# Analysis of the diffusive approximation of the Shallow Water equations

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May 17, 2007

**Abstract**

In this paper we study the properties of a doubly nonlinear diffusion equation arising in shallow water flow models. Existence, uniqueness, some regularity results and conditions for positivity of classical solutions are presented for the zero Dirichlet initial/boundary value problem. The Faedo Galerkin method is used to approximate the solution and the passing to the limit is done by means of the compactness and monotonicity properties of the equation in hand. Some basic facts about the solution such as boundedness, continuity in time of the $L^2$ norm are also obtained.

## 1 Introduction

In this paper we study the doubly nonlinear diffusion equation arising in shallow water flow models given by the following initial/boundary-value problem prescribed for any fixed $T > 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \left( u^{\alpha} \dfrac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{1}$$

where $\Omega$ is an open, bounded subset of $\mathbb{R}^n$ with $\partial\Omega \in C^1$, $f : \Omega \times (0,T] \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $0 < \gamma \leq 1$, $1 < \alpha < 2$ and $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ is the unknown. Here $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ refers to the Euclidean norm in $\mathbb{R}^n$.

Problem (1) is characterized as doubly nonlinear since the nonlinear behaviour appears inside the divergence term as a product of two nonlinearities involving $u$ and $\nabla u$, namely $u^{\alpha}$ and $\nabla u/|\nabla u|^{1-\gamma}$. In this paper, the key idea to study this problem is to introduce a change of variables that will separate such two nonlinearities. One will appear in the time derivative term and the other will remain in the divergence term. This idea simplifies the passing to the limit in the compactness argument to prove existence of solutions.

The outline of the paper is the following. In the introduction, we will discuss the relevance of the problem in the context of shallow water flow modeling, present a literature review

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of previous studies, and introduce the notation that is used throughout the paper. In the preliminaries, we will present the above mentioned change of variables that will tranform problem (1) into an alternative formulation. We will then study existence, some regularity and uniqueness of such an alternative formulation, in the subsequent sections. The consequences of the results obtained in this alternative formulation will translate in similar results for the original problem. These will be stated as corollaries in each section.

### 1.1 Motivation

Models for surface water flows are derived from the incompressible, three-dimensional Navier-Stokes equations, which consist of momentum equations for the three velocity components and a continuity equation. Depending on the physics of the flow, scaling arguments are used in order to obtain effective equations for the problem at hand. See [9]. Equation (1) is a simplified version of the two-dimensional shallow water equations[1] called the diffusive wave or zero-inertia approach, which neglects the acceleration terms in the horizontal momentum equations. Such an equation is a doubly nonlinear and degenerate parabolic equation for the water elevation, obtained from substituting particular forms of the depth averaged, two-dimensional horizontal velocities into the depth averaged continuity equation. Equation (1) is more commonly found in the literature written as

$$\frac{\partial H}{\partial t} - \nabla \cdot \left( \frac{(H-z)^\alpha}{c_f} \frac{\nabla H}{|\nabla H|^{\gamma-1}} \right) = f(t,x), \qquad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{2}$$

where $H(t,x)$ is the free surface elevation or hydraulic head, $z(x)$ is the bed surface or land elevation, $f(t,x)$ is a source/sink (such as rain or infiltration), $\alpha$ and $\gamma$ are non-negative parameters and $c_f(x)$ is a friction coefficient.

**Remark 1.1.** In this context, equation (2) makes sense physically only if $H - z \geq 0$. Note that in writing (1) we have assumed that $z \equiv 0$. In our study we will be interested in finding positive solutions of problem (1).

**Remark 1.2.** Whenever $H - z = 0$ (or alternatively $u = 0$ in (1)), equation (2) degenerates, *i.e.* it is no longer of parabolic type.

Equation (2) has proved to be suitable to model shallow water flow under uniform flow conditions, *i.e.* when the fluid motion is dominated by gravity and balanced by the boundary shear stress and has been used to simulate, for example, overland flow and flow in wetlands. See [11], [5], [12], [4].

An appropriate study of the existence, uniqueness and regularity of weak solutions to problem (1) has not been pursued to the best knowledge of the authors, although, relevant methods for the treatment of similar nonlinear parabolic equations have been developed in works by Friedman [3], Ladyzenskaja et al. [6], Lions [7], DiBenedetto [1] and Vazquez [8]. Note in particular that for the case when $\gamma = 1$, $c_f \equiv 1$, and $z \equiv 0$, equation (2) becomes the Porous Medium Equation (PME). One should expect similarities betwen the PME and the more general equation (2), although some differences may arise. A comprehensive study of the PME can be found in the book by Vázquez [8].

[1] In shallow water flows, the main scaling assumption consists in considering that the vertical scales are small relative to the horizontal ones. This approximation reduces the vertical momentum equation to the hydrostatic pressure relation $\partial_{x_3} p = \rho g$.

### 1.2 Notation

We will use the standard notation introduced in [2]. Let $X$ be a real Banach space, with norm $\|\cdot\|$. The symbol $L^p(0,T;X)$ will denote the Banach space of all measurable functions $u:[0,T]\to X$ such that

*(i)* $\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p\right)^{1/p} < \infty, \qquad$ for $1\le p<\infty$, and

*(ii)* $\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{0\le t\le T}\|u(t)\| < \infty.$

We will denote with $C([0,T];X)$ the space of all continuous functions $u:[0,T]\to X$ such that

$$\|u\|_{C(0,T;X)} := \max_{0\le t\le T}\|u(t)\| < \infty.$$

Let $u\in L^1(0,T;X)$, we say $v\in L^1(0,T;X)$ is the weak time derivative of $u$, denoted $u_t=v$, provided

$$\int_0^T \psi_t(t)\,u(t) = -\int_0^T \psi(t)\,v(t)$$

for all scalar test functions $\psi\in C_0^\infty(0,T)$. Throughout the paper, $W^{1,p}(0,T;X)$ will denote the space of all functions $u\in L^p(0,T;X)$ such that $u_t$ exists in the weak sense and $u_t\in L^p(0,T;X)$ with the norm

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p + \|u_t(t)\|^p\right)^{1/p} & (1\le p<\infty),\\ \operatorname*{ess\,sup}_{0\le t\le T}\left(\|u(t)\| + \|u_t(t)\|\right) & (p=\infty).\end{cases}$$

For $1\le p\le+\infty$, we will denote its conjugate as $p^*$ *i.e.*, $1/p+1/p^*=1$. For any measurable set $E\subset\Omega$ and real valued vector functions $u\in L^p(E)$ and $v\in L^{p^*}(E)$ we will denote for the duality pairing between $u$ and $v$ as

$$(u,v)_E := \int_E u\cdot v.$$

For simplicity, we use $(u,v):=(u,v)_\Omega$. Similarly, we will denote the duality pairing between $u\in W^{-1,p^*}(\Omega)$ and $v\in W_0^{1,p}(\Omega)$ as $\langle u,v\rangle$. Recall that the elements of $W^{-1,p^*}(\Omega)$ are the distributions that have continuous extention to $W_0^{1,p}(\Omega)$. These spaces are characterized in the following way: if $u\in W^{-1,p^*}(\Omega)$, then there exists functions $f^0,f^1,\cdots,f^n$ in $L^{p^*}(\Omega)$ such that

$$\langle u,v\rangle = (f^0,v) + \sum_{i=1}^n (f^i, v_{x_i}).$$

## 2 Preliminaries

Consider the following problem

$$\begin{cases} \dfrac{\partial\phi(v)}{\partial t} - \eta^\gamma\,\nabla\cdot\left(\dfrac{\nabla v}{|\nabla v|^{1-\gamma}}\right) = f & \text{on } \Omega\times(0,T]\\ v=0 & \text{on } \partial\Omega\times[0,T]\\ v=v_0 & \text{on } \Omega\times\{t=0\}\end{cases} \tag{3}$$

within the same context as problem (1). Where $\eta$ is a positive constant, and the function $\phi(x) \in C^{0,\eta}(\mathbb{R})$ is an *odd* function satisfying the following properties:

*(i)* $|\phi(x)| \leq |x|^\eta$ for $0 < \eta \leq \gamma < 1$, with equality for $|x| \geq R$ for some $R \geq 0$

*(ii)* $\phi(x)$ is a concave increasing function for $x \geq 0$,

Note that with the change of variables defined by $u = \phi(v)$, problem (3) is transformed into

$$\begin{cases} \dfrac{\partial u}{\partial t} - \eta^\gamma \, \nabla \cdot \left( \left((\phi^{-1})'(u)\right)^\gamma \dfrac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{4}$$

Now, choosing

$$0 < \eta = \frac{\gamma}{\alpha + \gamma} < 1, \qquad \text{and} \qquad \phi(x) = \frac{x}{|x|^{1-\eta}} \tag{5}$$

we can obtain the explicit expression for

$$(\phi^{-1})'(x) = (1+\theta)|x|^{\,\theta} \qquad \text{where} \qquad \theta = \frac{1-\eta}{\eta} = \frac{\alpha}{\gamma} \tag{6}$$

which yields the following equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \left( |u|^\alpha \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f. \tag{7}$$

The previous manipulations lead us to conclude that, at least formally, positive solutions of problem (3) are solutions of the original problem (1). Since we intend to characterize solutions of problem (1) that make physical sense, namely positive solutions, our focus will be to do a basic study of problem (3). Our findings will then be interpreted in terms of the original problem (1).

The additional property

*(iii)* $\phi'(0) < +\infty$

will be shown to be a sufficient condition in order to ensure the existence of a unique weak solution of problem (3). This property can be interpreted as a sort of ellipticity condition for the original problem (1) since it is a regularization for small values of $u$ where the degenerate character of problem (1) arises. To support such interpretation, we plotted in Figures 1, 2, and 3 functions $\phi(x)$, $\phi^{-1}(x)$ and $(\phi^{-1})'(x)$ without property *(iii)*. In Figures 4, 5, and 6 functions $\phi_{reg}(x)$, $\phi^{-1}_{reg}(x)$ and $(\phi^{-1})'_{reg}(x)$ with property *(iii)* are shown. Figure 6 shows how property *(iii)* indeed imposes ellipticity in the problem. We will continue using the symbol $\phi_{reg}$ whenever the function has the regularizing property *(iii)*.

## 2.1 Definitions of Weak Solution

**Definition 2.1.** *We say a function*

$$v \in L^{1+\gamma}(0,T; W_0^{1,(1+\gamma)}(\Omega)), \quad \textit{with} \;\; \phi(v)_t \in L^{(1+\gamma)^*}(0,T; W^{-1,(1+\gamma)^*}(\Omega)),$$

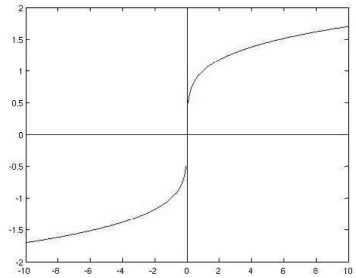

Figure 1: $\phi(x)$

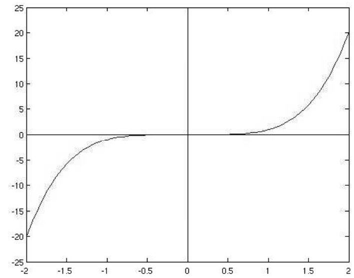

Figure 2: $\phi^{-1}(x)$

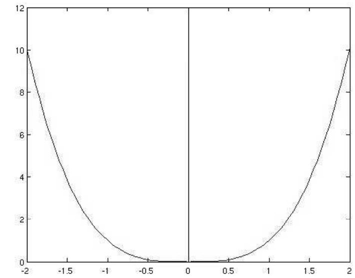

Figure 3: $(\phi^{-1})'(x)$

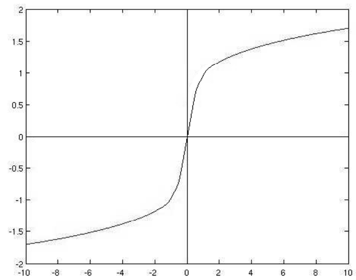

Figure 4: $\phi_{reg}(x)$

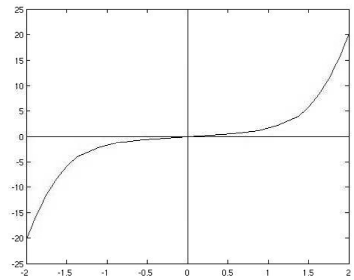

Figure 5: $\phi^{-1}_{reg}(x)$

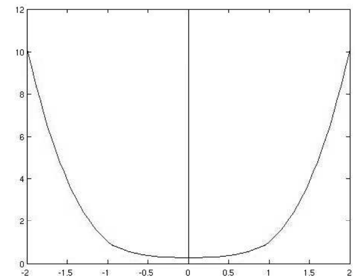

Figure 6: $(\phi^{-1})'_{reg}(x)$

*is a weak solution of the initial/boundary-value problem (3) provided*

$$\langle \phi(v)_t, w\rangle + \eta^\gamma \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w \right) = (f, w) \qquad a.e \text{ in time } 0 \le t \le T, \tag{8}$$

*for any* $w \in W_0^{1,(1+\gamma)}(\Omega))$ *and*

$$v(0) = v_0. \tag{9}$$

**Definition 2.2.** *We say a function* $u$*, with the properties*

$$\phi^{-1}(u) \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega)), \;\; \text{and} \;\; u_t \in L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)),$$

*is a weak solution of the initial/boundary-value problem (4) provided*

$$\langle u_t, w\rangle + \eta^\gamma \left( \left((\phi^{-1})'(u)\right)^\gamma \frac{\nabla u}{|\nabla u|^{1-\gamma}}, \nabla w \right) = (f, w) \qquad a.e \text{ in time } 0 \le t \le T, \tag{10}$$

*for any* $w \in W_0^{1,(1+\gamma)}(\Omega))$ *and*

$$u(0) = u_0. \tag{11}$$

**Remark 2.1.** It will be clear from the proof of existence that a consequence of equation (8) (resp. (10)) is that

$$\phi(v) \in C([0, T]; W^{-1,(1+\gamma)^*}) \qquad (\text{resp. } u \in C([0, T]; W^{-1,(1+\gamma)^*})$$

thus conditon (9) (resp. (11)) makes sense.

**Remark 2.2.** Observe that in Definition 2.2, $u$ need not be in any particular Sobolev space. Instead this regularity condition is imposed on $\phi^{-1}(u)$. Indeed, for some $v \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$, we require that $u = \phi(v)$. Since for a nonregular $\phi$ the function $\phi'(x)$ is not bounded at $x = 0$, integrability properties are lost for the distributional gradient of $u$. Therefore, it may be that the distribution $\nabla u$ is not a regular distribution and hence Definition 2.2 would not make sense. The following definition makes precise what we understand for $\nabla u$.

**Definition 2.3.** *Let $v \in W^{1,(1+\gamma)}(\Omega)$, and $\phi$ with the properties (i) and (ii). Set $u = \phi(v)$, then we define the pointwise gradient of $u$, denoted as $\nabla u$, as the measurable function*

$$\nabla u = \begin{cases} \phi'(v)\nabla v & \text{if} \quad |v| > 0 \\ 0 & \text{if} \quad v = 0. \end{cases}$$

It is worthwhile to observe that the weak gradient and the pointwise gradient coincide whenever $\phi$ has property *(iii)*.

# 3 Existence

**Theorem 3.1.** *Let $f$ and $v_0$ satisfy*

$$v_0 \in L^{1+\eta}(\Omega) \qquad \text{and} \qquad f \in L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)^*}(\Omega)), \tag{12}$$

*then, there exist a function $v$ with the properties*

$$v \in L^{(1+\gamma)}(0,T;W_0^{1,(1+\gamma)}(\Omega)), \tag{13}$$

*and*

$$\phi(v)_t \in L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)), \tag{14}$$

*such that it solves problem (3).*

*Proof.* In order to prove the existence of a weak solution of problem (3) we will use the method of compactness and monotonicity carefully explained in [7]. The method consists of four main steps:

*Step 1.* Constructing approximate solutions by the method of Faedo-Galerkin.

*Step 2.* Finding *a priori* estimates on such approximate solutions.

*Step 3.* Using the properties of compactness to extract a converging sub-sequence to pass to the limit.

*Step 4.* Using the monotonicity of the nonlinear operator $\mathcal{A}(x)$ (See Appendix A) to prove that the limit process indeed leads to the weak solution.

The proof continues as follows: *Step 1*, *Step 3* and *Step 4* will be performed for any regular $\phi_{reg}$. Computations in *Step 2* will be valid for $\phi(x) = x/|x|^{1-\eta}$, however identical estimates will hold for any $\phi_{reg}$ with constants independent of the function $\phi_{reg}$. Thus by this process, we will obtain a solution for the problem in any regularized case. In order to find a solution for a non regular $\phi$, a sequence of solutions of the regularized problem will be built and sent to the limit.

## Step 1. Approximate Solutions

Let $\{w_j\}_{j=1}^{\infty}$ be a basis of $V = W_0^{1,(1+\gamma)}(\Omega)$. Construct the Faedo-Galerkin approximate solution of problem (3), $v_m(t)$, the following way. For any fixed $t$

$$v_m(t) = \sum_{j=1}^{m} \zeta_j(t) w_j(x) \ \in [w_1, \ldots, w_m] = \text{the space generated by } \{w_j\}_{j=0}^{m}$$

and satisfying

$$(\phi(v_m)_t, w_j) + \eta^{\gamma} \left( \frac{\nabla v_m}{|\nabla v_m|^{1-\gamma}}, \nabla w_j \right) = (f, w_j) \qquad 1 \le j \le m, \tag{15}$$

$$v_m(0) = v_{0,m} \in [w_1, \ldots, w_m],$$

where $v_{0,m} \to v_0$ in $L^{1+\eta}(\Omega)$.

## Step 2. *A priori* Estimates

**Lemma 3.1.** *Set $\phi(x) = x/|x|^{1-\eta}$. Let $v_m$ be a Faedo-Galerkin approximate solution of problem (3), then the following estimates hold.*

$$\sup_{0 \le t \le T} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \le C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right) \tag{16}$$

*and*

$$\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma} \le C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right) \tag{17}$$

*where $(1+\eta)^* = (1+\eta)/\eta$.*

*Proof.* Multiply equation (15) by $\zeta_j(t)$ and sum for $1 \le j \le m$ to obtain (See Lemma A.3 in Appendix A for the first term)

$$\frac{d}{dt} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1+\eta}{\eta^{1-\gamma}} \int_{\Omega} |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta} (f, v_m) \tag{18}$$

and from Young's inequality

$$(f, v_m) \le \frac{\eta}{1+\eta} \|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{1+\eta} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}. \tag{19}$$

Now, since

$$\int_{\Omega} |\nabla v_m|^{1+\gamma} \ge 0$$

we get the inequality

$$\frac{d}{dt} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \le \|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{\eta} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}.$$

Using Gronwall's lemma we get that for all $t \in [0, T]$

$$\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \le C \left( \|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T \right)$$

which leads to the first estimate stated in (16).
*Note:* We have assumed, without loss of generality, that

$$\|v_{0,m}\|_{L^{1+\eta}(\Omega)} \leq \|v_0\|_{L^{1+\eta}(\Omega)}.$$

Integrating equation (18) in time

$$\|\phi(v_m)(T)\|^{(1+\eta)^*}_{L^{(1+\eta)^*}(\Omega)} + \frac{1+\eta}{\eta^{1-\gamma}} \int_0^T \int_\Omega |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta} \int_0^T (f, v_m) + \|v_{0,m}\|^{1+\eta}_{L^{1+\eta}(\Omega)}.$$

The above expression and inequality (19) imply that

$$\|\nabla v_m\|^{1+\gamma}_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))} \leq C\left(\|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T\right)$$

which finishes the proof. □

**Remark 3.1.** Note that by Poincaré inequality

$$\|v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))} \leq C(\Omega)\, \|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}$$

therefore the sequence $\{v_m\} \subset L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$ and it is uniformly bounded.

## Step 3. Passing to the limit

Let $v_m(t)$ be the Faedo-Galerkin sequence of approximate solutions of problem (3) defined by (15). Estimates (16) and (17) in Lemma 3.1 imply that there exists a convergent subsequence $\{v_\mu\}$ of $\{v_m\}$ such that

$$v_\mu \rightharpoonup v \quad \text{in} \quad L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega)) \quad \text{weakly,} \tag{20}$$
$$\phi(v_\mu)(T) \rightharpoonup \xi \quad \text{in} \quad L^{(1+\eta)^*}(\Omega) \quad \text{weakly,} \tag{21}$$

and as a consequence of (20) and the *Rellich-Kondrachov* compactness theorem we have that

$$v_\mu \to v \quad \text{in} \quad L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega)) \quad \text{strongly.} \tag{22}$$

In addition, inequality (17) implies

$$\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} \rightharpoonup \chi \quad \text{in} \quad L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)^*}(\Omega)) \text{ weakly.} \tag{23}$$

Integrating equation (15) in time and using the aforementioned convergence results, we can take the limit as $\mu \to \infty$ to find that for any $w \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$

$$\lim_{\mu\to\infty} \int_0^T (\phi(v_\mu)_t, w) = -\,\eta^\gamma \int_0^T (\chi, \nabla w) + \int_0^T (f\,, w). \tag{24}$$

We can conclude that

$$\phi(v_\mu)_t \rightharpoonup \vartheta \quad \text{in} \quad L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)) \quad \text{weakly,} \tag{25}$$

where the functional $\vartheta$ is defined by the right hand side of equation (24). Note, that for any regular $\phi$ the sequence $\{\phi(v_\mu)_t\}$ lies in $L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega))$ since $\phi(v_\mu)_t$ inherits all the integrability properties of $(v_\mu)_t$. Using (22) and Theorem A.2 in Appendix A we can conclude that

$$\phi(v)_t = \vartheta. \tag{26}$$

Therefore, for any $w \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$

$$\int_0^T \langle \phi(v)_t, w\rangle = -\,\eta^\gamma \int_0^T (\chi, \nabla w) + \int_0^T (f\ , w). \tag{27}$$

Note also that $L^{(1+\gamma)/\eta}(\Omega) \subset W^{-1,(1+\gamma)^*}(\Omega)$, hence we have

$$\phi(v) \in L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)/\eta}(\Omega)) \subset L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)).$$

Using the previous fact, together with (25) and (26)

$$\phi(v) \in W^{1,(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)).$$

So by Theorem A.1 in Appendix A we conclude that

$$\phi(v) \in C([0,T];W^{-1,(1+\gamma)^*}(\Omega))$$

and

$$\phi(v)(t) - \phi(v)(s) = \int_s^t \phi(v)_t \ \text{ for all } \ 0 \le s \le t \le T. \tag{28}$$

Multiply equation (28) by $w \in W^{1,1+\gamma}(\Omega)$ and integrate in $\Omega$ to obtain

$$\begin{aligned}
\langle \phi(v)(T) - \phi(v_0), w\rangle &= \int_0^T \langle \phi(v)_t, w\rangle \\
&= \lim_{\mu\to\infty} \int_0^T (\phi(v_\mu)_t, w) \\
&= \lim_{\mu\to\infty} (\phi(v_\mu)(T) - \phi(v_{0,\mu}), w) \\
&= \langle \xi - \phi(v_0), w\rangle\,.
\end{aligned}$$

Since $w$ is arbitrary, we conclude that

$$\phi(v)(T) = \xi. \tag{29}$$

## Step 4. Monotonicity argument

It only remains to show that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}$$

in equation (27). For that purpose, recall by the monotonicity Lemma A.1 in Appendix A that for any $w \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$

$$X_\mu \equiv \eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right) \ge 0$$

which we can rewrite as

$$X_\mu = T_{1,\mu} + T_{2,\mu}$$

where

$$T_{1,\mu} = \eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla v_\mu \right)$$

and

$$T_{2,\mu} = -\eta^\gamma \int_0^T \left( \frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla w \right) - \eta^\gamma \int_0^T \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right).$$

Note that

$$\limsup_\mu X_\mu = \limsup_\mu T_{1,\mu} + \limsup_\mu T_{2,\mu} \geq 0. \tag{30}$$

From (20) and (23) one can easily see that

$$\limsup_\mu T_{2,\mu} = \lim_\mu T_{2,\mu} = -\eta^\gamma \int_0^T (\chi, \nabla w) - \eta^\gamma \int_0^T \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w \right). \tag{31}$$

For the term $T_{1,\mu}$ one needs to be more careful. Using equation (15)

$$\begin{aligned} T_{1,\mu} &= -\int_0^T (\phi(v_\mu)_t, v_\mu) + \int_0^T (f\ , v_\mu) \\ &= -\frac{\eta}{\eta+1} \int_0^T \frac{d}{dt} \|\phi(v_\mu)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T (f\ , v_\mu) \\ &= \frac{\eta}{\eta+1} \|\phi(v_{0,\mu})\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \frac{\eta}{\eta+1} \|\phi(v_\mu)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T (f\ , v_\mu). \end{aligned}$$

Since by (29) and a well know property of weak limits

$$\|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)} = \|\xi\|_{L^{(1+\eta)^*}(\Omega)} \leq \liminf_\mu \|\phi(v_\mu)(T)\|_{L^{(1+\eta)^*}(\Omega)}.$$

Thus, we are lead to

$$\limsup_\mu T_{1,\mu} \leq \frac{\eta}{\eta+1} \left( \|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f\ , v).$$

Now, substitute $v$ for $w$ in (27). Perform the integration in time to find that

$$\eta^\gamma \int_0^T (\chi, \nabla v) = \frac{\eta}{\eta+1} \left( \|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f\ , v). \tag{32}$$

Thus, from (30), (31) and (32) we observe that

$$\int_0^T \left( \chi - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w \right) \geq 0$$

if we choose $w = v - \lambda\psi$ for $\lambda > 0$ and $\psi \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$ in the previous equation, then

$$\int_0^T \left( \chi - \frac{\nabla(v - \lambda\psi)}{|\nabla(v - \lambda\psi)|^{1-\gamma}}, \nabla\psi \right) \geq 0.$$

Taking the limit as $\lambda \to 0$ we finally obtain that

$$\int_0^T \left( \chi - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla\psi \right) \geq 0$$

which implies by Lebesgue's lemma that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}.$$

The previous fact completes the proof of Theorem 3.1 for any $\phi_{reg}$. Next, take $\{\phi_k\}_{k=1}^{\infty}$ to be a sequence of regularized functions converging uniformly to $\phi(x) = x/|x|^{1-\eta}$. Then,*a priori* estimates (16) and (17), which are independent of $k$, hold for the sequences $\{\phi_k(v_k)\}$ and $\{v_k\}$. Whence, *Step 3* and *Step 4* can be identically performed to find that $v$ defined as

$$v = \lim_{k\to\infty} v_k$$

is a weak solution of the problem for the non regular $\phi$. □

**Corollary 3.1.** *There exists a weak solution to problem (4), where the gradient of $u$ is understood as the pointwise gradient.*

*Proof.* Let $v$ be a weak solution of problem (3) with initial condition $v_0 = \phi^{-1}(u_0)$ and let $u = \phi(v)$. Immediately, the following holds:

*(i)* $u = 0$ in $(0,T) \times \partial\Omega$,

*(ii)* $u(0) = \phi(v(0)) = \phi(v_0) = \phi(\phi^{-1}(u_0)) = u_0$,

*(iii)* $\phi(v)_t = u_t$.

It only remains to show that the weak gradient of $v$ and the pointwise gradient of $u$ are related by

*(iv)* $\nabla v = (\phi^{-1})'(u)\nabla u \qquad$ a.e. in $(0,T)\times\Omega$.

For this purpose, observe that since $v \in L^{1+\gamma}(0,T; W_0^{1,(1+\gamma)}(\Omega))$ there exists a sequence $v_m \in L^{1+\gamma}(0,T; C^{\infty}(\Omega))$ such that

$$v_m \to v \quad \text{strongly in} \quad L^{1+\gamma}(0,T; L^{1+\gamma}(\Omega)) \qquad \text{and a.e. in } (0,T)\times\Omega.$$

Define the sequence $u_m = \phi(v_m)$. Since $v_m \in L^{1+\gamma}(0,T; C^{\infty}(\Omega))$, we have that the following relation holds true a.e.

$$\nabla u_m = \begin{cases} \phi'(v_m)\nabla v_m & \text{if} \quad |v_m| > 0 \\ 0 & \text{if} \quad v_m = 0. \end{cases}$$

Therefore,

$$u_m \to u \qquad \text{and} \qquad \nabla u_m \to \nabla u \qquad \text{a.e. in } (0,T)\times\Omega.$$

In addition, $v_m = \phi^{-1}(u_m)$, thus

$$\nabla v_m = (\phi^{-1})'(u_m)\,\nabla u_m \qquad \text{a.e. in } (0,T)\times\Omega.$$

Sending $m \to \infty$ in the previous expression, we find that

$$\nabla v = (\phi^{-1})'(u)\,\nabla u \qquad \text{in } L^{1+\gamma}(0,T; L^{1+\gamma}(\Omega)).$$

To conclude the proof, substitute *(iii)* and *(iv)* in equation (8) to obtain equation (10). □

**Remark 3.2.** As pointed out in equation (5) and (7) an immediate consequence of Corollary 3.1 is that if $u$ is a nonnegative solution of problem (3) then it solves problem (1) in the sense of Definition 2.2.

**Corollary 3.2.** *Let $v$ a weak solution of the initial/boundary value problem (3). Then for any $w \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$*

$$\langle \phi(v)_t, w\rangle + \left(\frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w\right) = (f,w) \quad a.e.\ in\ [0,T].$$

*Proof.* Fix $w \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$ and let $\{w_j\}$ be a basis for $W_0^{1,(1+\gamma)}(\Omega)$. Take a sequence $\{\psi_m\}$ of the form

$$\psi_m = \sum_{j=1}^{m} d_j^m(t) w_j \ \text{ with } \ d_j^m(t) \in L^\infty([0,T])$$

such that $\psi_m \to w$ strongly in $L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$. This is possible by density of such finite sums in the mentioned space.

Since $v$ is weak solution of problem (3) we get

$$\langle \phi(v)_t, \psi_m\rangle + \left(\frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla \psi_m\right) = (f,\psi_m) \quad \text{a.e. in } [0,T].$$

Send $m \to +\infty$ to conclude. □

# 4 Regularity

In this section we investigate basic regularity properties of solutions found in the existence Theorem (3.1). It is desirable to find more information on the time derivative of the function $\phi(v)$, in particular, it is worthwhile to find that it is a regular distribution. For this purpose let us first introduce what we understand by a Faedo-Galerkin solution.

**Definition 4.1.** *A weak solution of problem (3) is called a Faedo-Galerkin solution if it can be constructed by a sequence of Faedo-Galerkin approximates.*

Theorem (4.1) will give insight about the regularity of $v_t$ and $\phi(v)_t$. It shows in particular, that as long as $\phi$ is regular, we have in fact that $\phi(v)_t$ is a regular distribution. However, in the case when $\phi$ is not regular, it is not clear whether this result holds true.

**Theorem 4.1.** *Assume*

$$v_0 \in W_0^{1,1+\gamma}(\Omega), \quad and \quad f \in L^{(1+\eta)*}(0,T;L^{(1+\eta)*}(\Omega)).$$

*Let $v$ a Faedo-Galerkin solution of problem (3), then*

*(i) $v \in L^\infty(0,T;W_0^{1+\gamma}(\Omega))$,*

*(ii) $v_t$ exists as a regular distribution that lies in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$*

*with the estimate*

$$\int_{\{|v|>0\}} \left(\phi'(v)^{1/2} v_t\right)^2 + \sup_{[0,T]} \|\nabla v(t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \leq C\left(T,\ \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))},\ \|v_0\|_{W_0^{1,1+\gamma}(\Omega)}\right). \quad (33)$$

*Moreover, when $\phi$ is regular then $\phi(v)_t$ also lies in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$ and*

$$\phi(v)_t = \phi'(v)v_t. \quad (34)$$

*Proof.* Let $v_m(t)$ be a smooth Faedo-Galerkin approximation sequence of problem (3) with $v_{0,m} \to v_0$ in $W_0^{1,1+\gamma}(\Omega)$. Then, multiply equation (15) by $\zeta_j(t)_t$ and sum for $1 \leq j \leq m$ to obtain

$$(\phi(v_m)_t, (v_m)_t) + \eta^\gamma \left(\frac{\nabla v_m}{|\nabla v_m|^{1-\gamma}}, \nabla (v_m)_t\right) = (f, (v_m)_t).$$

Hence,

$$\left\|\frac{\phi(v_m)_t}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + \frac{\eta^\gamma}{1+\gamma}\frac{d}{dt}\|\nabla v_m\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} = (f, (v_m)_t).$$

In addition, note that

$$(f, (v_m)_t) \leq 1/2\left\|\frac{f}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + 1/2\left\|\frac{\phi(v_m)_t}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)}.$$

Thus, combining the last two relations we get

$$1/2\left\|\frac{\phi(v_m)_t}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + \frac{\eta^\gamma}{1+\gamma}\frac{d}{dt}\|\nabla v_m\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \leq 1/2\left\|\frac{f}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)}. \quad (35)$$

Integrating (35) in time from 0 to $T$, we obtain that

$$1/2\int_0^T \left\|\frac{\phi(v_m)_t}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + \frac{\eta^\gamma}{1+\gamma}\sup_{[0,T]}\|\nabla v_m(t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \leq 1/2\int_0^T \left\|\frac{f}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + \|\nabla v_{0,m}\|^{1+\gamma}_{L^{1+\gamma}(\Omega)}. \quad (36)$$

By the hypothesis imposed on $f$, the right hand side of (36) converges to

$$1/2\int_0^T \left\|\frac{f}{\phi'(v)^{1/2}}\right\|^2_{L^2(\Omega)} + \|\nabla v_0\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \quad \text{as} \quad m \to \infty.$$

This immediately implies that the right hand side is bounded. Because of the nonlinearities that occur in the left hand side of (36), it is not straightforward to send $m \to \infty$ to establish estimate (33). For this purpose, we will first establish a weak convergence result for the sequence $\{(v_m)_t\}$ in the following way.
Observe that since $\phi(v_m)_t = \phi'(v_m)(v_m)_t$ then

$$\int_0^T \|(v_m)_t\|^{1+\eta}_{L^{1+\eta}(\Omega)} \leq \frac{1+\eta}{2}\int_0^T \left\|\frac{\phi(v_m)_t}{\phi'(v_m)^{1/2}}\right\|^2_{L^2(\Omega)} + \frac{1-\eta}{2}\int_0^T \|1/\phi'(v_m)\|^q_{L^q(\Omega)} \quad (37)$$

where $q = (1+\eta)/(1-\eta)$. Note that

$$\phi'(x) = \frac{\eta}{|x|^{1-\eta}}, \tag{38}$$

therefore

$$\frac{1}{\phi'(x)} = \frac{|\phi(x)|^{\frac{1-\eta}{\eta}}}{\eta} \quad \text{and} \quad \int_0^T \left\|1/\phi'(v_m)\right\|^q_{L^q(\Omega)} = \frac{1}{\eta^{\,q}} \|\phi(v_m)\|^{(1+\eta)^*}_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}. \tag{39}$$

Hence, as a consequence of (36), (37), and (39) the sequence $\{(v_m)_t\}$ is bounded in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$. Thus, there exists a subsequence of $\{(v_m)_t\}$, labeled with the index $\mu$ such that

$$(v_\mu)_t \rightharpoonup v_t \ \text{ weakly in } L^{1+\eta}(0,T;L^{1+\eta}(\Omega)) \text{ as } \ \mu \to +\infty. \tag{40}$$

Second, define for all $\epsilon > 0$ and $k \geq 1$ the set

$$\Omega_{k,\epsilon} := \bigcap_{j\geq k}^{+\infty} \{[0,T]\times\Omega : |v_j| \geq \epsilon\}.$$

Thus,

$$\begin{aligned}\int_0^T \left\|\frac{\phi(v_\mu)_t}{\phi'(v_\mu)^{1/2}}\right\|^2_{L^2(\Omega)} &= \int_0^T \left\|\phi'(v_\mu)^{1/2}\ (v_\mu)_t\right\|^2_{L^2(\Omega)} \\ &\geq \int_{\Omega_{k,\epsilon}} \left(\phi'(v_\mu)^{1/2}\ (v_\mu)_t\right)^2. \end{aligned} \tag{41}$$

Now, in $\Omega_{k,\epsilon}$ we have the bound $\phi'(v_\mu)^{1/2} \leq \eta\ \epsilon^{\eta-1}$ for $\mu \geq k$ and clearly,

$$\phi'(v_\mu)^{1/2} \to \phi'(v)^{1/2} \ \text{ a.e. in } \Omega_{k,\epsilon}.$$

Using this fact with (40) we obtain

$$\phi'(v_\mu)^{1/2}\ (v_\mu)_t \rightharpoonup \phi'(v)^{1/2}\ v_t \ \text{ weakly in } L^{1+\eta}(\Omega_{k,\epsilon}).$$

Therefore, taking $\liminf_{\mu\to+\infty}$ in (41) and using the weakly lower semicontinuity property of convex functionals on $L^p$ it follows that

$$\int_{\Omega_{k,\epsilon}} \left(\phi'(v)^{1/2}\ v_t\right)^2 \leq \liminf_{\mu\to+\infty} \int_0^T \left\|\frac{\phi(v_\mu)_t}{\phi'(v_\mu)^{1/2}}\right\|^2_{L^2(\Omega)}. \tag{42}$$

As $v_j \to v$ a.e. in $[0,T]\times\Omega$, it follows that

$$\lim_{k\to\infty,\epsilon\to 0} \Omega_{k,\epsilon} = \{|v| > 0\}.$$

Hence, taking these limits in (42) we obtain

$$\int_{\{|v|>0\}} \left(\phi'(v)^{1/2} v_t\right)^2 \leq \liminf_{\mu\to+\infty} \int_0^T \left\|\frac{\phi(v_\mu)_t}{\phi'(v_\mu)^{1/2}}\right\|^2_{L^2(\Omega)}. \tag{43}$$

This takes care of the first term in (36). The second term of the left hand side is simpler to deal with. Note that by (36) there exist a subsequence of $\{v_m\}$, labeled again with the index $\mu$, such that

$$v_\mu \rightharpoonup \xi \text{ in } L^\infty(0,T;W_0^{1,1+\gamma}(\Omega)) \text{ weak}^*.$$

Since the sequence already converged weakly in $L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$ to $v$, we conclude that $\xi = v$. Therefore, we can take $\liminf_{\mu\to+\infty}$ in (36) to obtain

$$\begin{aligned} 1/2\int_{\{|v|>0\}} \left(\phi'(v)^{1/2}v_t\right)^2 + \frac{\eta^\gamma}{1+\gamma} \sup_{[0,T]} \|\nabla v(t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \\ \leq 1/2\int_0^T \left\|\frac{f}{\phi'(v)^{1/2}}\right\|^2_{L^2(\Omega)} + \|\nabla v_0\|^{1+\gamma}_{L^{1+\gamma}(\Omega)}. \quad (44) \end{aligned}$$

To get estimate (33), observe that using the first expression in (39) one can prove, using Hölder's inequality, that

$$\int_0^T \left\|\frac{f}{\phi'(v)^{1/2}}\right\|^2_{L^2(\Omega)} \leq \|f\|^2_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))} \|\phi(v)\|^{\frac{1-\eta}{\eta}}_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}$$

which together with estimate (16) prove *(i)*, *(ii)* and estimate (33). Finally when $\phi$ is regular, it is Lipchitz, then the chain rule formula in (34) follows by a standard result for Sobolev functions. □

**Corollary 4.1.** *Assume the conditions of Theorem (4.1). Then for any regular* $\phi$*,*

$$\phi(v)_t \in L^2(0,T;L^2(\Omega)),$$

*and the following estimate holds*

$$\|\phi(v)_t\|^2_{L^2(0,T;L^2(\Omega))} \leq C\left(T,\ \phi'(0),\ \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))},\ \|v_0\|_{W_0^{1,1+\gamma}(\Omega)}\right).$$

*Proof.* The conditions on $\phi(x)$ imply that for any $x \in \Re$

$$1 \leq \frac{\phi'(0)}{\phi'(x)}$$

thus, after applying the chain rule *(iii)* in Theorem 4.1, it follows that

$$\begin{aligned} \int_{\{|v|>0\}} \phi(v)_t^2 &= \int_{\{|v|>0\}} \left(\phi'(v)v_t\right)^2 \\ &\leq \phi'(0)\int_{\{|v|>0\}} \left(\phi'(v)^{1/2}v_t\right)^2. \end{aligned}$$

In addition, observe that in the set $\{v=0\}$ we have that $\phi(v) = 0$. Hence, a direct calculation shows that $\phi(v)_t = 0$ in the interior of this set. But $\phi(v)_t$ is measurable, therefore

$$\int_{\{v=0\}} \phi(v)_t^2 = 0.$$

Consequently,

$$\|\phi(v)_t\|^2_{L^2(0,T;L^2(\Omega))} \leq \phi'(0)\int_{\{|v|>0\}} \left(\phi'(v)^{1/2}v_t\right)^2.$$

Using estimate (33) in Theorem (4.1) we conclude the proof. □

**Theorem 4.2.** *Assume $v$ is a Faedo-Galerkin solution of problem (3), and additionally assume that*

$$v_0 \in L^\infty(\Omega) \qquad \text{and} \qquad f \in L^\infty(0,T;L^\infty(\Omega)),$$

*then*

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\left(\|v_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T\right). \tag{45}$$

*Proof.* In order to find an $L^\infty$ bound on $v$, we would like to uniformly control its $L^p$ norms. For this purpose, the key idea would be to multiply equation (3) by the test function $v/|v|^{1-s}$ for any $s \geq 1$ and use Gronwall's Lemma to establish the result. However, for a fixed time $t$, the test function $v/|v|^{1-s}$ does not necessarily belong to $W_0^{1,1+\gamma}(\Omega)$, so that we need to regularize it. For this end, let us introduce the family $\{\rho_\delta(x)\}_{\delta>0}$ approximating the function $x/|x|^{1-s}$

$$\rho_\delta(x) = \frac{1}{(1+\delta|x|)^s}\,\frac{x}{|x|^{1-s}}.$$

Note that $\rho_\delta(v)(t) \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$ since $\rho_\delta(x)$ is a $C^1([0,\infty))$ function with bounded derivative.

Using Corollary (3.2) we can chose $\rho_\delta(v)$ as a test function in equation (3). Observe that for any regular $\phi$, the Faedo-Galerkin solution $v$ has time derivative $v_t \in L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$ by Theorem (4.1), whence the chain rules applies,

$$\phi(v)_t = \phi'(v)v_t.$$

Therefore, the following relation holds immediately

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} = \langle \phi(v)_t, \Phi_\delta(v)\rangle$$

where

$$\Phi_\delta(x) = \int_0^x \phi'(z)\rho_\delta(z).$$

Thus, we obtain

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} + \eta^\gamma(|\nabla v|^{1+\gamma}, \rho_\delta'(v)) = (f, \rho_\delta(v)). \tag{46}$$

The second term in the left hand side of (46) is nonnegative, thus the following inequality holds

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} \leq \|f\|_{L^\infty(\Omega)}\|\rho_\delta(v)\|_{L^1(\Omega)}.$$

Using the fact that

$$|\rho_\delta(x)| \leq 1 + \frac{\eta+s}{\eta}\Phi_\delta(x),$$

we obtain from the previous relation that

$$\frac{d}{dt}X_\delta(t) \leq \|f(t)\|_{L^\infty(\Omega)}\left(|\Omega| + \frac{\eta+s}{\eta}X_\delta(t)\right),$$

where

$$X_\delta(t) = \|\Phi_\delta(v)(t)\|_{L^1(\Omega)}.$$

Using Gronwall's lemma we get

$$X_\delta(t) \leq \exp\left(\frac{\eta+s}{\eta}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right)\left\{X_\delta(0) + \|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right\}. \tag{47}$$

Inequality (47) is valid for any $\phi_{reg}$, thus, it is also valid for $\phi = x/|x|^{1-\eta}$. Similarly, observe that

$$\Phi_\delta(v)(t) \longrightarrow \frac{\eta}{\eta+s} v^{\eta+s}(t) \text{ pointwise as } \delta \to 0 \text{ in } [0,T]\times\Omega.$$

Thus, sending $\delta \to 0$ in (47) and using Fatou's Lemma it follows that

$$\frac{\eta}{\eta+s}\|v(t)\|^{\eta+s}_{L^{s+\eta}(\Omega)} \leq \exp\left(\frac{\eta+s}{\eta}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}\,T\right)\left\{\frac{\eta}{\eta+s}\|v_0\|^{s+\eta}_{L^{s+\eta}(\Omega)} + |\Omega|\,\|f\|_{L^\infty(0,T;L^\infty(\Omega))}\,T\right\}. \tag{48}$$

Taking the $\eta+s$ *root* in (48) and letting $s\to\infty$ we find that for $0\leq t\leq T$

$$\|v(t)\|_{L^\infty(\Omega)} \leq \exp\left(\eta^{-1}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}\,T\right)\max\left(1, \|v_0\|_{L^\infty(\Omega)}\right)$$

which proves the result. □

# 5 Uniqueness

Generally speaking, if $v$ is a weak solution of problem (3) some basic regularity on $\phi(v)_t$ must be obtained for pursuing a uniqueness result, otherwise this task can be very complex. Moreover, uniqueness may not be true. In Theorem (5.1) we will prove uniqueness under the assumption that

$$\phi(u)_t \in L^1(0,T;L^1(\Omega)). \tag{49}$$

In a hydrologic context, the previous assumption can be interpreted in the following way. Condition (49) implies that $u_t \in L^1(0,T;L^1(\Omega))$ in problem (1). Recall that $u$ represents the free surface elevation, or the column of water at a given point in the domain $\Omega$ in a physical system. Thus the volume $\mathcal{V}$ of water in $\Omega$ may be represented as

$$\mathcal{V}(\Omega,t) = \int_\Omega u(t).$$

Condition (49) implies that the the volume in the domain $\Omega$ changes continiuously in time. This is a natural condition when modeling hydrologic systems. The fact that the volume is a time continuous function follows from the observation that the time weak derivative of $\mathcal{V}$ exists and is given by

$$\mathcal{V}_t(\Omega,t) = \int_\Omega u_t(t) \quad \in L^1(0,T),$$

thus $\mathcal{V} \in W^{1,1}(0,T)$. As a consequence of Morrey's inequality, $\mathcal{V}$ is a Hölder continuous function in $[0,T]$.

**Theorem 5.1.** *Assume $u$ and $v$ are weak solutions of problem (3) with the additional property that*

$$\phi(u)_t,\ \phi(v)_t \in L^1(0,T;L^1(\Omega)), \tag{50}$$

*then $u=v$.*

*In particular, problem (3) has a unique Faedo-Galerkin solution as long as $\phi$ is regular.*

*Proof.* Since $u$ and $v$ are weak solutions of problem (3) then

$$\langle \phi(v)_t - \phi(u)_t, w\rangle + \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}} - \frac{\nabla u}{|\nabla u|^{1-\gamma}}, \nabla w \right) = 0 \tag{51}$$

for any $w \in W_0^{1,(1+\gamma)}(\Omega))$. Let $\{\beta_\delta(x)\}_{\delta>0}$ be the family of $C^1(\mathbb{R})$ increasing functions such that,

*(i)* $|\beta_\delta(x)| \le 1$, and

*(ii)* $\beta_\delta(x) \longrightarrow \mathrm{sgn}(x)$ as $\delta \to \infty$.

Substituting $w = \beta_\delta(v-u)$ in (51) we find that

$$\langle \phi(v)_t - \phi(u)_t, \beta_\delta(v-u)\rangle + \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}} - \frac{\nabla u}{|\nabla u|^{1-\gamma}}, \beta'_\delta(v-u)\nabla(v-u) \right) = 0.$$

Since $\beta'_\delta(v-u) \ge 0$, by Lemma A.1 in Appendix A, the second term in the previous expression is nonnegative, thus

$$\int_0^t \langle \phi(v)_t - \phi(u)_t, \beta_\delta(v-u)\rangle \le 0.$$

Note that $\{\beta_\delta(v-u)\} \subset L^\infty(0,T;L^\infty(\Omega))$. But $\phi(v)_t$ and $\phi(u)_t$ lie in $L^\infty(0,T;L^\infty(\Omega))^*$ by assumption, thus

$$\langle \phi(v)_t - \phi(u)_t, \beta_\delta(v-u)\rangle = \left(\phi(v)_t - \phi(u)_t, \beta_\delta(v-u)\right).$$

Using Lebesgue's Dominated Convergence Theorem we can take the limit as $\delta \to \infty$ in the above inequality to find that

$$\int_0^t (\phi(v)_t - \phi(u)_t, \mathrm{sgn}(v-u)) \le 0.$$

Observe that since $\mathrm{sgn}(v-u) = \mathrm{sgn}(\phi(v)-\phi(u))$, then for $0 \le t \le T$,

$$\begin{aligned} \int_0^t (\phi(v)_t - \phi(u)_t, \mathrm{sgn}(v-u)) &= \int_0^t ((\phi(v)-\phi(u))_t, \mathrm{sgn}(\phi(v)-\phi(u))) \\ &= \int_0^t \frac{d}{dt}\|\phi(v)-\phi(u)\|_{L^1(\Omega)} \\ &= \|\phi(v)(t)-\phi(u)(t)\|_{L^1(\Omega)} \le 0, \end{aligned} \tag{52}$$

from which we conclude that $u = v$ a.e. in $[0,T]\times\Omega$.

Finally, if $\phi$ is regular we know from Corollary (4.1) that Faedo-Galerkin solutions satisfy

$$\phi(u)_t,\ \phi(v)_t \in L^2(0,T;L^2(\Omega)) \subset L^1(0,T;L^1(\Omega)).$$

Hence, the previous result applies for them. □

**Remark 5.1.** By hypothesis $\phi(v) \in C([0,T];L^1(\Omega))$ since $\phi(v) \in W^{1,1}([0,T];L^1(\Omega))$. See Theorem A.1 in Appendix A. Thus, the last step in (52) can be safely performed.

## 6 Positivity

In this section we will study the conditions for which a classical solution mantains the sign of the initial condition. Recall that a classical solution to problem (3) is any function $v \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfying equation (3) pointwise. Here we have introduced, for the sake of simplicity, the rather standard notation

$$C_1^2(\Omega_T) = \{v : \Omega_T \to \mathbb{R}, \text{ such that } v,\ D_x v,\ D_x^2 v,\ v_t \in C(\overline{\Omega_T})\}$$

where $\Omega_T = \Omega \times (0, T]$ is usually known as the parabolic cylinder. We will also use in the sequel the parabolic boundary, defined as $\Gamma = \overline{\Omega_T} \setminus \Omega_T$.
We will first prove two auxiliary lemmas and then we will formulate the main theorem of the section.

**Lemma 6.1.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^n$ and $v \in C^2(\Omega) \cup C(\bar{\Omega})$. Then*

*(i) If $x_0 \in \Omega$ is a local maximum of $v$*

$$-\nabla \cdot \left( \frac{\nabla v(x_0)}{|\nabla v(x_0)|^{1-\gamma}} \right) \geq 0$$

*(ii) If $x_0 \in \Omega$ is a local minimum of $v$*

$$-\nabla \cdot \left( \frac{\nabla v(x_0)}{|\nabla v(x_0)|^{1-\gamma}} \right) \leq 0$$

*Proof.* Let $x_0$ be a local maximum, then the Hessian matrix of $v$ is negative definite in $x_0$, i.e.

$$D^2 v(x_0) \leq 0, \tag{53}$$

thus, in particular $\Delta v(x_0) \leq 0$.

A direct computation shows that,

$$\begin{aligned} -\nabla \cdot \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}} \right) &= \frac{1-\gamma}{|\nabla v|^{3-\gamma}} \sum_{i,j=1}^{n} v_{x_i} v_{x_i x_j} v_{x_j} - \frac{\Delta v}{|\nabla v|^{1-\gamma}} \\ &= \frac{1-\gamma}{|\nabla v|^{3-\gamma}} \nabla v^{\,T} \, D^2 v \, \nabla v - \frac{\Delta v}{|\nabla v|^{1-\gamma}}. \end{aligned} \tag{54}$$

Since $D^2 v(x_0)$ is negative definite we can use Lemma A.2 to conclude that

$$\nabla v(x_0)^{\,T} \, D^2 v(x_0) \, \nabla v(x_0) \geq |\nabla v(x_0)|^2 \Delta v(x_0).$$

Hence, after evaluating equation (54) at $x_0$ we get

$$-\nabla \cdot \left( \frac{\nabla v(x_0)}{|\nabla v(x_0)|^{1-\gamma}} \right) \geq - \frac{\gamma}{|\nabla v(x_0)|^{1-\gamma}} \, \Delta v(x_0) \geq 0.$$

For *(ii)* apply the previous result for the function $-v$. □

**Lemma 6.2.** *Let $w \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ and $\epsilon > 0$. Define the sets*

$$W = \{(t,x) \in \Omega_T \,:\, |w - \epsilon t| > 0\} \quad \textit{and} \quad \partial W_0 = \partial W \setminus \Omega \times \{T\}.$$

*If*

$$\phi'(w - \epsilon t) w_t - \nabla \cdot \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}} \right) > 0 \quad \textit{in} \quad W, \tag{55}$$

*then*

$$\min_{W} w = \min_{\partial W_0} w. \tag{56}$$

*Proof.* Assume that $(t_0, x_0) \in W$ with $0 < t < T$ is a local minimum of $w$, then at this point

$$\phi'(w - \epsilon t_0) w_t = 0,$$

and by Lemma (6.1)

$$-\nabla \cdot \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}} \right) \leq 0, \tag{57}$$

which is not possible by inequality (55). Therefore, any local minimum lies in $\partial W$.

Now suppose the local minimun lies in $(T, x_0) \in \partial W \cap \Omega \times \{T\}$, then we must have at this boundary point

$$\phi'(w - \epsilon T) w_t \leq 0.$$

Since we still have the inequality (57) at this point, we deduce once more by contradiction that this assumption is not possible. Therefore any local minimun must lie in $\partial W_0$. □

**Theorem 6.1.** *Let $v \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfying the equation*

$$\phi(v)_t - \nabla \cdot \left( \frac{\nabla v}{|\nabla v|^{1-\gamma}} \right) = f \quad \textit{in} \quad \Omega_T \tag{58}$$

*with*

*(i) $f \geq 0$ in $\Omega_T$,*

*(ii) $v(0,x) \geq 0$ with $x \in \Omega$ and,*

*(iii) $v(t,x) \geq 0$ on $\partial\Omega \times [0,T]$.*

*then*

$$v \geq 0 \quad \textit{in} \quad \overline{\Omega}_T.$$

*Proof.* Define $w = v + \epsilon t$ for a fixed $\epsilon > 0$, then by equation (58)

$$\phi(w - \epsilon t)_t - \nabla \cdot \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}} \right) = f \quad \text{in } \Omega_T.$$

Note in particular that in the set $W = \{(t,x) \in \Omega_T \,:\, |w - \epsilon t| > 0\}$ the chain rule can be used in the first term of the previous expression to obtain

$$\phi'(w - \epsilon t) w_t - \nabla \cdot \left( \frac{\nabla w}{|\nabla w|^{1-\gamma}} \right) = f + \epsilon\, \phi'(w - \epsilon t) > 0 \quad \text{in } W.$$

Therefore, $w$ satisfies the conditions of Lemma 6.2, then

$$\min_{W} w = \min_{\partial W_0} w.$$

Observe that $\Omega_T \setminus W = \{(t,x) \in \Omega_T : w - \epsilon t = 0\}$ and that

$$\partial W_0 \subseteq \partial(\Omega_T \setminus W) \cup \Gamma.$$

But on the one hand, assumptions *(ii)* and *(iii)* imply

$$w\,|_\Gamma \geq 0,$$

and on the other hand, in the set $\Omega_T \setminus W$, we have that $w = \epsilon t \geq 0$. Then

$$\min_{\Omega_T \setminus W} w \geq 0 \quad \text{and} \quad \min_{\partial W_0} w \geq 0.$$

The previous results imply together with (56) that

$$\min_{\Omega_T} w \geq 0,$$

thus, from the definition of $w$,

$$v = w - \epsilon t \geq -\epsilon t \quad \text{in} \quad \Omega_T.$$

The above inequality is valid for any $\epsilon > 0$, hence $v \geq 0$. This concludes the proof. ☐

**Remark 6.1.** Assume that conditions of Theorem (6.1) hold with the sign " $\geq$ " replaced by the sign " $\leq$ " in conditions *(i)*, *(ii)* and *(iii)*. Then we can apply Theorem (6.1) to the function $-v$ to readily conclude that

$$v \leq 0 \quad \overline{\Omega}_T.$$

# A Appendix

**Lemma A.1.** *The operator $\mathcal{A}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by*

$$\mathcal{A}(x) = \frac{x}{|x|^{1-\gamma}} \tag{59}$$

*is monotone, i.e., for any $x, y \in \mathbb{R}^n$*

$$(\mathcal{A}(x) - \mathcal{A}(y)) \cdot (x - y) \geq 0.$$

*Proof.* Define the function $\mathcal{B}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\mathcal{B}(x) = |x|^{\gamma+1} \qquad \text{where} \qquad |x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}$$

and note that

$$\frac{\partial}{\partial x_i}|x|^{\gamma+1} = (\gamma+1)|x|^{\gamma-1}x_i \qquad \Longrightarrow \qquad \frac{1}{\gamma+1}\nabla\mathcal{B}(x) = \mathcal{A}(x).$$

Since $\gamma + 1 > 1$, the function $\mathcal{B}(x)$ is strictly convex. The gradient of a convex function is strictly increasing in each and all of its components, thus the result of the lemma holds true. ☐

**Lemma A.2.** *Let $A$ be a symmetric positive definite matrix. Then for any $x \in \mathbb{R}^n$*

$$|x|^2 \ \mathit{trace}(A) \geq x^T A\, x$$

*Proof.* Since A is symmetric and positive definite matrix, there exists an orthonormal matrix $P$ such that

$$PAP^T = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), \quad PP^T = I$$

with $\lambda_i > 0$ for $i = 1, \cdots, n$. Therefore, for any $x \in \mathbb{R}^n$ we apply the change of variables $y = Px$ to get

$$x^T A\, x = y^T PAP^T y \leq |y|^2 \mathrm{trace}(PAP^T) = |x|^2 \mathrm{trace}(A).$$

□

**Lemma A.3.** *Set $\phi(x) = x/|x|^{1-\eta}$. Let $v_m$ be a Faedo-Galerkin approximate solution of problem (3), then the following holds a.e. in $(0,T) \times \Omega$*

$$\phi(v_m)_t = \begin{cases} \phi'(v_m)(v_m)_t & \text{if} \quad |v_m| > 0 \\ 0 & \text{if} \quad v_m = 0 \end{cases} \tag{60}$$

*and*

$$(\phi(v_m)_t, v_m) = \frac{\eta}{1+\eta}\frac{d}{dt}\|\phi(v_m)\|^{(1+\eta)^*}_{L^{(1+\eta)^*}(\Omega)} \tag{61}$$

*Proof.* Recall that

$$v_m(t) = \sum_{j=1}^{m} \zeta_j(t) w_j(x)$$

where the functions $\{\zeta_j(t)\}$ are absolutely continuous functions since they solve a first order one dimensional ODE. The previous statement implies that the family of approximate solutions $\{v_m(t)\}$ are absolutely continuous as well. Thus, to obtain formula (60) proceed as follows: in the set $\{(t,x) : |v_m| > 0\}$ apply the chain rule, and in the set $\{(t,x) : |v_m| > 0\}$ note that $\phi(v_m)$ is equal to the constant zero, hence, in the interior of this set $\phi(v_m)_t = 0$.

In order to prove (61), observe on one hand that in the set $\{(t,x) : |v_m| > 0\}$

$$\begin{aligned} \phi(v_m)_t\, v_m = \phi'(v_m)(v_m)_t\, v_m &= \frac{\eta}{1+\eta} \left(|v_m|^{1+\eta}\right)_t \\ &= \frac{\eta}{1+\eta} \left(|v_m|^{\eta(1+\eta)^*}\right)_t \\ &= \frac{\eta}{1+\eta}\frac{d}{dt}|\phi(v_m)|^{(1+\eta)^*}. \end{aligned}$$

On the other hand observe that in the set $\{v = 0\}$ both terms in (61) are equal to 0. □

**Theorem A.1.** *(Calculus in abstact space) Let $X$ a Banach space and let $u \in W^{1,p}(0,T;X)$ for some $1 \leq p \leq \infty$. Then*

*(i) $u \in C([0,T];X)$ (after possibly being redefined on a set of measure zero), and*

*(ii) $u(t_1) = u(t_0) + \int_{t_0}^{t_1} u_t(\tau)d\tau$ for all $0 \leq t_0 \leq t_1 \leq T$.*

*Proof.* See [2]. □

Assume that $\Omega$ is an open, bounded set, with smooth boundary, and $T > 0$. We have

**Theorem A.2.** *Let $\psi$ a continuous real valued function, and $0 < \eta \leq \gamma < 1$. Assume that*

*(i)* $|\psi(x)| \leq |x|^\eta$

*(ii)* $u_\mu \rightharpoonup u$ *in* $L^{1+\gamma}(0, T; W^{1,1+\gamma}(\Omega))$

*(iii)* $\psi(u_\mu)_t \rightharpoonup v$ *in* $L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega))$

*then,* $v = \psi(u)_t$.

*Proof.* Set $p = (1 + \gamma)/\eta$. Note that by (i) we have

$$\|\psi(u_\mu)\|^p_{L^p(0,T;L^p(\Omega))} \leq \|u_\mu\|^{1+\gamma}_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))} . \tag{62}$$

Since $L^p(0, T; L^p(\Omega))$ is a separable and reflexive Banach space (62) implies that

$$\psi(u_\mu) \rightharpoonup \xi \ \text{ weakly in } \ L^p(0, T; L^p(\Omega)). \tag{63}$$

Since

$$\frac{1+\gamma}{\eta} \geq \frac{1+\gamma}{\gamma} = (1+\gamma)^*,$$

it follows that

$$L^p(0, T; L^p(\Omega)) \subset L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)^*}(\Omega)) \subset L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)).$$

Then, for any $\varphi \in C_c^1(0, T)$ and $w \in W^{1,1+\gamma}(\Omega)$ we obtain

$$\int_0^T \langle \psi(u_\mu)_t, \varphi\, w\rangle = -\int_0^T (\psi(u_\mu), \varphi_t\, w) . \tag{64}$$

Next, using (ii) and the Rellich-Kondrachow Compactness Theorem it is possible to obtain a subsequence $\{u_{\mu'}\}$ of $\{u_\mu\}$ such that

$$u_{\mu'} \to u \text{ strongly in } \ L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)).$$

Therefore,

$$\psi(u_{\mu'}) \to \psi(u) \ \text{ a.e. in } \ [0, T] \times \Omega.$$

Combine this with (63) to conclude that $\xi = \psi(u)$. In this way we can take $\mu \to \infty$ in (64) to conclude that $v = \psi(u)_t$. □

**Theorem A.3.** *Assume that $\Omega$ is measurable and $|\Omega| < \infty$. Assume also that $f \in L^p(\Omega)$ for any $1 \leq p < \infty$ and $\|f\|_{L^p(\Omega)} \leq M$ for some $M > 0$. Then*

$$f \in L^\infty(\Omega) \qquad \textit{and} \qquad \|f\|_{L^\infty(\Omega)} \leq M. \tag{65}$$

*Proof.* See [10, p. 126] for a version of this result. A slight modification of this proof will work for this version. □

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