega_4))^{1/2} \tag{4.7}$$

$$= C\varepsilon^{-2} \cdot |\varepsilon \ln \varepsilon|^{1/2} = C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|. \tag{4.8}$$

Similarly, we have

$$||p_{x^2y}||_{0,\Omega_6} \le ||p_{x^2y}||_{\infty,\Omega_6} \cdot (\text{meas}(\Omega_6))^{1/2} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|.$$

(ii) Note that

$$\begin{aligned} &\int_{\Omega_5} p_{x^2y} \cdot p_{x^2y}\,dxdy \\ &= \int_{\sigma_x}^{1-\sigma_x} (p_{x^2y} \cdot p_{x^2})|_{y=\sigma_y}^{y=1-\sigma_y} dx - \int_{\Omega_5} p_{x^2y^2} \cdot p_{x^2}\,dxdy \\ &\le C\varepsilon^{-2}, \end{aligned}$$

from which we have

$$||p_{x^2y}||_{0,\Omega_5} \le C\varepsilon^{-1}.$$

(iii) While for i=1,3,7,9, $(\text{meas}(\Omega_i))^{1/2} = O(\varepsilon|\ln\varepsilon|)$, we have

$$||p_{x^2y}||_{0,\Omega_i} \le ||p_{x^2y}||_{\infty,\Omega_i} \cdot (\text{meas}(\Omega_i))^{1/2} \tag{4.9}$$

$$= C\varepsilon^{-3} \cdot |\varepsilon \ln\varepsilon| = C\varepsilon^{-2}|\ln\varepsilon|. \tag{4.10}$$

□

By symmetry, we can obtain:

LEMMA 4.4. *For the solution p of (4.1)-(4.2), we have*

$$\begin{aligned} &(i) \quad ||p_{xy^2}||_{0,\Omega_i} \le C\varepsilon^{-3/2}|\ln^{1/2}\varepsilon|, \quad i = 2,4,6,8, \\ &(ii) \quad ||p_{xy^2}||_{0,\Omega_i} \le C\varepsilon^{-1}, \quad i = 5, \\ &(iii) \quad ||p_{xy^2}||_{0,\Omega_i} \le C\varepsilon^{-2}|\ln\varepsilon|, \quad i = 1,3,7,9. \end{aligned}$$

4.2. The error estimates in the isotropic case. Note that in this case, $\alpha = (\varepsilon^{-2}, \varepsilon^{-2})$, hence $\mathbf{u} = -a\nabla p = -\varepsilon^2(p_x, p_y)$. The equation (2.18) becomes

$$\varepsilon^{-2}((\mu^1,\mu^1) + (\mu^2,\mu^2)) + (b\tau,\tau) \tag{4.11}$$

$$= \varepsilon^{-2}((\Pi_h u^1 - u^1, \mu^1) + (\Pi_h u^2 - u^2, \mu^2)) + (b(P_h p - p), \tau) \tag{4.12}$$

LEMMA 4.5. *Let* $\mathbf{u} = -(\varepsilon^2 p_x, \varepsilon^2 p_y)$ *and* $\mathbf{u}_h = -(\varepsilon^2 p_x^h, p_y^h)$ *be the solutions of (4.1)-(4.2) and (2.3)-(2.4) respectively, then*

$$\varepsilon(||\Pi_h p_x - p_x^h||_{0,\Omega} + ||\Pi_h p_y - p_y^h||_{0,\Omega}) + ||P_h p - p_h||_{0,\Omega} \le CN^{-2}. \tag{4.13}$$

Proof. Since the proof is almost the same as the anisotropic case, we just sketch some key steps. For any i, by Lemma 2.2, we have

$$M_{1,\Omega_i} \equiv \varepsilon^{-2}\int_{\Omega_i} (\Pi_h u^1 - u^1)\mu^1 \le Ch_x^2 ||p_{x^3}||_{\infty,\Omega_i} (\text{meas}(\Omega_i))^{1/2} ||\mu^1||_{0,\Omega_i}, \tag{4.14}$$

$$M_{2,\Omega_i} \equiv \varepsilon^{-2}\int_{\Omega_i} (\Pi_h u^2 - u^2)\mu^2 \le Ch_y^2 ||p_{y^3}||_{\infty,\Omega_i} (\text{meas}(\Omega_i))^{1/2} ||\mu^2||_{0,\Omega_i}. \tag{4.15}$$

By Lemma 2.1 and the property of the projection P_h, we have

$$M_{3,\Omega_i} \equiv \int_{\Omega_i} b(P_hp-p)\tau = \int_{\Omega_i}(b-P_hb)(P_hp-p)\tau$$
$$\leq CN^{-1}[h_x||p_x||_{\infty,\Omega_i} + h_y||p_y||_{\infty,\Omega_i}](\mathrm{meas}(\Omega_i))^{1/2}||\tau||_{0,\Omega_i}.$$

By the construction of T_h, we know that $h_x = O(\varepsilon|\ln\varepsilon|/N)$ on $\Omega_i, i = 1,3,4,6,7,9, h_x = O(N^{-1})$ on $\Omega_i, i = 2,5,8, h_y = O(N^{-1})$ on $\Omega_i, i = 4,5,6,$ $h_y = O(\varepsilon|\ln\varepsilon|/N)$ on $\Omega_i, i = 1,2,3,7,8,9$.

For i=1,3,7,9, by Lemma 4.1, we have

$$M_{1,\Omega_i} \leq C(\varepsilon|\ln\varepsilon|/N)^2\cdot\varepsilon^{-3}\cdot(\varepsilon|\ln\varepsilon|)\cdot||\mu^1||_{0,\Omega_i} \tag{4.16}$$
$$= CN^{-2}\cdot\varepsilon|\ln^3\varepsilon|\cdot\varepsilon^{-1}||\mu^1||_{0,\Omega_i}, \tag{4.17}$$
$$M_{2,\Omega_i} \leq CN^{-2}\cdot\varepsilon|\ln^3\varepsilon|\cdot\varepsilon^{-1}||\mu^2||_{0,\Omega_i}, \tag{4.18}$$
$$M_{3,\Omega_i} \leq CN^{-1}\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-1}\cdot(\varepsilon|\ln\varepsilon|)||\tau||_{0,\Omega_i} \tag{4.19}$$
$$\leq CN^{-2}\cdot\varepsilon|\ln^2\varepsilon|\cdot||\tau||_{0,\Omega_i}. \tag{4.20}$$

For i=2,8, we have

$$M_{1,\Omega_i} \leq CN^{-2}\cdot\varepsilon^{-1}\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\mu^1||_{0,\Omega_i} \tag{4.21}$$
$$= CN^{-2}\cdot\varepsilon^{1/2}|\ln^{1/2}\varepsilon|\cdot\varepsilon^{-1}||\mu^1||_{0,\Omega_i}, \tag{4.22}$$
$$M_{2,\Omega_i} \leq C(\frac{\varepsilon|\ln\varepsilon|}{N})^2\cdot\varepsilon^{-3}\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\mu^2||_{0,\Omega_i} \tag{4.23}$$
$$= CN^{-2}\cdot\varepsilon^{1/2}|\ln^{5/2}\varepsilon|\cdot\varepsilon^{-1}||\mu^2||_{0,\Omega_i}, \tag{4.24}$$
$$M_{3,\Omega_i} \leq CN^{-1}(N^{-1}+\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-1})\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\tau||_{0,\Omega_i} \tag{4.25}$$
$$= CN^{-2}(1+|\ln\varepsilon|)\cdot\varepsilon^{1/2}|\ln^{1/2}\varepsilon|\cdot||\tau||_{0,\Omega_i}. \tag{4.26}$$

For i=4,6, we have

$$M_{1,\Omega_i} \leq C(\frac{\varepsilon|\ln\varepsilon|}{N})^2\cdot\varepsilon^{-3}\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\mu^1||_{0,\Omega_i} \tag{4.27}$$
$$= CN^{-2}\cdot\varepsilon^{1/2}|\ln^{5/2}\varepsilon|\cdot\varepsilon^{-1}||\mu^1||_{0,\Omega_i}, \tag{4.28}$$
$$M_{2,\Omega_i} \leq CN^{-2}\cdot\varepsilon^{-1}\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\mu^2||_{0,\Omega_i} \tag{4.29}$$
$$= CN^{-2}\cdot\varepsilon^{1/2}|\ln^{1/2}\varepsilon|\cdot\varepsilon^{-1}||\mu^2||_{0,\Omega_i}, \tag{4.30}$$
$$M_{3,\Omega_i} \leq CN^{-1}(N^{-1}+\frac{\varepsilon|\ln\varepsilon|}{N}\cdot\varepsilon^{-1})\cdot(\varepsilon|\ln\varepsilon|)^{1/2}\cdot||\tau||_{0,\Omega_i} \tag{4.31}$$
$$= CN^{-2}(1+|\ln\varepsilon|)\cdot\varepsilon^{1/2}|\ln^{1/2}\varepsilon|\cdot||\tau||_{0,\Omega_i}. \tag{4.32}$$

For i=5, we have

$$M_{1,\Omega_i} \leq CN^{-2}\cdot\varepsilon^{-1}||\mu^1||_{0,\Omega_i}, \tag{4.33}$$
$$M_{2,\Omega_i} \leq CN^{-2}\cdot\varepsilon^{-1}||\mu^2||_{0,\Omega_i}, \tag{4.34}$$
$$M_{3,\Omega_i} \leq CN^{-2}||\tau||_{0,\Omega_i}. \tag{4.35}$$

The proof completes by combining the above estimates along with the fact that $\varepsilon^{1/2}|\ln^{5/2}\varepsilon| < 4.6$, for $0 < \varepsilon \leq 1$. □

THEOREM 4.6. *Under the assumptions of Lemma 4.5, we have*

$$\varepsilon(||p_x - p_x^h||_{0,\Omega} + ||p_y - p_y^h||_{0,\Omega}) + ||p - p_h||_{0,\Omega} \le CN^{-1}. \tag{4.36}$$

Furthermore, we have the following interior error estimates:

$$||p - p_h||_{\infty,\Omega_5} \le CN^{-1}. \tag{4.37}$$

Proof. By Lemma 2.2, we can obtain

$$Q_{1,\Omega_i} \equiv \varepsilon||p_x - \Pi_h p_x||_{0,\Omega_i} \le C\varepsilon(h_x||p_{x^2}||_{\infty,\Omega_i} \cdot (\text{meas}(\Omega_i))^{1/2} + h_y||p_{xy}||_{0,\Omega_i}),$$
$$Q_{2,\Omega_i} \equiv \varepsilon||p_y - \Pi_h p_y||_{0,\Omega_i} \le C\varepsilon(h_x||p_{yx}||_{0,\Omega_i} + h_y||p_{y^2}||_{\infty,\Omega_i}(\text{meas}(\Omega_i))^{1/2}),$$
$$Q_{3,\Omega_i} \equiv ||p - \Pi_h p||_{0,\Omega_i} \le C(h_x||p_x||_{\infty,\Omega_i} + h_y||p_y||_{\infty,\Omega_i}) \cdot (\text{meas}(\Omega_i))^{1/2}.$$

For i=1,3,7,9, by Lemmas 4.1-4.2, we have

$$\begin{aligned} Q_{1,\Omega_i} &\le C\varepsilon[\frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-2} \cdot |\varepsilon\ln\varepsilon| + \frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-3/2}|\ln^{1/2}\varepsilon|] \\ &= CN^{-1}[\varepsilon|\ln^2\varepsilon| + \varepsilon^{1/2}|\ln^{3/2}\varepsilon|]. \end{aligned}$$

Similarly, $Q_{2,\Omega_i} \le CN^{-1}[\varepsilon|\ln^2\varepsilon| + \varepsilon^{1/2}|\ln^{3/2}\varepsilon|]$, and

$$Q_{3,\Omega_i} \le C\frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-1} \cdot \varepsilon|\ln\varepsilon| = CN^{-1} \cdot \varepsilon|\ln^2\varepsilon|.$$

For i=2,8, we have

$$Q_{1,\Omega_i} \le C\varepsilon[N^{-1} \cdot (\varepsilon|\ln\varepsilon|)^{1/2} + \frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-3/4}|\ln^{1/4}\varepsilon|] \tag{4.38}$$
$$= CN^{-1}(\varepsilon^{3/2} \cdot |\ln^{1/2}\varepsilon| + \varepsilon^{5/4}|\ln^{5/4}\varepsilon|), \tag{4.39}$$
$$Q_{2,\Omega_i} \le C\varepsilon[N^{-1} \cdot (\varepsilon^{-3/4}|\ln^{1/4}\varepsilon|) + \frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-2} \cdot (\varepsilon|\ln\varepsilon|)^{1/2}] \tag{4.40}$$
$$= CN^{-1}(\varepsilon^{1/4}|\ln^{1/4}\varepsilon| + \varepsilon^{1/2}|\ln^{3/2}\varepsilon|), \tag{4.41}$$
$$Q_{3,\Omega_i} \le C[N^{-1} + \frac{\varepsilon|\ln\varepsilon|}{N} \cdot \varepsilon^{-1}] \cdot (\varepsilon\ln\varepsilon|)^{1/2} \tag{4.42}$$
$$= CN^{-1} \cdot \varepsilon^{1/2}|\ln^{1/2}\varepsilon| \cdot (1 + |\ln\varepsilon|). \tag{4.43}$$

Note that for any $\varepsilon \in (0, 1]$, we have

$$\varepsilon^{1/2}|\ln^{3/2}\varepsilon| < 1.2, \quad \varepsilon|\ln^2\varepsilon| < 0.6, \quad \varepsilon^{1/4}|\ln^{1/4}\varepsilon| < 0.8,$$

which ensures that for any $\varepsilon \in (0, 1]$, we have

$$Q_{k,\Omega_i} \le CN^{-1}, k = 1, 2, 3, i = 1, \cdots, 9.$$

By symmetry, similar estimates hold true for $i = 4, 6$.
For i=5, it is easy to obtain that

$$Q_{1,\Omega_i} \le C\varepsilon N^{-1}, \quad Q_{2,\Omega_i} \le C\varepsilon N^{-1}, \quad Q_{3,\Omega_i} \le CN^{-1}.$$

This along the above inequalities completes the proof. ∎

4.3. Postprocessing and superconvergence at Gaussian points. By using Lemmas 4.1, 4.3, 4.4, and going through the similar proofs as for the anisotropic case, we can obtain:

THEOREM 4.7. *Under the assumptions of Lemma 4.5, we have*

$$\varepsilon(||p_x - K_h(p_x^h)||_{0,\Omega} + ||p_y - K_h(p_y^h)||_{0,\Omega}) + ||p - K_h(p_h)||_{0,\Omega} \le CN^{-2}. \tag{4.44}$$

THEOREM 4.8. *Under the assumptions of Lemma 4.5, we have*

$$\varepsilon(|||p_x - p_x^h|||_1 + |||p_y - p_y^h|||_2) + |||p - p_h|||_0 \le CN^{-2}. \tag{4.45}$$

5. Numerical results. In this section we present some numerical tests on the cell-centered finite difference method [29, 24, 35, 3, 2], since the cell-centered finite difference scheme is equivalent to the rectangular RT_0 MFEM with special numerical quadrature formulae [29]. Similar theoretical analysis can be obtained by proper modifications of the proofs in [24, 35, 3, 2].

To check our theoretical analysis, we tested an example problem (3.1)-(3.2) where $b = 2$ and f is chosen properly such that we have a known analytic solution

$$p(x, y) = (1 - \frac{\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon)}{1 + \exp(-1/\varepsilon)} + x(1-x))y(1-y).$$

This solution contains typical boundary layers at sides x=0 and x=1 plus a regular part. A piecewise uniform anisotropic mesh is constructed as § 3.1. In our test, we take $N/3$ subintervals on each subdomain, where $\beta = 1, N = 12, 24, 48, 96$. We tested two different meshes, i.e., the global uniform mesh (denoted as U-mesh) and the anisotropic mesh (denoted as A-mesh).

First we run the problem for $\varepsilon = 1$ (i.e., non-singular perturbation problem) on the U-mesh. The standard convergence rates [26, 8, 6] are observed in Tables 5.1-5.8, i.e., $O(N^{-1})$ for p and $a^{1/2} \bigtriangledown p$ in the L^2-norm and L^∞-norm, and superconvergence rates $O(N^{-2})$ for p in the norm $|||\cdot|||_0$ and $\bigtriangledown p$ in the norm $|||\cdot||| = (|||\cdot|||_1^2 + |||\cdot|||_2^2)^{1/2}$.

		N			
	ε	12	24	48	96
U-mesh	1.0	0.905301E-02	0.451916E-02	0.225864E-02	0.112920E-02
	10^{-2}	0.608737E-01	0.366082E-01	0.200071E-01	0.102777E-01
	10^{-3}	0.616223E-01	0.405709E-01	0.275444E-01	0.189503E-01
	10^{-4}	0.616232E-01	0.405734E-01	0.275524E-01	0.190640E-01
	10^{-5}	0.616232E-01	0.405735E-01	0.275525E-01	0.190642E-01
A-mesh	10^{-2}	0.318419E-01	0.159143E-01	0.795931E-02	0.398109E-02
	10^{-3}	0.313440E-01	0.156468E-01	0.782003E-02	0.390955E-02
	10^{-4}	0.311878E-01	0.155621E-01	0.777714E-02	0.388803E-02
	10^{-5}	0.311548E-01	0.155434E-01	0.776749E-02	0.388331E-02

TABLE 5.1
Global errors for p in the L^2-norm

		N			
	ε	12	24	48	96
U-mesh	1.0	0.287658E-01	0.143102E-01	0.714587E-02	0.357177E-02
	10^{-2}	0.171838	0.107187	0.593422E-01	0.305915E-01
	10^{-3}	0.174426	0.120538	0.842534E-01	0.588929E-01
	10^{-4}	0.174429	0.120546	0.842791E-01	0.592583E-01
	10^{-5}	0.174429	0.120546	0.842793E-01	0.592589E-01
A-mesh	10^{-2}	0.483468E-01	0.243909E-01	0.122418E-01	0.613499E-02
	10^{-3}	0.429943E-01	0.216831E-01	0.108622E-01	0.543325E-02
	10^{-4}	0.416191E-01	0.209666E-01	0.105025E-01	0.525326E-02
	10^{-5}	0.413478E-01	0.208171E-01	0.104260E-01	0.521584E-02

TABLE 5.2
Global errors for $a^{1/2}\nabla p$ in the L^2-norm

		N			
	ε	12	24	48	96
U-mesh	1.0	0.410306E-03	0.103474E-03	0.259267E-04	0.648534E-05
	10^{-2}	0.139205E-02	0.380036E-03	0.304127E-03	0.284128E-03
	10^{-3}	0.141093E-02	0.353743E-03	0.878537E-04	0.291362E-04
	10^{-4}	0.141132E-02	0.354492E-03	0.887148E-04	0.221607E-04
	10^{-5}	0.141132E-02	0.354499E-03	0.887299E-04	0.221888E-04
A-mesh	10^{-2}	0.143765E-02	0.397305E-03	0.114788E-03	0.319861E-04
	10^{-3}	0.141930E-02	0.364079E-03	0.980868E-04	0.264123E-04
	10^{-4}	0.141644E-02	0.356064E-03	0.906718E-04	0.233904E-04
	10^{-5}	0.141617E-02	0.354938E-03	0.890296E-04	0.224977E-04

TABLE 5.3
Global super-errors for p in the norm $|||\cdot|||_0$

	ε	N 12	24	48	96
U-mesh	1.0	0.891844E-03	0.230327E-03	0.580601E-04	0.145453E-04
	10^{-2}	0.226417E-02	0.585426E-03	0.893942E-03	0.887309E-03
	10^{-3}	0.219491E-02	0.553963E-03	0.147541E-03	0.787511E-04
	10^{-4}	0.219340E-02	0.550885E-03	0.137930E-03	0.346084E-04
	10^{-5}	0.219339E-02	0.550856E-03	0.137871E-03	0.344788E-04
A-mesh	10^{-2}	0.232789E-02	0.789512E-03	0.271153E-03	0.818078E-04
	10^{-3}	0.221200E-02	0.604019E-03	0.189522E-03	0.566224E-04
	10^{-4}	0.220142E-02	0.558471E-03	0.149493E-03	0.415304E-04
	10^{-5}	0.220089E-02	0.552080E-03	0.139633E-03	0.363830E-04

TABLE 5.4

Global super-errors for $a^{1/2}\nabla p$ in the norm $|||\cdot|||$

		Boundary blocks		Center block	
	ε	p	$a^{1/2}\nabla$p	p	$a^{1/2}\nabla$p
U-mesh	1.0	0.01505	0.10982E+00	0.01496	0.43016E-01
U-mesh		0.25509	0.93522E+00	0.05183	0.10970E+00
A-mesh	10^{-2}	0.11316	0.41374E+00	0.05136	0.11988E+00
U-mesh		0.25915	0.94997E+00	0.05183	0.10970E+00
A-mesh	10^{-3}	0.14657	0.53621E+00	0.05127	0.12200E+00
U-mesh		0.25916	0.94998E+00	0.05183	0.10970E+00
A-mesh	10^{-4}	0.17215	0.63019E+00	0.05125	0.12234E+00
U-mesh		0.25916	0.94999E+00	0.05183	0.10970E+00
A-mesh	10^{-5}	0.19136	0.70084E+00	0.05125	0.12238E+00

TABLE 5.5

Local errors for p and $a^{1/2}\nabla p$ in the L^∞-norm with $N = 12$

		Boundary blocks		Center block	
	ε	p	$a^{1/2}\nabla$p	p	$a^{1/2}\nabla$p
U-mesh	1.0	0.00759	0.58082E-01	0.00754	0.22655E-01
U-mesh		0.22428	0.85942E+00	0.02601	0.53550E-01
A-mesh	10^{-2}	0.06557	0.25011E+00	0.02595	0.56218E-01
U-mesh		0.25488	0.97697E+00	0.02601	0.53550E-01
A-mesh	10^{-3}	0.09042	0.34494E+00	0.02594	0.56772E-01
U-mesh		0.25490	0.97702E+00	0.02601	0.53550E-01
A-mesh	10^{-4}	0.11204	0.42780E+00	0.02594	0.56861E-01
U-mesh		0.25490	0.97702E+00	0.02601	0.53550E-01
A-mesh	10^{-5}	0.13067	0.49933E+00	0.02594	0.56873E-01

TABLE 5.6

Local errors for p and $a^{1/2}\nabla p$ in the L^∞-norm with $N = 24$

Then we tested the singular perturbation problem for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5}. Convergence rates $O(N^{-1})$ for p and $a^{1/2}\nabla p$ in the L^2-norm are clearly shown in Tables 5.1-5.2 on the A-mesh, which agrees with Theorem 3.12. Global superconvergence rates $O(N^{-2})$ for p and $a^{1/2}\nabla p$ in the norm $|||\cdot|||$ are obtained in Tables 5.3-5.4 on the A-mesh, which is consistent with Theorem 3.17. Note that the errors in Tables 5.1-5.4 on the A-mesh decrease a little as ε decreases. The reason is that

		Boundary blocks		Center block	
	ε	p	$a^{1/2}\nabla$p	p	$a^{1/2}\nabla$p
U-mesh	1.0	0.00380	0.29778E-01	0.00378	0.11557E-01
U-mesh		0.16627	0.64992E+00	0.01302	0.26419E-01
A-mesh	10^{-2}	0.03515	0.13700E+00	0.01301	0.32710E-01
U-mesh		0.25245	0.98877E+00	0.01302	0.26419E-01
A-mesh	10^{-3}	0.05054	0.19672E+00	0.01301	0.28956E-01
U-mesh		0.25253	0.98904E+00	0.01302	0.26419E-01
A-mesh	10^{-4}	0.06491	0.25267E+00	0.01301	0.28784E-01
U-mesh		0.25253	0.98904E+00	0.01302	0.26419E-01
A-mesh	10^{-5}	0.07816	0.30435E+00	0.01301	0.28792E-01

TABLE 5.7

Local errors for p and $a^{1/2}\nabla p$ in the L^{∞}-norm with $N = 48$

		Boundary blocks		Center block	
	ε	p	$a^{1/2}\nabla$p	p	$a^{1/2}\nabla$p
U-mesh	1.0	0.00190	0.15064E-01	0.00189	0.58291E-02
U-mesh		0.10553	0.41578E+00	0.00651	0.13116E-01
A-mesh	10^{-2}	0.01800	0.71092E-01	0.00651	0.19810E-01
U-mesh		0.24971	0.98857E+00	0.00651	0.13116E-01
A-mesh	10^{-3}	0.02648	0.10433E+00	0.00651	0.15571E-01
U-mesh		0.25128	0.99465E+00	0.00651	0.13116E-01
A-mesh	10^{-4}	0.03480	0.13694E+00	0.00651	0.15048E-01
U-mesh		0.25128	0.99466E+00	0.00651	0.13116E-01
A-mesh	10^{-5}	0.04281	0.16838E+00	0.00651	0.15009E-01

TABLE 5.8

Local errors for p and $a^{1/2}\nabla p$ in the L^{∞}-norm with $N = 96$

those ε-related terms, which controls the coefficients in the convergence rates, becomes smaller as ε decreases. From (3.49), the interpolation properties and the derivative estimates of the solution, we have the follow error estimate

$$||\mathbf{u}-\mathbf{u}_h||_{0,\Omega}+||p-p_h||_{0,\Omega}\leq Ch(\varepsilon||p||_{2,\Omega}+||p||_{1,\Omega})\leq Ch\varepsilon^{-1}, \tag{5.1}$$

for RT_0 on a generic quasi-uniform mesh. Hence the convergence rates will degenerate as ε becomes smaller for fixed h, this phenomenon is observed in Tables 5.1-5.8 for the U-mesh.

The explanation for the nice behaviours on the U-mesh obtained in Tables 5.3-5.4 is that the boundary layers are so thin that all the interior nodes are far away from the boundary layers. Hence the obtained estimates are actually interior estimates. To further compare the behaviours between the U-mesh and A-mesh, we presented the blockwise L^{∞} errors on both meshes in Tables 5.5-5.8, where we denote the "Boundary blocks" for Ω_1, Ω_3 and "Center block" for Ω_2. Tables 5.5-5.8 show that our anisotropically mesh performs much better than the uniform mesh inside the boundary blocks, though they have almost the same behaviours away from the boundary layers. $O(N^{-1})$ convergence rates for p and $a^{1/2}\bigtriangledown p$ in the L^{∞}-norm are observed in Tables 5.5-5.8 on subdomains away from the boundary layers, which is proved for p in Theorem 3.12. However, the rigorous proof for $a^{1/2}\bigtriangledown p$ is still open. Furthermore, global pointwise convergence rates $O(N^{-1+\delta})$, $\delta\in(0,1)$ for p and $a^{1/2}\bigtriangledown p$ on the A-mesh are observed in Tables 5.5-5.8, while the theoretical justification is unavailable at this moment.

REFERENCES

[1] S. Adjerid, M. Aiffa and J. E. Flaherty, *High-order finite element methods for singularly perturbed elliptic and parabolic problems*, SIAM J. Appl. Math. 55 (1995), pp. 520–543.

[2] T. Arbogast, C.N. Dawson, P.T. Keenan, M.F. Wheeler and I. Yotov, *Enhanced cell-centered finite differences for elliptic equations on general geometry*, SIAM J. Sci. Comput. 19 (1998), pp. 404–425.

[3] T. Arbogast, M.F. Wheeler and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[4] J.H. Bramble and J. Xu, *Some estimates for a weighted L^2 projection*, Math. Comp. 56 (1991), pp. 463-476.

[5] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

[6] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[7] L. Demkowicz and J.T. Oden, *An adaptive characteristic Petrov-Galerkin finite element method for convection-dominated linear and nonlinear parabolic problems in one space variable*, J. Comput. Phys. 67 (1986), pp. 188–213.

[8] J. Douglas and J.E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp. 44 (1985), pp. 39–52.

[9] R. Duran, *Superconvergence for rectangular mixed finite elements*, Numer. Math. 58 (1990), pp. 287–298.

[10] R.E. Ewing, R.D. Lazarov and J. Wang, *Superconvergence of the velocity along the Gauss lines in mixed finite element methods*, SIAM J. Numer. Anal. 28 (1991), pp. 1015-1029.

[11] G. Fairweather, *Finite Element Galerkin Methods for Differential Equations*, Marcel Dekker, New York, 1978.

[12] W. Hackbusch, *Multigrid Methods and Applications*, Springer-Verlag, Berlin, 1985.

[13] H. Han and R.B. Kellogg, *Differentiability properties of solutions of the equation $-\varepsilon^2\triangle u + ru = f(x,y)$ in a square*, SIAM J. Math. Anal. 21 (1990), pp. 394–408.

[14] X. C. Hu, T. A. Manteuffel, S. MacCormick and T. F. Russell, *Accurate discretization for singular perturbations: the one-dimensional case*, SIAM J. Numer. Anal. 32 (1995), pp. 83–109.

[15] R. B. Kellogg and M. Stynes, *Optimal approximability of solutions of singularly perturbed two-point boundary value problems*, SIAM J. Numer. Anal. 34 (1997), pp. 1808–1816.

[16] J. Li, *Quasi-optimal uniformly convergent finite element methods for the elliptic boundary layer problem*, Comput. Math. Appl. 33 (1997), pp. 11–22.

[17] J. Li and I. M. Navon, *Uniformly convergent finite element methods for singularly perturbed elliptic boundary value problems I: reaction-diffusion type*, Comput. Math. Appl. 35 (1998), pp. 57–70.

[18] J. Li and I. M. Navon, *Uniformly convergent finite element methods for singularly perturbed elliptic boundary value problems II: convection-diffusion type*, Comput. Methods Appl. Mech. Engrg. 162 (1998), pp. 49–78.

[19] J. Li and I. M. Navon, *Global uniformly convergent finite element methods for singularly perturbed elliptic boundary value problems: higher-order elements*, Comput. Methods Appl. Mech. Engrg. (in press).

[20] Q. Lin, J. Li and A. Zhou, *A rectangle test for Ciarlet-Raviart and Hermann-Miyoshi scheme*, in Proc. Sys. Sci. & Sys. Engrg., Great Wall Culture Publ. Co., Hong Kong, 1991, pp. 230–233.

[21] Q. Lin and N.N. Yan, *The Construction and Analysis of High Accurate Finite Element Methods*, Hebei University Press, Hebei, China, 1996 (in Chinese).

[22] J.J.H. Miller, E. O'Riordan and G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1995.

[23] K.W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman and Hall, 1996.

[24] M. Nakata, A. Weiser and M.F. Wheeler, *Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains*, in The Mathematics of Finite Elements and Applications V, J.R. Whiteman, ed., Academic Press, London, 1985.

[25] J.T. Oden, *Optimal h-p finite element methods*, Comput. Methods Appl. Mech. Engrg. 112

(1994), pp. 309–331.

[26] R.A. Raviart and J.M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Math., 606, Springer-Verlag, New York, 1977, pp. 292–315.

[27] H.-G. Roos, *Layer-adapted grids for singular perturbation problems*, Z. Angew. Math. Mech. 78 (1998), pp. 291–310.

[28] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin 1996.

[29] T.F. Russell and M.F. Wheeler, *Finite element and finite difference methods for continuous flows in porous media*, Chapter II, in The Mathematics of Reservoir Simulation, Frontiers Appl. Math. 1, R.E. Ewing, ed., SIAM, Philadelphia, PA, 1983, pp. 35–106.

[30] A. H. Schatz and L. B. Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems in two and one dimensions*, Math. Comp., 40 (1983), pp. 47–89.

[31] C. Schwab and M. Suri, *The p and hp versions of the finite element method for problems with boundary layers*, Math. Comp. 65 (1996), pp. 1403–1429.

[32] W.G. Szymczak and I. Babuska, *Adaptivity and error estimation for the finite element method applied to convection diffusion problem*, SIAM J. Numer. Anal. 21 (1984), pp. 910–953.

[33] R. Verfürth, *Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation*, Numer. Math. 78 (1998), pp. 479–493.

[34] J. Wang, *Superconvergence and extrapolation for mixed finite element methods on rectangular domains*, Math. Comp. 56 (1991), pp. 477–503.

[35] A. Weiser and M.F. Wheeler, *On convergence of block-centered finite-differences for elliptic problems*, SIAM J. Numer. Anal. 25 (1988), pp. 351-375.

[36] A. Zhou and J. Li, *The full approximation accuracy for the Stream-Function-Vorticity-Pressure Method*, Numer. Math. 68 (1994), pp. 427–435.