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ERROR CONTROL IN FINITE ELEMENT APPROXIMATIONS OF NONLINEAR PROBLEMS IN MECHANICS

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Summary. *This work describes extensions of goal-oriented methods for a posteriori error estimation and control of numerical approximation to a class of highly-nonlinear problems in computational solid mechanics. To apply the theory of goal-oriented error estimation, a backward-in-time dual formulation of these problems is derived, and residual error estimators for meaningful quantities of interest are established. The target problem class is that of axisymmetric deformations of layered elastomer-reinforced shells-of-revolution subjected to shock loading.*

1 INTRODUCTION

The objective in this work is to extend goal-oriented *a posteriori* error estimation [3, 1] to highly nonlinear dynamic simulations of the deformation of submerged bodies subjected to shock loading. While the developments to date provide an abstract mathematical framework for error estimation in highly nonlinear problems, few applications to important problems in nonlinear continuum mechanics appear to have been made, owing to the inherent complexities in such problems. To capture features of the nonlinear dynamics of solid bodies and structures under shock loading involves a host of complicated features and has been the focus of research in computational solid mechanics for many decades (see the recent treatise [2]). The analysis of the evaluation of approximation error of quantities of interest in such applications involves solving first a forward-in-time problem for the system response, and then a backward-in-time problem for the dual solution associated with the particular quantity of interest. In the present investigation, *a posteriori* error estimates for key quantities of interest are derived for a class of complex and highly-nonlinear problems in computational solid mechanics: the dynamical behavior of a heterogeneous, layered shells subjected to shock loading. The models considered here involve axisymmetric deformations of thick bodies-of-revolution undergoing very high strains and strain rates, and large elastic and inelastic deformations.

2 MODEL PROBLEM

Basic elements of the formulation and much of our notation are standard. We use an updated Lagrange formulation of the field equations of nonlinear continuum mechanics [2]). Let the open bounded domain of $\mathbb{R}^3$ occupied by the body of interest be denoted by Ω_0 in the reference configuration and Ω_t in the current configuration, $t \in (0,T)$. Let $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ be the spatial position of material particles that were located at position $\mathbf{X}$ in the reference configuration, and let $\rho = \rho(\mathbf{x},t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$, $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$, and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x},t)$ denote the current mass density, the displacement field, the velocity field, and the Cauchy stress tensor, respectively.

2.1 Primal Problem

The governing equations for the velocity and displacement fields are given by:

$$\rho\frac{d\mathbf{v}}{dt} - \nabla\cdot\boldsymbol{\sigma} = 0, \qquad \frac{d\mathbf{u}}{dt} = \mathbf{v}, \qquad \text{in } \Omega$$

subjected to the boundary and initial conditions:

$$\begin{aligned}
&\text{(B.C.)} \quad \mathbf{u}(\mathbf{x},t) = \mathbf{0} \text{ on } \Gamma_t^D, \quad \mathbf{n}\cdot\nabla\boldsymbol{\sigma} = \mathbf{g} \text{ on } \Gamma_t^N, \quad \forall t \in (0,T)\\
&\text{(I.C.)} \quad \mathbf{u}(\mathbf{X},0) = \mathbf{u}_0(\mathbf{X}), \quad \mathbf{v}(\mathbf{X},0) = \mathbf{v}_0(\mathbf{X}), \quad \forall \mathbf{X} \in \Omega_0
\end{aligned}$$

The density at time t is obtained from $\rho J = \rho_0$, ρ_0 given, and the energy e, which is used for updating the yield stress of the materials, is computed by solving $\rho\frac{de}{dt} = \boldsymbol{\sigma} : \mathbf{D}$, subjected to the initial condition $e(\mathbf{X},0) = e_0(\mathbf{X})$.

A weak formulation of the primal problem governing the motion of the material body reads:

$$\text{Find } (\mathbf{u},\mathbf{v}) \in \mathcal{V} \text{ such that } \quad B((\mathbf{u},\mathbf{v});(\mathbf{z},\mathbf{w})) = F(\mathbf{z},\mathbf{w}), \qquad \forall(\mathbf{z},\mathbf{w}) \in \mathcal{V} \tag{1}$$

where $\mathcal{V} = \mathcal{Z}\times\mathcal{W}$ defines a suitable product space of admissible displacement-velocity pairs and $B(\cdot;\cdot)$ and $F(\cdot)$ are the semilinear and linear forms:

$$\begin{aligned}
B((\mathbf{u},\mathbf{v});(\mathbf{z},\mathbf{w})) = \int_0^T\int_{\Omega_t} &\left(\rho\frac{d\mathbf{v}}{dt}\cdot\mathbf{w} + \boldsymbol{\sigma} : \nabla_x\mathbf{w}\right) + \left(\frac{d\mathbf{u}}{dt}\cdot\mathbf{z} - \mathbf{v}\cdot\mathbf{z}\right)\,dxdt\\
&+ \int_{\Omega_0}\left(\rho_0\mathbf{v}(\mathbf{X},0)\cdot\mathbf{w}(\mathbf{X},0) + \mathbf{u}(\mathbf{X},0)\cdot\mathbf{z}(\mathbf{X},0)\right)\,dX\\
F(\mathbf{z},\mathbf{w}) = \int_0^T\int_{\Gamma_t^N} \mathbf{g}\cdot\mathbf{w}\,dAdt &+ \int_{\Omega_0}\left(\rho_0\mathbf{v}_0(\mathbf{X})\cdot\mathbf{w}(\mathbf{X},0) + \mathbf{u}_0(\mathbf{X})\cdot\mathbf{z}(\mathbf{X},0)\right)\,dX
\end{aligned}$$

Note that the initial conditions for the displacement and velocity are weakly enforced in this formulation.

2.2 Discretization in time and space

The equations are discretized in space by the finite element method to obtain piecewise bilinear approximations $\mathbf{u}_h$ and $\mathbf{v}_h$ for the displacement $\mathbf{u}$ and velocity $\mathbf{v}$, respectively. The time interval $[0,T]$ is decomposed into subintervals $[t^n, t^{n+1}]$ and the time step Δt is determined by the Courant condition. The equations are discretized in time using an explicit *leap frog* method in which displacements and velocities are computed half a time step apart. The time-discrete momentum equation then reads, for all $\mathbf{w}_h$,

$$\int_{\Omega_{t^n}} \rho_h^n \mathbf{v}_h^{n+\frac{1}{2}} \cdot \mathbf{w}_h \, dx = \int_{\Omega_{t^n}} \rho_h^n \mathbf{v}_h^{n-\frac{1}{2}} \cdot \mathbf{w}_h \, dx + \Delta t \left(\int_{\Gamma_{t^n}^N} \mathbf{g}^n \cdot \mathbf{w}_h \, dA - \int_{\Omega_{t^n}} \boldsymbol{\sigma}_h^n : \nabla_x \mathbf{w}_h \, dx \right)$$

while the displacement field is simply updated as

$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n + \Delta t \, \mathbf{v}_h^{n+\frac{1}{2}}$$

We present in the next section a methodology to estimate the errors in the approximations for the displacements and velocities with respect to quantities of interest.

3 ERROR ESTIMATION

We suppose that we are interested here in quantities of the form:

$$Q(\mathbf{u}, \mathbf{v}) = \int_{\Omega_T} k_\varepsilon(\mathbf{x})(\mathbf{u} \cdot \boldsymbol{\alpha} + \mathbf{v} \cdot \boldsymbol{\beta}) \, dx \tag{2}$$

where $k_\varepsilon(\mathbf{x})$ is a kernel function defined on the deformed region underneath the shock loading material and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given unit vectors. The kernel function $k_\varepsilon(\mathbf{x})$ and vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will be constructed here such that the quantity of interest represents a local average of the vertical displacement u_z over a sub-domain of Ω_T (i.e. $\boldsymbol{\alpha} = (0,0,1)$ and $\boldsymbol{\beta} = (0,0,0)$).

3.1 The dual problem

Following [3, 1], the weak form of the dual problem of (1) consists in solving for $(\mathbf{p}, \mathbf{q}) \in \mathcal{V}$ such that

$$B'((\mathbf{u}, \mathbf{v}); (\mathbf{z}, \mathbf{w}), (\mathbf{p}, \mathbf{q})) = Q'((\mathbf{u}, \mathbf{v}); (\mathbf{z}, \mathbf{w})), \qquad \forall (\mathbf{z}, \mathbf{w}) \in \mathcal{V} \tag{3}$$

where it is straightforward to show that $Q'((\mathbf{u}, \mathbf{v}); (\mathbf{z}, \mathbf{w})) = Q(\mathbf{z}, \mathbf{w})$ since Q is a linear functional, and

$$\begin{aligned} B'((\mathbf{u}, \mathbf{v}); (\mathbf{z}, \mathbf{w}), (\mathbf{p}, \mathbf{q})) = & \int_0^T \int_{\Omega_t} \left(-\frac{d\mathbf{p}}{dt} \cdot \mathbf{z} + \nabla_x \mathbf{q} : \delta\boldsymbol{\sigma}^u : \nabla_x \mathbf{z} \right) dxdt + \int_{\Omega_T} \mathbf{p} \cdot \mathbf{z} \, dx \\ & + \int_0^T \int_{\Omega_t} \left(-\rho \frac{d\mathbf{q}}{dt} \cdot \mathbf{w} + \nabla_x \mathbf{p} : \delta\boldsymbol{\sigma}^v : \nabla_x \mathbf{w} - \mathbf{p} \cdot \mathbf{w} \right) dxdt + \int_{\Omega_T} \rho \mathbf{q} \cdot \mathbf{w} \, dx \end{aligned}$$

where $\delta\boldsymbol{\sigma}^u$ and $\delta\boldsymbol{\sigma}^v$ denote the first derivatives of the Cauchy stress with respect to $\mathbf{u}$ and $\mathbf{v}$. We observe that the dual problem (3) involves two coupled first-order partial differential equations with data given at time T and thus needs to be solved backwards in time.

3.2 Error estimator

The error estimator η for the error in the quantity of interest is computed as follows:

$$Q(\mathbf{u}, \mathbf{v}) - Q(\mathbf{u}_h^n, \mathbf{v}_h^n) \approx \eta \equiv \mathcal{R}((\mathbf{u}_h^n, \mathbf{v}_h^n); (\tilde{\mathbf{p}}, \tilde{\mathbf{q}})) \tag{4}$$

where $\mathcal{R}$ is the residual functional with respect to the primal problem, i.e. $\mathcal{R}((\mathbf{u},\mathbf{v});(\mathbf{z},\mathbf{w})) = F(\mathbf{z},\mathbf{w}) - B((\mathbf{u},\mathbf{v});(\mathbf{z},\mathbf{w}))$, and $(\tilde{\mathbf{p}},\tilde{\mathbf{q}}))$ is an approximation of the dual problem using second-order serendipity elements. The dual problem is computed here using an implicit algorithm in time with fewer time iterations.

4 NUMERICAL EXAMPLE

We consider a problem in which a shock-like pressure loading is applied to the entire outside surface of a shell with a rotational axis of symmetry. The shell is composed of two layers, one of steel and the other made of an elastomeric material. The goal of the study is to study the effectiveness of the elastomeric layer as a reinforcement of the steel structure. The quantity of interest is the displacement of a region of the layered shell under the shock loading source. The Cauchy stress tensor for each material is decomposed into its dilatational and deviatoric components, $\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{S}$, p being the mechanical pressure and $\mathbf{S}$ the deviatoric stress. A Mie-Gruneisen equation of state to model the pressure is assumed for both materials. The deviatoric stress is computed from the equation of state for the steel and is assumed negligible for the elastomer. Preliminary distributions of the error estimates with respect to time were obtained for two configurations, one with the steel in the outer layer (steel on top), the other with the elastomer in the outer layer (elastomer on top), see Figure 1.

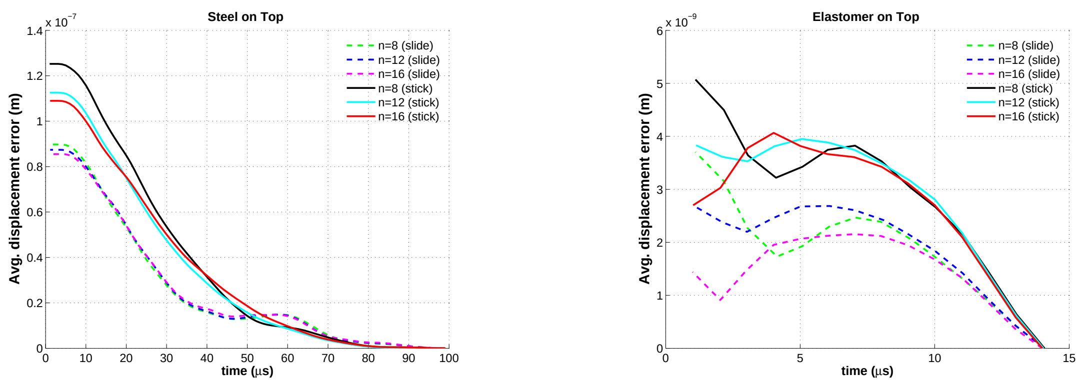

Figure 1: Temporal distribution of the error estimator: steel on top (left), elastomer on top (right).

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