Analysis of a Multiscale Discontinuous Galerkin Method for Convection Diffusion Problems

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Abstract

We study a multiscale discontinuous Galerkin method introduced in [10] that reduces the computational complexity of the discontinuous Galerkin method, seemingly without adversely affecting the quality of results. For a stabilized variant we are able to obtain the same error estimates for the advection-diffusion equation as for the usual discontinuous Galerkin method. We assess the stability of the unstabilized case numerically and find that the inf-sup constant is positive, bounded uniformly away from zero, and very similar to that for the usual discontinuous Galerkin method.

1 Introduction

The discontinuous Galerkin method has undergone rapid development in recent years (see, e.g., [6] and [5]). Although it has been shown to possess advantageous properties in a number of circumstances, its practical utility has been limited by the much larger number of degrees-of-freedom it requires compared with continuous Galerkin methods [8]. This problem has persisted since the inception of the method and has only been recently addressed with the development of a multiscale discontinuous Galerkin method [10] that has the computational structure of a continuous method. The new method utilizes local, element-wise problems to develop a transformation between the parameterization of the discontinuous space and a related, smaller, continuous space. The transformation enables a direct construction of the global matrix problem in terms of the degrees-of-freedom of the continuous space. In the multiscale interpretation, the continuous field is viewed as the coarse scales and the discontinuous field is viewed as the sum of the coarse and fine scales. The discontinuous part of the solution can be determined by element-wise post-processing of the continuous solution. In [10] it was shown numerically that the new method at least retains the quality of the discontinuous Galerkin method, and in some instances improves upon it, while at the same time it has the potential to significantly reduce computational cost. A more general framework encompassing the ideas is presented in [3].

In this paper we initiate the mathematical analysis of the method developed in [10]. In Section 2 we present the boundary-value problem under consideration, namely, advection-diffusion, and give general definitions necessary for subsequent developments. In Section 3 we introduce a discontinuous Galerkin (DG) method that employs interior penalty stabilization and allows for symmetric, neutral, and skew-symmetric treatment of element interface terms corresponding to the diffusion operator. We also introduce a stabilized variant (SDG) that accounts for control of the streamline derivative on element interiors. The DG method is shown to be coercive with respect to the norm induced by its bilinear form, referred to as the DG-norm, and, likewise, the SDG method is shown to be coercive with respect to the SDG-norm induced by its bilinear form. However, the DG-norm is weak in that, in the advective limit, it only controls jumps on element interfaces. In [7], convergence of the DG method in the DG-norm was proved by utilizing the L^2-interpolant, circumventing the need for a stronger stability condition. Here we prove that the DG method is inf-sup stable with respect to the SDG-norm and this enables us to prove its convergence in the SDG norm by standard means.

In Section 4 we present the multiscale generalizations of DG and SDG, referred to as MDG and SMDG, respectively. We define the local, element-wise problems, which amount to the DG method on individual elements with weakly-enforced boundary conditions specified by the shared degrees-of-freedom of the continuous representation, and we define the "interscale transfer spaces" which emanate from the solutions of the local problems. We prove the inf-sup stability of the local problems in term of the SDG-norm, without streamline-derivative stabilization in the local problems. We also establish the approximation properties of the interscale transfer spaces. With these, and the fact that SDG is coercive on the discontinuous space, we are able to prove convergence and establish the same error estimates for SMDG as SDG (and DG). However, the proof for MDG poses an additional obstacle, namely, DG is inf-sup stable with respect to the SDG-norm on the entire discontinuous space but not necessarily inf-sup stable on the interscale transfer subspace. This problem remains open. However, a numerical assessment of the situation is made in Section 5 where the inf-sup constant is calculated for a class of boundary-value problems over a broad range of advection and diffusion parameters, and meshes. For the cases considered, we find that the MDG method is inf-sup stable with respect to the SDG-norm, and the values of the inf-sup constant are very similar to those for the DG method. These results are consistent with the numerical evaluations performed in [10]. We also assess the stability behavior of the methods in terms of the interior penalty parameter and confirm that MDG behaves in similar fashion to DG. Results for SMDG are analogous to those for MDG and thus are omitted for brevity. Conclusions are drawn in Section 6.

2 Preliminaries

2.1 Problem description

Let Ω be a bounded *polygonal* domain in $\mathbb{R}^{n_d}$. The strong form of the boundary value problem we are interested in is the following:

$$\begin{aligned} -\kappa\Delta\phi + \boldsymbol{a}\cdot\nabla\phi &= f \quad \text{in } \Omega \\ \phi &= g \quad \text{on } \Gamma \end{aligned} \tag{1}$$

where $\kappa \geq 0$ is the diffusion coefficient, $\boldsymbol{a}$ is the solenoidal velocity vector field defined on $\overline{\Omega}$ and $\Gamma = \partial\Omega$ is the boundary on which Dirichlet conditions are imposed. More general boundary conditions may be considered as well, see [9] and [10]. We assume that the values of the diffusion coefficient κ and the velocity field $\boldsymbol{a}$ ensure wellposedness of (1). Additional assumptions on these coefficients will be set later.

2.2 General definitions

We introduce the following partition of the boundary:

$$\Gamma^- = \{x \in \Gamma : \boldsymbol{a}(x)\cdot\boldsymbol{n}(x) \leq 0\} \tag{2}$$

$$\Gamma^+ = \{x \in \Gamma : \boldsymbol{a}(x)\cdot\boldsymbol{n}(x) > 0\} \tag{3}$$

where $\boldsymbol{n}$ is the outward unit normal with respect to Γ. Γ^- will be referred to as the *inflow* boundary and Γ^+ as the *outflow* boundaries.

Let $\{\mathcal{T}_h\}_h$ be a family of partitions of Ω into elements T. Each $\mathcal{T}_h$ is assumed to be *admissible* (i.e., non-overlapping elements, their union reproduces the domain, etc.), and *shape regular* (i.e., the elements verify a minimum angle condition, uniformly with respect to h). The elements $T \in \mathcal{T}_h$ are either triangles/quadrilaterals in two dimensions or tetrahedra/hexaedra in three dimensions. Let h_T denote the diameter of T and $h = \max_{T\in\mathcal{T}_h} h_T$. We denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$ (*including* edges on the boundary Γ) and by $\mathcal{E}_h^o$ the set of internal edges (*excluding* edges on the boundary Γ) and, by abuse of notation, we denote by Γ both the boundary $\partial\Omega$ and the collection of edges lying on it.

We also define a partition of the element boundary ∂T:

$$\Gamma_T^- = \{x \in \partial T : \boldsymbol{a}(x)\cdot\boldsymbol{n}(x) \leq 0\} \tag{4}$$

$$\Gamma_T^+ = \{x \in \partial T : \boldsymbol{a}(x)\cdot\boldsymbol{n}(x) > 0\} \tag{5}$$

Here $\Gamma_T^\mp$ represent the element inflow/outflow boundary, respectively, so that $\partial T = \Gamma_T = \Gamma_T^+ \cup \Gamma_T^-$.

In order to derive a discontinuous Galerkin formulation, following [1], *jumps* and *averages* for scalar and vector fields have to be defined on the edges in $\mathcal{E}_h$. Therefore, consider an interior edge $e \in \mathcal{E}_h^o$, and denote by T^+ and T^- the downwind and upwind elements that share it, respectively, and by $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ their respective outward-pointing unit normals. Given a scalar field ν, possibly discontinuous across e, we set $\nu^\pm = \nu|_{T^\pm}$ on e and define

$$\langle\nu\rangle = \frac{1}{2}(\nu^+ + \nu^-) \qquad [\![\nu]\!] = \nu^+\boldsymbol{n}^+ + \nu^-\boldsymbol{n}^-. \tag{6}$$

Analogously, for a vector field $\boldsymbol{\tau}$ we set $\boldsymbol{\tau}^{\pm} = \boldsymbol{\tau}|_{T^{\pm}}$ on e and define

$$\langle \boldsymbol{\tau} \rangle = \frac{1}{2}(\boldsymbol{\tau}^{+} + \boldsymbol{\tau}^{-}) \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^{+} \cdot \boldsymbol{n}^{+} + \boldsymbol{\tau}^{-} \cdot \boldsymbol{n}^{-}. \tag{7}$$

The previous definitions are specialized on the edges on Γ as:

$$\langle \nu \rangle = \nu, \qquad [\![\nu]\!] = \nu\, \boldsymbol{n}, \qquad \langle \boldsymbol{\tau} \rangle = \boldsymbol{\tau}, \qquad \forall e \in \Gamma. \tag{8}$$

We will extensively make use of the following *biased* identity (based on [1, formula (3.3)]):

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \int_{\partial T} \boldsymbol{\tau} \cdot \boldsymbol{n}\, \nu = & \sum_{e\in\mathcal{E}_h^o} \left(\int_e \nu^{\pm} [\![\boldsymbol{\tau}]\!] + \int_e [\![\nu]\!] \cdot \boldsymbol{\tau}^{\mp} \right) \\ & + \sum_{e\in\Gamma} \int_e \nu\, \boldsymbol{\tau} \cdot \boldsymbol{n}.\end{aligned} \tag{9}$$

In what follows, C is a constant, possibly different at each occurrence, which is independent of h and of the coefficients κ and $\boldsymbol{a}$. Moreover, $\alpha \lesssim \beta$ means $\alpha \le C\beta$, while $\alpha \sim \beta$ means $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

We suppose that κ and $\boldsymbol{a}$ are constant on each element $T \in \mathcal{T}_h$. We make use of the following notation: $\kappa_T = \kappa|_T$, $\boldsymbol{a}_T = \boldsymbol{a}|_T$ and $a_T = |\boldsymbol{a}_T|$. Finally, we assume that for any pair of elements T^+ and T^- sharing an edge,

$$\kappa_{T^+} \sim \kappa_{T^-}. \tag{10}$$

3 The Discontinuous Galerkin method

3.1 Method description

Given a positive index k, the following approximation space is introduced:

$$V_h = \{ v \in L^2(\Omega) : v|_T \in \mathcal{P}^k(T), \quad \forall T \in \mathcal{T}_h \} \tag{11}$$

where $\mathcal{P}^k(T)$ is the space of polynomials of degree at most k supported on T.

A possible DG (*discontinuous Galerkin*) formulation for (1) is: find $\phi^{DG} \in V_h$ such that

$$B^{DG}(\phi^{DG}, \mu) = L^{DG}(g, f; \mu) \qquad \forall\, \mu \in V_h, \tag{12}$$

where

$$\begin{aligned} B^{DG}(\nu, \mu) = & - \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\nabla} \mu \cdot (\boldsymbol{a}\nu - \kappa \boldsymbol{\nabla} \nu) \\ & + \sum_{e\in\mathcal{E}_h^o} \int_e \left([\![\mu]\!] \cdot (\boldsymbol{a}\nu^- - \kappa^- \nabla \nu^-) + s\kappa^- \nabla \mu^- [\![\nu]\!] \right) \\ & + \sum_{e\in\Gamma} \int_e s\kappa \nabla \mu \cdot \boldsymbol{n} \nu + \kappa \nabla \nu \cdot \boldsymbol{n} \mu \\ & + \sum_{e\in\Gamma^+} \int_e \mu \nu \boldsymbol{a} \cdot \boldsymbol{n} + \varepsilon \sum_{e\in\mathcal{E}_h} \int_e \frac{\langle \kappa \rangle}{h_\perp} [\![\mu]\!] \cdot [\![\nu]\!], \end{aligned}$$

and

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e \frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\kappa\nabla\mu\cdot\boldsymbol{n}g\right) - \sum_{e\in\Gamma^-}\int_e \boldsymbol{a}\cdot\boldsymbol{n}\mu g;$$

s is either -1, 0, or 1, (corresponding to symmetric, neutral and skew-symmetric interior penalty methods) and for each $e \in \mathcal{E}_h^o$, we set $h_\perp = \dfrac{|T^+|+|T^-|}{2\,|e|}$, while for $e \in \Gamma$ we set $h_\perp = \dfrac{|T|}{|e|}$.

Remark 3.1 *Notice that on each internal edge $e \in \mathcal{E}_h^o$ the normal component of the velocity field $\boldsymbol{a}$ is continuous, owing to the assumption* $\mathrm{div}(\boldsymbol{a}) = 0$.

It will be useful to write the bilinear form $B^{DG}(\cdot,\cdot)$ as a sum of two contributions: the "diffusive" part, and the "convective" part,

$$B^{DG}(\nu,\mu) = B^{DG}_{\mathfrak{D}}(\nu,\mu) + B^{DG}_{\mathfrak{C}}(\nu,\mu), \tag{13}$$

where

$$\begin{aligned} B^{DG}_{\mathfrak{D}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}\mu\cdot\kappa\boldsymbol{\nabla}\nu - \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\kappa^-\nabla\mu^-[\![\nu]\!] \\ &\quad + \sum_{e\in\Gamma}\int_e s\kappa\nabla\mu\cdot\boldsymbol{n}\nu + \kappa\nabla\nu\cdot\boldsymbol{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!] \end{aligned} \tag{14}$$

$$B^{DG}_{\mathfrak{C}}(\nu,\mu) = \sum_{T\in\mathcal{T}_h} -\int_T \boldsymbol{\nabla}\mu\cdot\boldsymbol{a}\nu + \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\boldsymbol{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\boldsymbol{a}\cdot\boldsymbol{n}. \tag{15}$$

We also define the DG-norm

$$\|\nu\|^2_{DG} = \|\nu\|^2_{\mathfrak{D}} + \|\nu\|^2_{\mathfrak{C}}, \tag{16}$$

where

$$\begin{aligned} \|\nu\|^2_{\mathfrak{D}} &= \sum_{T\in\mathcal{T}_h}\left(\kappa_T|\nu|^2_{H^1(T)} + h_T^2\kappa_T|\nu|^2_{H^2(T)}\right) + \varepsilon\sum_{e\in\mathcal{E}_h}\left(h_\perp^{-1}\|\langle\kappa\rangle[\![\nu]\!]\|^2_{L^2(e)}\right), \\ \|\nu\|^2_{\mathfrak{C}} &= \sum_{e\in\mathcal{E}_h}\||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}[\![\nu]\!]\|^2_{L^2(e)}. \end{aligned} \tag{17}$$

The DG formulation is consistent: let ϕ be the solution of (1), then it is easy to verify that

$$B^{DG}(\phi,\mu) = L^{DG}(g,f;\mu) \qquad \forall\,\mu\in V_h.$$

As far as the stability is concerned, we first recall that the form $B^{DG}(\cdot,\cdot)$ is coercive with respect to the DG-norm, as stated in the next proposition.

Proposition 3.2 *For each value of s, there exists positive $\bar{\varepsilon}$ such that, for all $\varepsilon > \bar{\varepsilon}$, there exists $\alpha_{DG} > 0$ such that $B^{DG}(\mu, \mu) \geq \alpha_{DG} \|\mu\|^2_{DG}$, for all $\mu \in V_h$. Moreover, α_{DG} is independent of the mesh-size h, and the coefficients κ and $\boldsymbol{a}$.*

Proof: The coercivity of the convection term easily follow by integration by parts:

$$B^{DG}_{\mathfrak{C}}(\mu, \mu) \geq \frac{1}{2} \|\mu\|^2_{\mathfrak{C}}.$$

Moreover, analogously to the stability proof provided in [1], there exists $\bar{\varepsilon}$, such that, under the assumption $\varepsilon > \bar{\varepsilon}$, the coercivity of the diffusive term holds, that is,

$$B^{DG}_{\mathfrak{D}}(\mu, \mu) \geq \tilde{\alpha}_{DG} \|\mu\|^2_{\mathfrak{D}}. \tag{18}$$

Actually, when $s = 1$ (skew-symmetric case) the result holds for any $\varepsilon > 0$. □

The coercivity as given in Proposition 3.2 is enough to provide an estimate of the form

$$\|\phi - \phi^{DG}\|^2_{DG} \leq C \sum_{T \in \mathcal{T}_h} \left[\left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right], \tag{19}$$

which can be obtained reasoning as in [7], for example. On the other hand, if the convection dominates and the exact solution ϕ is smooth, the quantity $\|\phi - \phi^{DG}\|_{DG}$ is basically a measure of the jumps of the discrete solution. In this case the estimate (19) gives very little information on the error $\phi - \phi^{DG}$.

In order to improve the control of the error, we can add an SUPG like stabilization to the DG formulation (as it was first done in [11] for linear hyperbolic problems). Then, we set

$$B^{SDG}(\nu, \mu) = B^{DG}(\nu, \mu) + \sum_{T \in \mathcal{T}_h} \tau_T \int_T (\mathcal{L}_T \nu)(\boldsymbol{a} \cdot \nabla \mu), \tag{20}$$

$$L^{SDG}(g, f; \mu) = L^{DG}(g, f; \mu) + \sum_{T \in \mathcal{T}_h} \tau_T \int_T f(\boldsymbol{a} \cdot \nabla \mu), \tag{21}$$

where $\mathcal{L}_T \nu = -\kappa \Delta \nu + \boldsymbol{a} \cdot \nabla \nu$ on T and τ_T is a stabilization parameter. For the purpose of the error analysis, the required asymptotic behavior of τ_T is $\tau_T \sim \dfrac{h_T}{a_T}$ in the convection-dominated regime (i.e., when $\dfrac{\kappa_T}{h_T\, a_T} \lesssim 1$) and $\tau_T \sim \dfrac{h_T^2}{\kappa_T}$ in the diffusion-dominated regime (i.e., when $\dfrac{h_T a_T}{\kappa_T} \lesssim 1$). We simply set

$$\tau_T = \tau \min \left\{ \frac{h_T}{a_T}, \frac{h_T^2}{\kappa_T} \right\}, \tag{22}$$

where τ is a positive real number at our disposal.

The SDG (*stabilized discontinuous Galerkin*) formulation reads: find $\phi^{SDG} \in V_h$ such that

$$B^{SDG}(\phi^{SDG}, \mu) = L^{SDG}(g, f; \mu) \qquad \forall\, \mu \in V_h. \tag{23}$$

For the theoretical analysis of the SDG scheme (23), we will need the following SDG-norm

$$\|\nu\|_{SDG}^2 = \|\nu\|_{DG}^2 + \sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T)}^2, \tag{24}$$

and the related

$$|\!|\!|\nu|\!|\!|_{SDG}^2 = \|\nu\|_{SDG}^2 + \sum_{e\in\mathcal{E}_h^0} \||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu^-\|_{L^2(e)}^2 + \sum_{T\in\mathcal{T}_h} \tau_T^{-1}\|\nu\|_{L^2(T)}^2. \tag{25}$$

It is immediate that the SDG formulation is consistent. Moreover, the problem (23) admits a unique solution under suitable assumptions, as a consequence of the following known result.

Proposition 3.3 *For each value of s, there exist positive $\overline{\tau}$ and $\overline{\varepsilon}$ such that, for all $\tau < \overline{\tau}$ and $\varepsilon > \overline{\varepsilon}$, there exists $\alpha_{SDG} > 0$ such that*

$$B^{SDG}(\mu,\mu) \geq \alpha_{SDG}\|\mu\|_{SDG}^2, \qquad \forall\, \mu \in V_h, \tag{26}$$

where α_{SDG} is independent of the mesh-size h, and the coefficients κ and $\boldsymbol{a}$. Moreover

$$B^{SDG}(\nu,\mu) \lesssim |\!|\!|\nu|\!|\!|_{SDG}\|\mu\|_{SDG}, \qquad \forall\,\nu \in V_h + H^1(\Omega)\,, \mu \in V_h. \tag{27}$$

Proof: We first note that, due to (22),

$$\sum_{T\in Th} \tau_T\|\kappa\Delta\mu\|_{L^2(T)}^2 \lesssim \sum_{T\in Th} \tau\kappa_T h_T^2 |\mu|_{H^2(T)}^2 \lesssim \|\mu\|_{\mathfrak{D}}^2. \tag{28}$$

Thanks to Proposition 3.2, when ε is greater than a suitable $\overline{\varepsilon}$ we have

$$B^{SDG}(\mu,\mu) \geq \alpha_{DG}\|\mu\|_{SDG}^2 - \sum_{T\in\mathcal{T}_h} \tau_T \int_T (\kappa\Delta\mu)(\boldsymbol{a}\cdot\nabla\mu).$$

By the Cauchy-Schwarz inequality and (28), (26) is proved by choosing τ sufficiently small. In order to prove (27), we proceed in a standard way as follows.

$$B^{SDG}(\nu,\mu) = B_{\mathfrak{D}}^{DG}(\nu,\mu) + B_{\mathfrak{C}}^{DG}(\nu,\mu) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T (\boldsymbol{a}\cdot\nabla\mu)(\mathcal{L}_T\nu) = I + II + III.$$

We estimate the three terms separately. First, reasoning similarly to [1],

$$I = B_{\mathfrak{D}}^{DG}(\nu,\mu) \lesssim \|\nu\|_{\mathfrak{D}}\|\mu\|_{\mathfrak{D}}. \tag{29}$$

Second, by the Cauchy-Schwarz inequality:

$$\begin{aligned} II &= \sum_{T\in\mathcal{T}_h} - \int_T \boldsymbol{a}\cdot\nabla\mu\,\nu + \sum_{e\in\mathcal{E}_h^o} \int_e [\![\mu]\!]\cdot\boldsymbol{a}\nu^- + \sum_{e\in\Gamma^+} \int_e \mu\nu\boldsymbol{a}\cdot\boldsymbol{n} \\ &\lesssim \|\mu\|_{SDG}\left(\sum_{e\in\mathcal{E}_h^0\cup\Gamma^+} \||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu^-\|_{L^2(e)}^2 + \sum_{T\in\mathcal{T}_h}\tau_T^{-1}\|\nu\|_{L^2(T)}^2\right)^{1/2} \\ &\lesssim \|\mu\|_{SDG}|\!|\!|\nu|\!|\!|_{SDG}. \end{aligned} \tag{30}$$

Third, again by the Cauchy-Schwarz inequality, and (28):

$$III = \sum_{T\in\mathcal{T}_h} \tau_T \int_T (\boldsymbol{a}\cdot\nabla\mu)(-\kappa\Delta\nu + \boldsymbol{a}\cdot\nabla\nu) \lesssim \|\mu\|_{SDG}\|\nu\|_{SDG}. \tag{31}$$

□

3.2 Error estimate

We first provide an error estimate for the SDG method (23).

Proposition 3.4 *Let ϕ be the solution of* (1)*, and assume $\phi \in H^{k+1}(\Omega)$. Let ϕ^{SDG} be given by (23). Under the assumption of Proposition 3.3, the following error estimate holds:*

$$\|\phi - \phi^{SDG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2}. \tag{32}$$

Proof: Let $\phi^I \in V_h$ be the usual nodal interpolant of ϕ. Using coercivity and continuity, (26) and (27), together with consistency, we get

$$\begin{aligned} \alpha_{SDG} \|\phi^{SDG} - \phi^I\|^2_{SDG} &\leq B^{SDG}(\phi^{SDG} - \phi^I, \phi^{SDG} - \phi^I) \\ &= B^{SDG}(\phi - \phi^I, \phi^{SDG} - \phi^I) \lesssim |\!|\!| \phi - \phi^I |\!|\!|_{SDG} \|\phi^{SDG} - \phi^I\|_{SDG}. \end{aligned} \tag{33}$$

For the usual local estimates on the interpolation error $\phi - \phi^I$ we readily obtain

$$|\!|\!| \phi - \phi^I |\!|\!|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(\kappa_T h_T^{2k} + \tau_T a_T^2 h_T^{2k} + \tau_T^{-1} h_T^{2k+2} + \tau_T \kappa_T^2 h_T^{2k-2} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2}. \tag{34}$$

When choosing the stabilization parameter τ_T according to (22), by direct comparison, we see that (32) follows. □

For the pure discontinuous Galerkin method (12), a suitable control on the streamline derivative can be obtained, as was first studied in [12] for the pure convection (scalar hyperbolic) equation. In the following result, we prove an inf-sup condition for the bilinear form $B^{DG}(\cdot,\cdot)$ with respect to the SDG-norm. This improves the stability result stated in Proposition 3.2.

Theorem 3.5 *There exists $\bar{\varepsilon}$ such that for all $\varepsilon \geq \bar{\varepsilon}$,*

$$\inf_{\nu \in V_h} \sup_{\mu \in V_h} \frac{B^{DG}(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}} \geq \beta_{DG} > 0; \tag{35}$$

where β_{DG} is independent of h, κ, $\boldsymbol{a}$, and the domain.

Proof: Given $\nu \in V_h$, we choose $\mu = \nu + \gamma \sum_{\mathcal{T}_h} \tau_T (\boldsymbol{a} \cdot \nabla \nu)_{|T} = \nu + \gamma \mu_2$ where γ is a positive parameter at our disposal. Note that $\mu \in V_h$, as the velocity field is piecewise constant on $\mathcal{T}_h$. We prove the following:

$$\|\mu\|_{SDG} \lesssim \|\nu\|_{SDG}, \tag{36}$$

$$B(\nu, \mu) \geq \beta \|\nu\|^2_{SDG}. \tag{37}$$

We start by proving (36). To this end, we need to estimate the different terms of $\|\mu_2\|_{SDG}$. Recall that, from (22),

$$\tau_T \leq \tau \frac{h_T^2}{\kappa_T}, \tag{38}$$

and

$$\tau_T \le \tau \frac{h_T}{a_T}. \tag{39}$$

Using (39), we have

$$\tau_T \|\boldsymbol{a}\cdot\nabla(\tau_T\, \boldsymbol{a}\cdot\nabla\nu)\|^2_{L^2(T)} \le \tau_T^3 a_T^2 \|\nabla(\boldsymbol{a}\cdot\nabla\nu)\|^2_{L^2(T)} \le (\tau C_{inv})^2 \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)}, \tag{40}$$

where C_{inv} is the constant of the local inverse inequality, giving

$$\sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\mu_2\|^2_{L^2(T)} \lesssim \|\nu\|^2_{SDG}.$$

Consider an internal edge $e \in \mathcal{E}_h^o$, and denote by T^- and T^+ the adjacent upwind and downwind elements. We have

$$\begin{aligned}\|[\![\mu_2]\!]\|^2_{L^2(e)} &\lesssim \|\mu_{2|T^-}\|^2_{L^2(e)} + \|\mu_{2|T^+}\|^2_{L^2(e)} \\ &\le \tau_{T^-}^2 \|(\boldsymbol{a}\cdot\nabla\nu)_{|T^-}\|^2_{L^2(e)} + \tau_{T^+}^2 \|(\boldsymbol{a}\cdot\nabla\nu)_{|T^+}\|^2_{L^2(e)}.\end{aligned} \tag{41}$$

Using the trace inequality, $\|\xi\|_{L^2(e)} \lesssim \|\xi\|_{L^2(T)}\|\nabla\xi\|_{L^2(T)}$, which holds for all $\xi \in H^1(T)$, and then using the inverse inequality, and we also have

$$\|(\boldsymbol{a}\cdot\nabla\nu)_{|T^\pm}\|^2_{L^2(e)} \lesssim C_{inv} h_{T^\pm}^{-1} \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T^\pm)}. \tag{42}$$

From (41) and (42) we obtain

$$\||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}[\![\mu_2]\!]\|_{L^2(e)} \lesssim \tau_{T^+}^{1/2} \|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T^+)} + \tau_{T^-}^{1/2} \|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T^-)}.$$

Similarly, for a boundary edge $e \in \Gamma$, if $e \subset \partial T$ then

$$\||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}[\![\mu_2]\!]\|_{L^2(e)} \lesssim \tau_T^{1/2} \|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T)}.$$

Summarizing, we have proved

$$\|\mu_2\|^2_{\mathfrak{c}} \lesssim \sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)}. \tag{43}$$

By the inverse inequality, as in (40), we have

$$\kappa_T \|\nabla\mu_2\|^2_{L^2(T)} \le \kappa_T a_T^2 \tau_T^2 |\nu|^2_{H^2(T)} \lesssim C_{inv}^2 \kappa_T \|\nabla\nu\|^2_{L^2(T)}. \tag{44}$$

On the other hand, recalling (10), (41)–(42) implies that, for each $e \in \mathcal{E}_h^o$:

$$\frac{\langle\kappa\rangle}{h_\perp} \|[\![\mu_2]\!]\|^2_{L^2(e)} \lesssim \kappa_{T^+} \|\nabla\nu\|^2_{L^2(T^+)} + \kappa_{T^-} \|\nabla\nu\|^2_{L^2(T^-)}, \tag{45}$$

or, for $e \in \Gamma$,

$$\frac{\langle\kappa\rangle}{h_\perp} \|[\![\mu_2]\!]\|^2_{L^2(e)} \lesssim \kappa_T \|\nabla\nu\|^2_{L^2(T)}. \tag{46}$$

This proves that

$$\|\mu_2\|_{\mathfrak{D}} \leq C_{\mathfrak{D}} \|\nu\|_{\mathfrak{D}} \tag{47}$$

where $C_{\mathfrak{D}}$ is a constant indenpendent of the mesh size and the problem parameters.

We turn now to the proof of (37). First of all, we have

$$B_{\mathfrak{C}}^{DG}(\nu,\nu) = \frac{1}{2} \sum_{e\in\mathcal{E}_h} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|^2_{L^2(e)}$$

On the other hand, by integration by parts,

$$B_{\mathfrak{C}}^{DG}(\nu,\mu_2) = \sum_{T\in\mathcal{T}_h} \left(\tau_T \int_T |\boldsymbol{a}\cdot\nabla\nu|^2 - \int_{\partial T^-} \tau_T \left(\boldsymbol{a}\cdot\nabla\nu\right)^+ [\![\boldsymbol{a}\nu]\!] \right), \tag{48}$$

and, using (42),

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \int_{\partial T^-} \tau_T \left(\boldsymbol{a}\cdot\nabla\nu\right)^+ [\![\boldsymbol{a}\nu]\!] &\leq \sum_{T\in\mathcal{T}_h} \tau_T \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} (\boldsymbol{a}\cdot\nabla\nu)^+ \|_{L^2(\partial T^-)} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|_{L^2(\partial T^-)} \\
&\leq \frac{1}{2\lambda} \sum_{T\in\mathcal{T}_h} \tau_T^2 \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} (\boldsymbol{a}\cdot\nabla\nu)^+ \|^2_{L^2(\partial T^-)} \\
&\quad + \frac{\lambda}{2} \sum_{T\in\mathcal{T}_h} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|^2_{L^2(\partial T^-)} \\
&\leq \frac{C\,C_{inv}}{2\lambda} \sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)} + \frac{\lambda}{2} \sum_{e\in\mathcal{E}_h} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|^2_{L^2(e)}.
\end{aligned}$$

Using these estimates, we have:

$$B_{\mathfrak{C}}^{DG}(\nu,\mu) \geq \left(1 - \frac{\gamma\,\lambda}{2}\right) \sum_{e\in\mathcal{E}_h} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|^2_{L^2(e)} + \gamma \left(1 - \frac{C\,C_{inv}}{2\lambda}\right) \sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)}. \tag{49}$$

For the estimation of the diffusion part $B_{\mathfrak{D}}^{DG}(\nu,\mu)$, we use coercivity (18), continuity (e.g., see (29)) of $B_{\mathfrak{D}}^{DG}(\cdot,\cdot)$ and the estimate (47), to obtain:

$$B_{\mathfrak{D}}^{DG}(\nu,\mu) = B_{\mathfrak{D}}^{DG}(\nu,\nu) + \gamma B_{\mathfrak{D}}^{DG}(\nu,\mu_2) \geq \beta_1 \|\nu\|^2_{\mathfrak{D}} - \gamma\tilde{\beta}_2 \|\mu_2\|_{\mathfrak{D}} \|\nu\|_{\mathfrak{D}} \geq (\beta_1 - \gamma C_{\mathfrak{D}} \tilde{\beta}_2) \|\nu\|^2_{\mathfrak{D}}. \tag{50}$$

Summing equations (49) and (50), and setting $\beta_2 = C_{\mathfrak{D}}\tilde{\beta}_2$ we obtain:

$$\begin{aligned}
B^{DG}(\nu,\mu) &\geq (\beta_1 - \gamma\beta_2) \|\nu\|^2_{\mathfrak{D}} \\
&\quad + \left(1 - \frac{\gamma\,\lambda}{2}\right) \sum_{e\in\mathcal{E}_h} \| |\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2} [\![\nu]\!] \|^2_{L^2(e)} + \gamma \left(1 - \frac{C\,C_{inv}}{2\lambda}\right) \sum_{T\in\mathcal{T}_h} \tau_T \|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)}.
\end{aligned}$$

The theorem is then proved by choosing $\lambda = CC_{inv}$ and $\gamma = \min\left\{\lambda^{-1}, \dfrac{\beta_1}{2\beta_2}\right\}$. □

From Theorem 3.5 we deduce the following error estimate for the DG scheme.

Corollary 3.6 *Let ϕ be the solution of* (1), *and assume $\phi \in H^{k+1}(\Omega)$; let ϕ^{DG} be the solution of* (12). *We have*

$$\|\phi - \phi^{DG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2}. \tag{51}$$

Proof: Let $\phi^I \in V_h$ be the nodal interpolant of ϕ and let $\zeta = \phi^{DG} - \phi^I$. Let $\mu \in V_h$ be the test function provided by (36)–(37). Using consistency and Proposition 3.3,

$$\begin{aligned}
\beta \|\zeta\|^2_{SDG} &\leq B^{DG}(\zeta, \mu) = B^{DG}(\phi - \phi^I, \mu) \\
&\lesssim \|\!|\phi - \phi^I|\!\|_{SDG} \|\mu\|_{SDG} \\
&\lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2} \|\zeta\|_{SDG}.
\end{aligned}$$

We deduce (51) by the triangle inequality. □

4 The Multiscale Discontinuous Galerkin method

In this section, we present a reduction technique, referred to as the MDG (*multiscale discontinuous Galerkin*) method, which was first introduced in [10]. Furthermore, a stabilized variant of this method, referred as SMDG (*stabilized multiscale discontinuous Galerkin*), will be introduced subsequently.

The main idea is the following: *(i)* Solve (12) or (23) on a suitable subspace of V_h preserving the stability and approximation properties, *(ii)* Use a multiscale paradigm and local problems to perform the elimination of degrees-of-freedom for both the test and trial spaces.

4.1 Method description

We introduce the space

$$\overline{V}_h = V_h \cap H^1(\Omega).$$

The *local problems* read: $\forall \bar{\nu} \in \overline{V}_h$, find $\nu \in V_h$ such that for all $T \in \mathcal{T}_h$,

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu), \tag{52}$$

where we have set:

$$\begin{aligned}
b_T(\nu, \mu) &= \int_T \kappa \nabla \nu \cdot \nabla \mu - \int_{\Gamma_T} (\kappa \nabla \nu \cdot \boldsymbol{n} \mu - s\kappa \boldsymbol{\nabla} \mu \cdot \boldsymbol{n} \nu) + \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \nu \\
&\quad - \int_T \boldsymbol{\nabla} \mu \cdot \boldsymbol{a} \nu + \int_{\Gamma_T^+} (1 + \delta)\, \mu \nu \boldsymbol{a} \cdot \boldsymbol{n}, \\
\ell_T(\bar{\nu}, f; \mu) &= -\int_{\Gamma_T^-} \mu \bar{\nu} \boldsymbol{a} \cdot \boldsymbol{n} + \delta \int_{\Gamma_T^+} \mu \bar{\nu} \boldsymbol{a} \cdot \boldsymbol{n} + \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \bar{\nu} \\
&\quad + \int_{\Gamma_T} s\kappa \boldsymbol{\nabla} \mu \cdot \boldsymbol{n} \bar{\nu} + \int_T f\, \mu.
\end{aligned} \tag{53}$$

Observe that (52) is a DG formulation for the local problem $\mathcal{L}_T\nu = f$ on T, with $\nu = \bar{\nu}$ on the boundary ∂T. Comparing the local DG formulation (52) with the global DG formulation (12), notice that the former has an extra term, which depends on a new parameter $\delta > 0$. This new term is needed for implementation purposes (see [10]).

We denote by $\mathfrak{T}_h : \overline{V}_h \times L^2(\Omega) \to V_h$ the operator which associates to each $(\bar{\nu}, f) \in \overline{V}_h \times L^2(\Omega)$ the solution ν of the local problems (52) on each element $T \in \mathcal{T}_h$. The stability of (52), which is stated below (in Proposition 4.4), implies that the problems (52) admit unique solutions on each element $T \in \mathcal{T}_h$, that is, the operator $\mathfrak{T}_h$ is well defined. $\mathfrak{T}_h$ represents the "interscale transfer operator", and the associated "interscale transfer spaces" are the (affine) manifold

$$\mathfrak{T}_h(\overline{V}_h, f) = \left\{\mathfrak{T}_h(\bar{\nu}, f) \mid \bar{\nu} \in \overline{V}_h\right\},$$

and the (linear) manifold

$$\mathfrak{T}_h(\overline{V}_h, 0) = \left\{\mathfrak{T}_h(\bar{\nu}, 0) \mid \bar{\nu} \in \overline{V}_h\right\}.$$

With this notation, the MDG method reads: find $\phi^{MDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that:

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \tag{54}$$

Its stabilized version SMDG reads: find $\phi^{SMDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that:

$$B^{SDG}(\phi^{SMDG}, \mu) = L^{SDG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \tag{55}$$

Notice that SMDG is an SUPG stabilization of MDG.

Remark 4.1 *The spaces $\mathfrak{T}_h(\overline{V}_h, f)$ and $\mathfrak{T}_h(\overline{V}_h, 0)$ can be parameterized by means of the degrees-of-freedom of $\overline{V}_h$ lying on the "skeleton" $\Sigma = \cup_{e \in \mathcal{E}_h} e$.*

Remark 4.2 *The MDG method can be interpreted as a multiscale technique. Both trial and test discontinuous functions $\nu \in V_h$ can be split into a continuous* coarse *scale $\bar{\nu}$ plus a discontinuous* fine *scale $\nu' = \nu - \bar{\nu}$. Performing integration by parts in (52), we find that ν' satisfies*

$$b_T(\nu', \mu) = \int_T (f - \mathcal{L}_T\bar{\nu})\mu\,, \quad \forall \mu \in V_h(T). \tag{56}$$

Equation (56) *suggests a relationship between the MDG approach and the RFB (*Residual-Free Bubble*) approach (see, e.g., [4]). Consider, for the sake of simplicity, the case of lowest order approximation $k = 1$. Actually ν' in* (56) *can be understood as the DG approximation of the* exact *residual-free bubble ν^{bubble}, which satisfies $\mathcal{L}_T\nu^{bubble} = f - \mathcal{L}_T\bar{\nu}$ on T, with $\nu^{bubble} = 0$ on the boundary ∂T. A DG approximation of the* exact *residual-free bubble has been used also in the DB (*discontinuous bubble*) implementation of the RFB formulation (see [13]). The major difference between MDG and DB is that for the latter the space of test functions was $\overline{V}_h$ instead of $\mathfrak{T}_h(\overline{V}_h, 0)$. The relation between those two approaches deserves further investigation.*

4.2 Approximation properties of $\mathfrak{T}_h(\overline{V}_h, f)$

The first step in the analysis of problems (54) and (55) is the study of the approximation properties of the interscale transfer affine space $\mathfrak{T}_h(\overline{V}_h, f)$.

Theorem 4.3 (Approximation) *Let ϕ be the solution of (1); then there exists $\nu \in \mathfrak{T}_h(\overline{V}_h, f)$ such that*

$$\|\phi - \nu\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2}. \tag{57}$$

Before proving Theorem 4.3, we need some lemmas. On each element $T \in \mathcal{T}_h$, we introduce the following local norm:

$$\begin{aligned} \|\nu\|^2_{SDG(T)} := \kappa_T |\nu|^2_{H^1(T)} + h_T^2 \kappa_T |\nu|^2_{H^2(T)} + \tau_T \|\boldsymbol{a} \cdot \nabla \nu\|^2_{L^2(T)} \\ + \varepsilon h_T^{-1} \kappa_T \|\nu\|^2_{L^2(\partial T)} + \||\boldsymbol{a} \cdot \boldsymbol{n}|^{1/2} \nu\|^2_{L^2(\partial T)}. \end{aligned} \tag{58}$$

In what follows, we set $V_h(T) = V_h|_T$ (note that this is nothing other than the space of degree k polynomials on T). The first lemma states that the local problems (52) are stable.

Lemma 4.4 (Local Stability) *There exists positive $\overline{\varepsilon}$ and $\overline{\delta}$ such that for all $\varepsilon \geq \overline{\varepsilon}$ and $\delta \leq \overline{\delta}$,*

$$\inf_{\nu \in V_h(T)} \sup_{\mu \in V_h(T)} \frac{b_T(\nu, \mu)}{\|\nu\|_{SDG(T)} \|\mu\|_{SDG(T)}} \geq \beta_b > 0, \qquad \forall T \in \mathcal{T}_h, \tag{59}$$

and the constant β_b is independent of T, κ and $\boldsymbol{a}$.

Proof: If $\delta = 0$, then (59) is a particular case of (35) where the domain is T, (endowed with a one-element mesh) instead of Ω. Then, for $\delta = 0$, given $\nu \in V_h(T)$ there exists $\mu \in V_h(T)$ such that

$$\begin{aligned} \|\mu\|_{SDG(T)} \leq \|\nu\|_{SDG(T)}, \\ b_T(\nu, \mu) \geq \beta_{DG} \|\nu\|^2_{SDG(T)}. \end{aligned} \tag{60}$$

If $\delta \neq 0$, given $\nu \in V_h(T)$ and for the same $\mu \in V_h(T)$ of in (60), we have

$$\begin{aligned} \|\mu\|_{SDG(T)} \leq \|\nu\|_{SDG(T)}, \\ b_T(\nu, \mu) \geq \beta_{DG} \|\nu\|^2_{SDG(T)} + \int_{\Gamma_T^+} \delta \mu \nu \boldsymbol{a} \cdot \boldsymbol{n}. \end{aligned} \tag{61}$$

Moreover

$$\left| \int_{\Gamma_T^+} \delta \mu \nu \boldsymbol{a} \cdot \boldsymbol{n} \right| \leq \delta \|\mu\|_{SDG(T)} \|\nu\|_{SDG(T)}.$$

Then, for $\delta \leq \overline{\delta} = \beta_{DG}/2$, from (61) we get

$$\begin{aligned} \|\mu\|_{SDG(T)} \leq \|\nu\|_{SDG(T)}, \\ b_T(\nu, \mu) \geq \frac{\beta_{DG}}{2} \|\nu\|^2_{SDG(T)}, \end{aligned} \tag{62}$$

which gives (59), for $\beta_b = \beta_{DG}/2$. □

The local problems are consistent: let ϕ be the solution of (1), then

$$b_T(\phi,\mu) = \ell_T(\phi_{|\Gamma_T}, f, \mu) \qquad \forall \mu \in V_h, \forall T \in \mathcal{T}_h. \tag{63}$$

In the following lemma we state a Poincaré-like estimate for the norm $\|\cdot\|_{SDG(T)}$.

Lemma 4.5 *For each element $T \in \mathcal{T}_h$, and each function $\nu \in H^1(T)$ the following estimate holds:*

$$\tau_T^{-1}\|\nu\|^2_{L^2(T)} \lesssim \|\nu\|^2_{SDG(T)}. \tag{64}$$

Proof: Fix an element $T \in \mathcal{T}_h$. Because of the definition (22) of τ_T, (64) is a consequence of the two Poincaré estimates

$$\frac{a_T}{h_T}\|\nu\|^2_{L^2(T)} \lesssim \frac{h_T}{a_T}\|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)} + \||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu\|^2_{L^2(\partial T)}, \tag{65}$$

$$\frac{\kappa_T}{h_T^2}\|\nu\|^2_{L^2(T)} \lesssim \kappa_T|\nu|^2_{H^1(T)} + h_T^{-1}\|\kappa^{1/2}\nu\|^2_{L^2(\partial T)}. \tag{66}$$

The inequality (66) is a consequence of the standard Poincaré inequality plus a scaling argument. Therefore, we concentrate on the less common (65). Let η be the solution of the problem:

$$\boldsymbol{a}\cdot\nabla\eta = 1 \quad \text{on } T, \quad \text{and} \quad \eta|_{\Gamma_T^-} = 0.$$

It is easy to verify that $\|\eta\|_{L^\infty(T)} \le \dfrac{h_T}{a_T}$. Given $v \in H^1(T)$, we estimate $\|\cdot\|_{L^2(T)}$ as follows:

$$\begin{aligned}
\|\nu\|^2_{L^2(T)} &= \int_T \nu^2\,\boldsymbol{a}\cdot\nabla\eta = -\int_T \boldsymbol{a}\cdot\nabla(\nu^2)\,\eta + \int_{\Gamma_T^+}\boldsymbol{a}\cdot\boldsymbol{n}\,\eta\,\nu^2 \\
&\le \|\eta\|_{L^\infty(T)}(2\|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T)}\|\nu\|_{L^2(T)} + \||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu\|^2_{L^2(\partial T)}) \\
&\le \frac{h_T}{a_T}(2\|\boldsymbol{a}\cdot\nabla\nu\|_{L^2(T)}\|\nu\|_{L^2(T)} + \||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu\|^2_{L^2(\partial T)}) \\
&\le \frac{2h_T^2}{a_T^2}\|\boldsymbol{a}\cdot\nabla\nu\|^2_{L^2(T)} + \frac{1}{2}\|\nu\|^2_{L^2(T)} + \frac{h_T}{a_T}\||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}\nu\|^2_{L^2(\partial T)}.
\end{aligned}$$

The inequality (65) follows, dividing both sides by $\dfrac{h_T}{a_T}$.

□

Proof of Theorem 4.3. Let $\phi^I \in V_h$ be the nodal interpolant of ϕ and let ν be the solution of the following local problems:

$$b_T(\nu_{|T},\mu) = \ell_T(\phi^I_{|\Gamma_T}, f, \mu) \qquad \forall\mu \in V_h, \forall T \in \mathcal{T}_h.$$

We have $\nu \in \mathfrak{T}_h(\overline{V}_h, f)$ and we will show that ν verifies the estimate (57). First, we prove that

$$\|\phi-\nu\|^2_{SDG} \lesssim \sum_{T\in\mathcal{T}_h}\|\phi-\nu\|^2_{SDG(T)}. \tag{67}$$

It is immediate that

$$\|\phi-\nu\|_{SDG}^2+\sum_{e\in\mathcal{E}_h^0\cup\Gamma^+}\||\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}(\phi-\nu)^-\|_{L^2(e)}^2\lesssim\sum_{T\in\mathcal{T}_h}\|\phi-\nu\|_{SDG(T)}^2, \tag{68}$$

and, making use of (64), we also have

$$\sum_{T\in\mathcal{T}_h}\tau_T^{-1}\|\nu\|_{L^2(T)}^2\lesssim\sum_{T\in\mathcal{T}_h}\|\phi-\nu\|_{SDG(T)}^2. \tag{69}$$

Therefore, from (67) and the usual triangle inequality, we get

$$\|\phi-\nu\|_{SDG}^2\lesssim\sum_{T\in\mathcal{T}_h}\|\phi-\phi^I\|_{SDG(T)}^2+\sum_{T\in\mathcal{T}_h}\|\phi^I-\nu\|_{SDG(T)}^2=I+II.$$

Let us concentrate on II first. Fix a generic $T\in\mathcal{T}_h$. Consistency (63) implies:

$$b_T(\phi-\nu,\mu)=\ell_T(\phi-\phi^I,0;\mu)\qquad\forall\mu\in V_h(T). \tag{70}$$

By Lemma 4.4, there exists $\tilde{\mu}\in V_h(T)$ such that $\|\tilde{\mu}\|_{SDG(T)}\lesssim\|\phi^I-\nu\|_{SDG(T)}$ and

$$\begin{aligned}\|\phi^I-\nu\|_{SDG(T)}^2&\lesssim b_T(\phi^I-\nu,\tilde{\mu})\\&=b_T(\phi^I-\phi,\tilde{\mu})+b_T(\phi-\nu,\tilde{\mu})\\&=b_T(\phi^I-\phi,\tilde{\mu})+\ell_T(\phi-\phi^I,0;\tilde{\mu}).\end{aligned} \tag{71}$$

We have

$$\begin{aligned}&b_T(\phi^I-\phi,\tilde{\mu})\lesssim(\|\phi^I-\phi\|_{SDG(T)}+\tau_T^{-1}\|\phi^I-\phi\|_{L^2(T)})\|\tilde{\mu}\|_{SDG(T)},\\&\ell_T(\phi-\phi^I,\tilde{\mu})\lesssim\|\phi^I-\phi\|_{SDG(T)}\|\tilde{\mu}\|_{SDG(T)}.\end{aligned} \tag{72}$$

Thanks to (71)–(72), and the Poincaré estimate (64), we obtain

$$\begin{aligned}\|\phi^I-\nu\|_{SDG(T)}&\lesssim\|\phi^I-\phi\|_{SDG(T)}+\tau_T^{-1}\|\phi^I-\phi\|_{L^2(T)}\\&\lesssim\|\phi^I-\phi\|_{SDG(T)}.\end{aligned}$$

Squaring and summing over all the elements, we end up with

$$II\lesssim I.$$

Finally, observe that, by using the standard estimates for the interpolation error, we easily get

$$I\lesssim\left(\sum_{T\in\mathcal{T}_h}\left(a_Th_T^{2k+1}+\kappa_Th_T^{2k}\right)|\phi|_{H^{k+1}(T)}^2\right)^{1/2}.$$

This gives (57). □

4.3 Error estimate

An optimal error estimate for the SMDG method readily follows from Theorem 4.3 and Proposition 3.3:

Theorem 4.6 *Let ϕ and ϕ^{SMDG} be the solutions of (1) and (55) respectively. Under the same assumption of Proposition 3.3,*

$$\|\phi - \phi^{SMDG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2}. \tag{73}$$

Proof: Let $\nu \in \mathfrak{T}_h(\overline{V}_h, f)$ be the approximant of ϕ given by Theorem 4.3, and $\zeta = \phi^{SMDG} - \nu$. Linearity ensures that $\zeta \in \mathfrak{T}_h(\overline{V}_h, 0)$, that is, it is an admissible test function for (55). Repeating the same steps as in Proposition 3.4, we obtain the estimate:

$$\|\phi - \phi^{SMDG}\|_{SDG} \lesssim \|\!|\phi - \nu|\!\|_{SDG}$$

The statement is then proved by using Theorem 4.3. □

Remark 4.7 *The problem of providing an optimal error estimate for MDG remains open. The error estimate* (19) *for DG, proved in [7], makes use of an interpolant which is the L^2-projection of ϕ onto V_h, which is not generally available in $\mathfrak{T}_h(\overline{V}_h, f)$. On the other hand, the stronger error estimate* (51) *we have proved in Section 3, still for DG, relies on the validity of the inf-sup condition* (35)*. A similar error analysis for the MDG method would need the following inf-sup condition:*

$$\inf_{\nu \in \mathfrak{T}_h(\overline{V}_h, 0)} \sup_{\mu \in \mathfrak{T}_h(\overline{V}_h, 0)} \frac{B^{DG}(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}} \geq \beta_{MDG} > 0; \tag{74}$$

which is not *a consequence of* (35)*. One of the objectives of the next section is the numerical evaluation of the inf-sup constant β_{MDG} in* (74)*.*

5 Selection of parameters

The stability of the numerical schemes we have considered depends on the parameters ε (which specifies the amount of *interior penalty* stabilization, in all the formulations) and τ (which specifies the amount of streamline stabilization, in SDG and SMDG). In this section, we want to investigate more in detail the relation between the stability of the schemes and the value of the parameters for a specific model problem. Moreover, we investigate numerically the validity of (74) and we demonstrate that (74) holds, at least for the cases covered by our numerical experiments.

We consider a square domain $\Omega = [0, 1]^2$, and a uniform partition $\mathcal{T}_h$ of $N \times N$ square elements. Then, we select bilinear finite element spaces, discontinuous for V_h and globally continuous for $\bar{V}_h$. We restrict ourselves to the simplest case of constant coefficients κ and $\boldsymbol{a}$.

Numerical testing of this configuration has been performed in [10]. Here, we want to measure the stability of the schemes by a numerical evaluation of the inf-sup constant

$$\inf_{\nu \in V_h} \sup_{\mu \in V_h} \frac{B(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}}, \tag{75}$$

for the DG and SDG formulations (where $B(\cdot,\cdot) \equiv B^{DG}(\cdot,\cdot)$ and $B(\cdot,\cdot) \equiv B^{SDG}(\cdot,\cdot)$, resp.) and

$$\inf_{\nu \in \mathfrak{T}_h(\overline{V}_h, 0)} \sup_{\mu \in \mathfrak{T}_h(\overline{V}_h, 0)} \frac{B(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}}, \tag{76}$$

for the MDG and SMDG formulations (where $B(\cdot,\cdot) \equiv B^{DG}(\cdot,\cdot)$ and $B(\cdot,\cdot) \equiv B^{SDG}(\cdot,\cdot)$, resp.). The evaluation of (75) and (76) can be performed through a generalized eigenvalue computation (see, e.g., [2] for details). In the sequel we assume $\delta = 0$. Very similar results are obtained with the choice $\delta = 0.01$, proposed in [10], which has advantages from the implementation standpoint.

5.1 The interior penalty parameter

First, we study the effect of ε, the amount of interior penalty stabilization. We focus on the diffusion-dominated regime, where the interior penalty term plays a role, setting $\kappa = 1$ and $\|\boldsymbol{a}\| = 10^{-10}$. The values of (75)–(76) are plotted in Figure 1–2, respectively, for the DG and MDG schemes (similar result are obtained for the stabilized SDG and SMDG schemes), and for a partition of 10×10 elements ($N = 10$). The symmetric version ($s = -1$), the skew-symmetric version ($s = 1$), as well as the neutral version ($s = 0$) are considered. We confirm that the skew-symmetric version is stable for all positive ϵ, while the other two formulations are unstable if the interior penalty stabilization is too small. Nevertheless, the symmetric version attains more accurate numerical solutions and is preferred (see [10]). We also observe that the MDG scheme needs less interior penalty stabilization than the DG scheme. This is not surprising: indeed, roughly speaking, in the diffusive regime, $\mathfrak{T}_h(\overline{V}_h, 0)$ is composed of functions that are *almost* continuous, and therefore the interior penalty stabilization is only needed on the boundary of Ω.

5.2 The SUPG parameter and the inf-sup stability of MDG

Second, we analyze the role of the streamline stabilization. We select, from now on, the symmetric version ($s = -1$) and we take $\varepsilon = 6$ (this gives sufficient interior penalty stabilization to both DG and SDG, as seen in Section 5.1). We know, from Theorem 3.5, that there is no need of streamline stabilization in the DG method. This is confirmed in Figure 3, where (75) is plotted for different κ and $\boldsymbol{a} = [\cos(\theta), \sin(\theta)]$, on a grid of 10×10. We have set $\tau = 1/2$ in the definition of $\|\cdot\|_{SDG}$. The values of (75) are bounded away from zero, uniformly with respect to the operator coefficients. In Figure 5 we focus the attention on the convection-dominated regime, which is now the most interesting case: we set $\kappa = 10^{-6}$ and compute (75) for different $\boldsymbol{a} = [\cos(\theta), \sin(\theta)]$, on different uniform meshes of $N \times N$ elements. We confirm that the inf-sup condition holds uniformly with respect to the mesh-size.

The major result of this section is the evaluation of the stability of the MDG scheme. Actually, the MDG scheme turns out to be stable with respect to the $\|\cdot\|_{SDG}$, for the model case here considered: in Figure 4 we plot the inf-sup constant (76) for different κ and $\boldsymbol{a} = [\cos(\theta), \sin(\theta)]$, on the uniform 10×10 grid, while in Figure 6 we plot (76) in the convection-dominated regime ($\kappa = 10^{-6}$) for different directions of the convective field $\boldsymbol{a}$ and different uniform meshes. Our conclusion is that, at least for this model case, the MDG scheme is inf-sup stable, that is, condition (74) holds with β_{MDG} independent of the problem coefficients and the mesh-size. From this, and reasoning as in Theorem 4.6, we can infer the optimal error estimate for the MDG scheme:

$$\|\phi - \phi^{MDG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2} . \tag{77}$$

Similar plots and results are obtained for the stabilized SDG and SMDG methods, in accordance with Proposition 3.3, and are omitted.

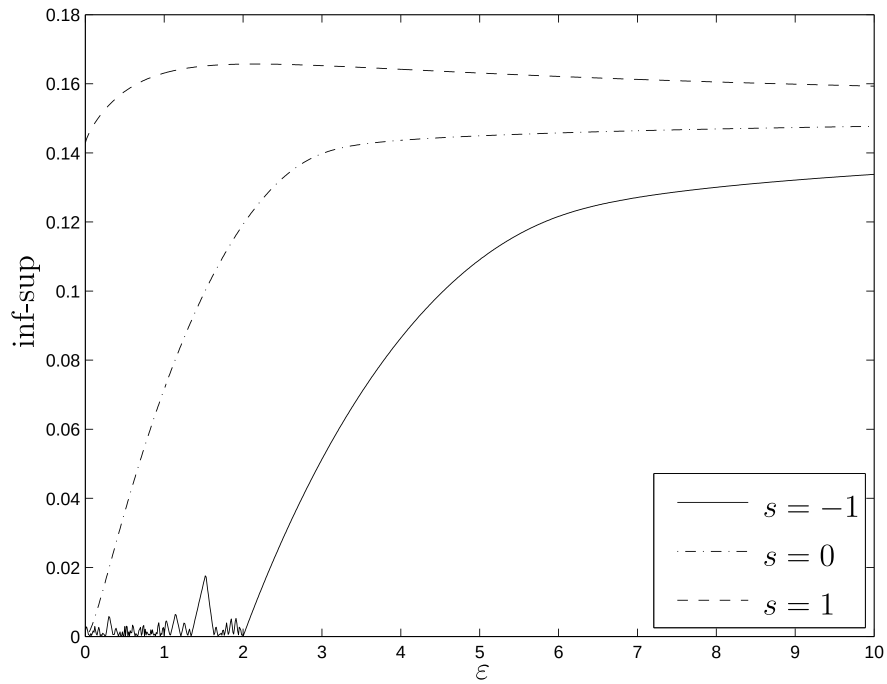

Figure 1: Inf-sup constant of the DG method vs. ε

6 Conclusions

The mathematical analysis of the multiscale discontinuous Galerkin MDG method introduced in [10] was initiated. This method alleviates a long-standing drawback of discontinuous Galerkin methods, namely, the large size of the solution space. It utilizes local, element-wise

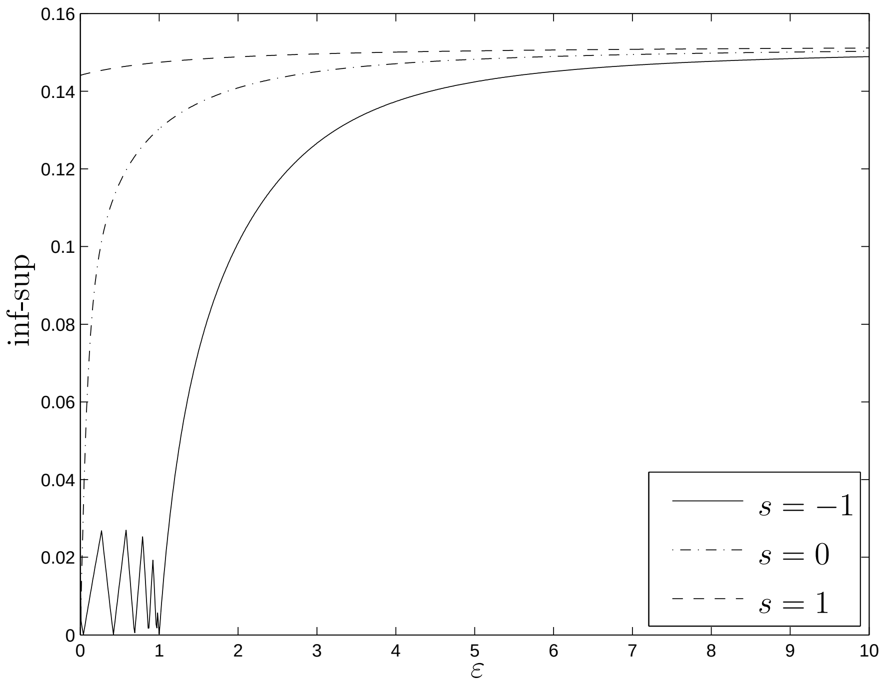

Figure 2: Inf-sup constant of the MDG method vs. ε.

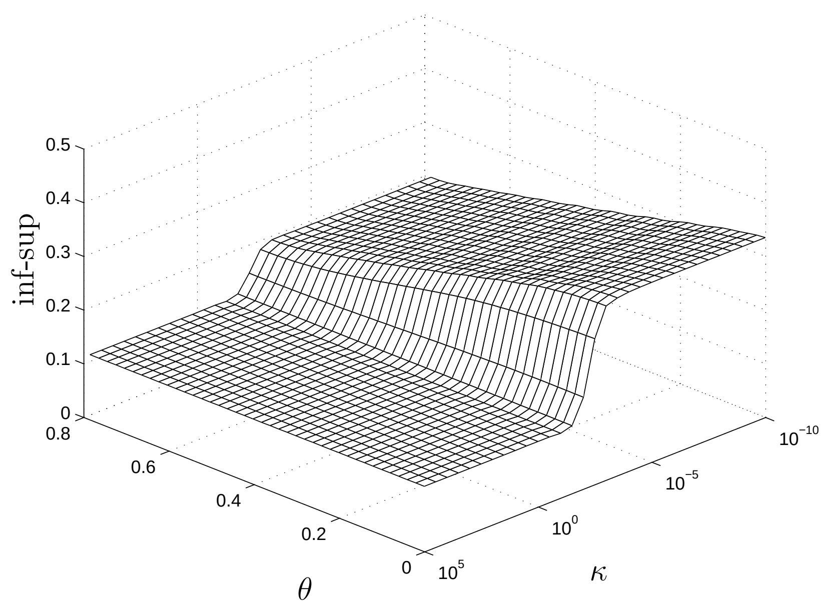

Figure 3: Inf-sup constant of the DG method vs. $\boldsymbol{a} = [\cos\theta, \sin\theta]$ and κ.

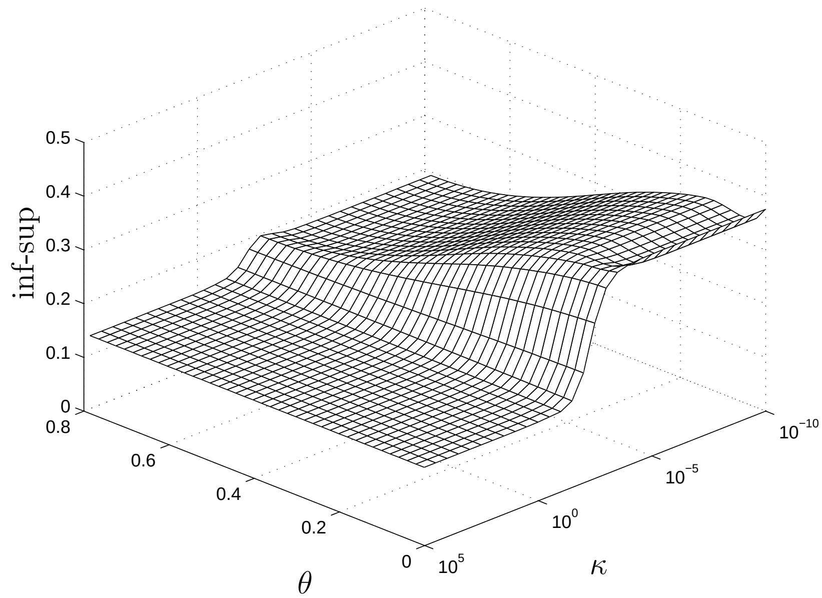

Figure 4: Inf-sup constant of the MDG method vs. $\boldsymbol{a} = [\cos\theta, \sin\theta]$ and κ.

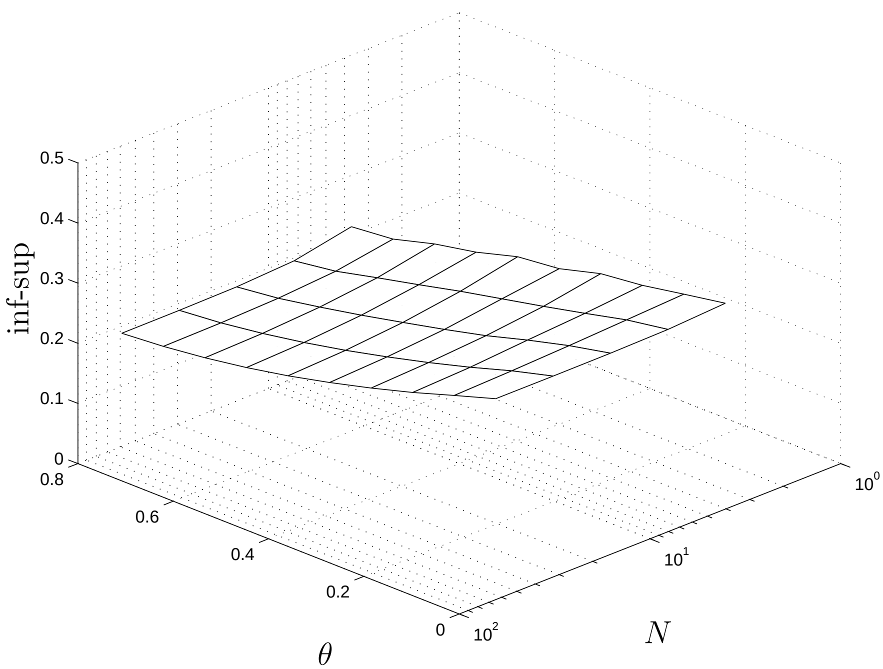

Figure 5: Inf-sup constant of the DG method vs. $\boldsymbol{a} = [\cos\theta, \sin\theta]$ and N.

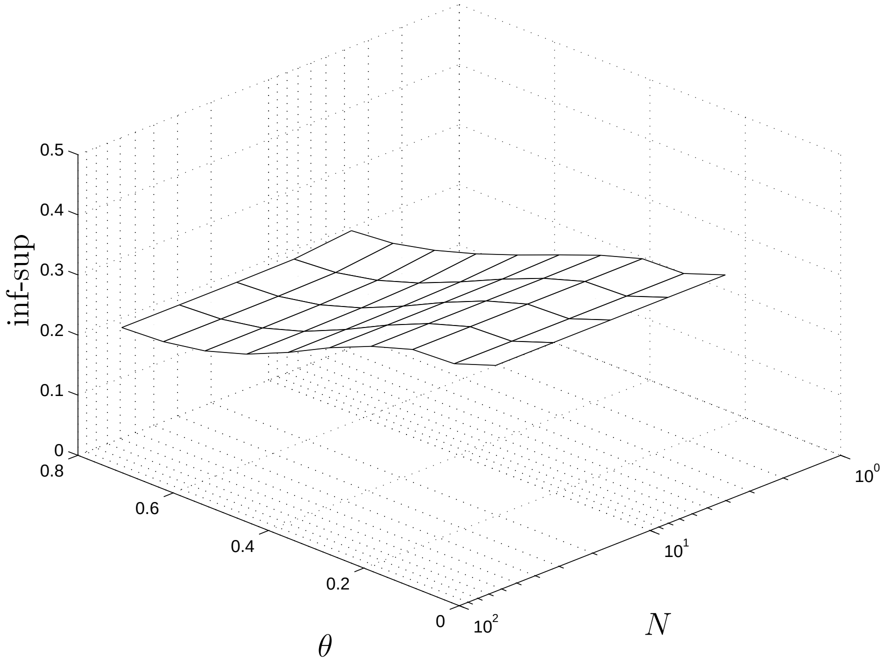

Figure 6: Inf-sup constant of the MDG method vs. $\boldsymbol{a} = [\cos\theta, \sin\theta]$ and N.

problems to generate an interscale transfer operator, enabling the size of the matrix problem to be significantly reduced, apparently without degradation in the quality of results.

We studied MDG and a stabilized version, SMDG. We were able to characterize the approximation properties of the interscale transfer spaces. The corresponding global discontinuous Galerkin methods, DG and SDG, are inf-sup stable and coercive, respectfully, with respect to the norm induced by the bilinear form of SDG. Coercivity is inherited by the interscale transfer subspaces, but not necessarily inf-sup stability. Consequently, we were able to obtain the same error estimates for SMDG as for DG and SDG but the situation for MDG remains open. Numerical evaluations of the inf-sup constant for MDG indicated that it was positive, bounded uniformly away from zero, and very similar to that for DG. These results are consistent with the numerical calculations performed in [10].

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