Computational and Applied Mathematics
Area A Program of Study

I. Basic Operational Mathematics

1. **Advanced calculus (2 weeks)**

1.1 Differentiation: directional and partial derivatives, Frechet differentials of first and higher order, gradient, Jacobian, Hessian, Implicit Function Theorem, differentiation of implicitly defined functions, and Taylor expansion.
1.2 Geometry: scalar, cross and mixed products in $\mathbf{R}^3$, affine and curvilinear coordinates, and the operators grad, div, and curl. Integration of line and surface integrals of the first and second kind. Integration by parts: Fundamental Green Identity, Green Identities, Gauss Theorem, and Stokes Theorem.

2. **Complex Analysis (5 weeks)**

2.1 Complex numbers, Euler formula, functions of a complex argument, complex differentiability, Cauchy-Riemann conditions, elementary functions, multi-valued functions, and branch cuts.
2.2 Integration: definitions, Fundamental Cauchy Theorem, Cauchy Integral Theorem, Taylor and Laurent expansions, and radius of convergence.
2.3 The Residue Theorem: classification of isolated singularities, evaluation of residues at poles, and Residue Theorem applications to singular integrals and Cauchy principle value contour integrals.
2.4 Fourier and Laplace transforms: definitions, properties, evaluation of the inverse Laplace transform using the Residue Theorem, and applications to linear ordinary differential equations with constant coefficients.

3. **Linear Algebra and Eigenvalue Problems (2 weeks)**

3.1 Finite dimensional vector spaces: eigenvalues, eigenspaces, diagonalization of matrices, self-adjoint and normal operators, generalized eigenvalues, Jordan decomposition, resonance, Fredholm alternative, Rayleigh quotient, and the power method.
3.2 Sturm-Liouville theory: self-adjoint two-point boundary value problems, the Sturm-Liouville Theorem, and examples of L^2-orthonormal bases.

4. **Ordinary Differential Equations (ODE's) (3 weeks)**

4.1 Elementary theory: particular and general solutions, and integral curves. Methods of solution: separation of variables, substitution, exact differential, and integrating factors. Cauchy's Theorem on local existence and uniqueness of solutions.

4.2 Linear ODE's: particular and general solutions, Wronskian, constant coefficients case, Cauchy-Euler equation, undetermined coefficients, variation of parameters, systems of ODE's with constant coefficients, and relation to generalized eigenvalues.
4.3 Equations with variable coefficients: regular and singular points, Frobenius Theorem, and Bessel and Hankel functions.

5. **Partial Differential Equations (PDE's) (3 weeks)**

5.1 Laplace, heat and wave equations, separation of variables, Fourier and Laplace methods, and relation to Sturm-Liouville theory.
5.2 Second order PDE's: initial-value problems, systems of two first order equations, characteristics, classification of second order PDE's, and relation to prime integrals.

Suggested Courses (one of the following)

1. EM 386L / ASE 380P, Mathematical Methods in Applied Mechanics, II.
2. CAM 381M / PHY 381M, Methods of Mathematical Physics.

II. Foundations of Modern Analysis

1. **Preliminaries (3 weeks)**

1.1 Set theory, logic, and relations.
1.2 Functions and cardinality of Sets.
1.3 Elementary topology in $\mathbf{R}^n$.

2. **Vector Spaces (2 weeks)**

2.1 Hamel basis, dimension, linear transformations, and matrix representations.
2.2 Algebraic dual, dual basis, algebraic transpose, inner product, and adjoint operators.

3. **Elements of Lebesgue Measure and Integration Theory (3 weeks)**

3.1 Abstract measure theory: sigma algebras, measure, and Lebesgue measure.
3.2 Lebesgue measurable and integrable functions, Fatou's Lemma, and the Dominated Convergence Theorem.
3.3 Fubini's Theorem, Lebesgue sums, Riemann versus Lebesgue integrability, Holder and Minkowski inequalities, essential supremum, and L^p-spaces.

4. **Topological and Metric Spaces (5 weeks)**

4.1 Bases, filters, and definition of topology.
4.2 Topological subspaces and sequences.
4.3 Continuity and compactness.
4.4 Normed and metric spaces and completeness.
4.5 Bolzano-Weierstrass Theorem and Contraction Mapping Theorem.

5. **Introduction to Banach Spaces [optional] (2 weeks)**

5.1 Topological vector spaces and the Hahn-Banach Extension Theorem.
5.2 Bounded linear operators on normed spaces.
5.3 The Closed Range Theorem.

Suggested Course
CAM 386M / EM 386M, Functional Analysis in Theoretical Mechanics.

III. Methods of Applied Mathematics 1

1. **Banach Spaces (6 weeks)**

1.1 Normed linear spaces and convexity.
1.2 Convergence, completeness, and Banach spaces.
1.3 Continuity, open sets, and closed sets.
1.4 Continuous Linear Transformations.
1.5 Hahn-Banach Extension Theorem.
1.6 Linear functionals, dual and reflexive spaces, and weak convergence.
1.7 The Baire Theorem and uniform boundedness.
1.8 Open Mapping and Closed Graph Theorems.
1.9 Closed Range Theorem.
1.10 Compact sets and Ascoli-Arzela Theorem.
1.11 Compact operators and the Fredholm alternative.

2. **Hilbert Spaces (4 weeks)**

2.1 Basic geometry, orthogonality, bases, projections, and examples.
2.2 Bessel's inequality and the Parseval Theorem.
2.3 The Riesz Representation Theorem.
2.4 Compact and Hilbert-Schmidt operators.
2.5 Spectral theory for compact, self-adjoint and normal operators.
2.6 Sturm-Liouville Theory.

3. **Distributions (4 weeks)**

3.1 Seminorms and locally convex spaces.
3.2 Test functions and distributions.
3.3 Calculus with distributions.

Suggested Course

CAM 385C / Math 383C, Methods of Applied Mathematics.

IV. Methods of Applied Mathematics 2

1. **The Fourier Transform and Sobolev Spaces (4 weeks)**

1.1 The Schwartz space and tempered distributions.
1.2 The Fourier transform.
1.3 The Plancherel Theorem.
1.4 Convolutions.
1.5 Fundamental solutions of PDE's.
1.6 Sobolev spaces.
1.7 Imbedding Theorems.
1.8 The Trace Theorem.

2. **Variational Boundary Value Problems (BVP) (3 weeks)**

2.1 Weak solutions to elliptic BVP's.
2.2 Variational forms.
2.3 Lax-Milgram Theorem.
2.4 Galerkin approximations.
2.5 Green's functions.

3. **Differential Calculus in Banach Spaces and Calculus of Variations (4 weeks)**

3.1 The Frechet derivative.
3.2 The Chain Rule and Mean Value Theorems.
3.3 Higher order derivatives and Taylor's Theorem.
3.4 Banach's Contraction Mapping Theorem and Newton's Method.
3.5 Inverse and Implicit Function Theorems, and applications to nonlinear functional equations.
3.6 Extremum problems, Lagrange multipliers, and problems with constraints.
3.7 The Euler-Lagrange equation.
3.8 Convex functions, lower semicontinuous functions, and existence of minima.
3.9 Applications to classical mechanics and geometry.

4. **Asymptotic Analysis (3 weeks)**

4.1 Definitions and fundamental properties.
4.2 Examples of transcendental equations and initial-value problems.
4.3 Boundary layers in regular and singular perturbations.
4.4 Perturbation methods for linear eigenvalue problems.

Suggested Course

CAM 385D / Math 383D, Methods of Applied Mathematics.

V. Advanced Topics

Possible advanced topics include

1. CAM 381D / M 381D, Complex Analysis.
2. CAM 381R / M 381C, Real Analysis.
3. CAM 381N / PHY 381N, Methods of Mathematical Physics.
4. CAM 381S / M 381E, Functional Analysis.
5. CAM 384K / M 385C, Theory of Probability.
6. CAM 384R / M 384C, Mathematical Statistics.
7. CAM 386N / EM 386N, Qualitative Methods in Nonlinear Mechanics.
8. CAM 391, Introductory Dynamical Systems.
9. CAM 391C / M 391C, Topics in Analysis.
    - Analytic Number Theory.
    - Banach and Hilbert Spaces; Functional Analysis.
    - Conference Course in Analysis.
    - Fourier Analysis.
    - Number Theory.
    - Riemann Surfaces.
    - Theory of Wavelets.
10. CAM 393C / M 393C, Topics in Applied Mathematics.
    - Quantum Mechanics; Quantum Field Theory.
    - Ergodic Theory.
    - Group Representations.
    - Statistical Mechanics; Statistical Physics.
    - Introductory Partial Differential Equations.
    - Monotone Operators and Partial Differential Equations.
    - Hilbert Space Methods for Partial Differential Equations.
    - Hamiltonian Dynamics.
    - Nonlinear Functional Analysis.
    - Euler and Navier-Stokes Equations.
    - Microlocal Calculus and Spectral Asymptotics.
    - Calculus of Variations.
11. CAM 394C / M 394C, Topics in Probability and Statistics.
    - Advanced Probability and Random Processes Sampling.
    - Conference Course in Probability and Statistics.

Bibliography

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2. C. Caratheodory, *Calculus of Variations and Partial Differential Equations of the First Order*, 2nd English Edition, Chelsea, 1982.
3. E.W. Cheney and H.A. Koch, *Notes on Applied Mathematics*, Department of Mathematics, University of Texas at Austin.
4. L. Debnath and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, Academic Press, 1990.
5. I.M. Gelfand and S.V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963.

6. M.D. Greenberg, *Foundations of Applied Mathematics*, Prentice-Hall, 1978.
7. D.H. Griffel, *Applied Functional Analysis*, Halsted-Wiley, 1981.
8. J.T. Oden and L.F. Demkowicz, *Applied Functional Analysis*, CRC Press, 1996.
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10. W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.
11. W. Rudin, *Real and Complex Analysis*, 3rd Edition, McGraw-Hill, 1987.
12. R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, available at http://ejde.math.swt.edu//mono-toc.html.
13. K. Yosida, *Functional Analysis*, Springer-Verlag, 1980.

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webman@ticam.utexas.edu