CAM AREA A QUALIFYING EXAM

May 28, 1999, 9:00 a.m.–12:00 noon

Work any 6 of the following 8 problems.

1. Prove that if two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, defined on a common vector space V, are equivalent, i.e., $\exists\, C_{12}, C_{21} > 0$ such that

$$\|\boldsymbol{v}\|_1 \le C_{12}\|\boldsymbol{v}\|_2, \quad \|\boldsymbol{v}\|_2 \le C_{21}\|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{v} \in V,$$

then the corresponding topologies are identical.

2. Let $D = \{(x,y,z) : x^2+y^2+z^2 < a^2, z > 0\}$, for $a > 0$, be a hemisphere in $\mathbb{R}^3$. Let $\boldsymbol{v} = (xy, y^2, 0)$. Compute the surface integral

$$\int_{\partial D} \boldsymbol{v} \times \boldsymbol{n}\, dS,$$

where ∂D is the (entire) boundary of D, and $\boldsymbol{n}$ denotes the outward normal unit vector on ∂D.

3. Solve the following initial-value problem using the Laplace transform. Use the Residue Theorem to compute the inverse Laplace transform.

$$\begin{cases} \ddot{x} + \dot{x} + x = 0 \quad t > 0, \\ x(0) = 1, \quad \dot{x}(0) = 1. \end{cases}$$

4. Define what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space B to converge weakly to x. Now suppose that $x_n \overset{w}{\rightharpoonup} x$. Prove that $x_n \to x$ (strongly) if, and only if, $\|x_n\| \to \|x\|$.

5. Let H be a Hilbert space and $U \in B(H,H)$. Show that the following are equivalent.

(i) $U^*U = I$ (identity map on H).

(ii) $(Ux, Uy) = (x, y)$ for all $x, y \in H$.

(iii) $\|Ux\| = \|x\|$ for all $x \in H$.

6. Let $\varphi \in \mathcal{D}$, $\varphi(x) \ge 0$ for all $x \in \mathbb{R}$, $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$, and $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$ for all $\epsilon > 0$.

(a) If $\psi \in \mathcal{D}$, show that $\psi * \varphi_\epsilon \overset{\mathcal{D}}{\to} \psi$ as $\epsilon \to 0$.

(b) If $u \in \mathcal{D}'$, show that $u * \varphi_\epsilon \overset{\mathcal{D}'}{\to} u$ as $\epsilon \to 0$.

7. Find the form of the curve in the plane (*not* the curve itself), of minimal length, joining $(0,0)$ to $(1,0)$ such that the area bounded by the curve, the x and y axes, and the line $x = 1$ has area $\pi/8$.

8. Let $f(y)$ be continuous and strictly increasing on the interval $[a, \infty)$. Suppose that $f(y) \sim y$ as $y \to \infty$ and $x > f(a)$. Let $y = y(x) \in (a, \infty)$ be the root of the equation $f(y) = x$. Show that $y(x) \sim x$ as $x \to \infty$.