

COMPUTATIONAL AND APPLIED MATHEMATICS PRELIMINARY EXAM
Area A: Applicable Mathematics **May 29, 1998, 9:00-12:00 noon**

Work any 6 of the following 8 problems.

1. Solve the following problem by using the Laplace Transform:

$$x'' + 2x' = H(t - 1), \quad x(0) = 0, \quad x'(0) = 1,$$

where H is the Heaviside function. (Use the Residue Theorem to calculate the inverse transform.)

2. Solve the following *loaded* vibrating string problem using the method of separation of variables:

$$\begin{aligned} y_{tt} &= a^2 y_{xx} + x, & 0 < x < \pi, & 0 < t, \\ y(0, t) &= y(\pi, t) = 0, \\ y(x, 0) &= y_t(x, 0) = 0, \end{aligned}$$

where $a > 0$ is constant.

3. Prove that the integral of a nonnegative function f over a measurable set A is zero if, and only if, f is zero almost everywhere on A .

4. Let Y be a closed subspace of a Banach space X . Recall that for any $x \in X$, $[x] = x + Y \subset X$ is an equivalence class of elements of X . Define

$$\|[x]\|_{X/Y} = \inf_{z \in [x]} \|z\|_X.$$

Prove that this is a norm on the quotient space X/Y .

5. State the Hahn-Banach Theorem and use it to prove that if X is a normed linear space and $x \in X$, then

$$\|x\|_X = \sup_{f \in X^*, f \neq 0} \frac{|f(x)|}{\|f\|_{X^*}}.$$

6. Let $\{\varphi_\epsilon\}_{\epsilon > 0}$ be an approximation to the identity on \mathbb{R}^d :

$$(i) \varphi \in \mathcal{D}; \quad (ii) \varphi \geq 0; \quad (iii) \int \varphi(x) dx = 1; \quad (iv) \varphi_\epsilon(x) = \epsilon^{-d} \varphi(\epsilon^{-1}x).$$

(a) If $\psi \in \mathcal{D}$, show that $\psi * \varphi_\epsilon \xrightarrow{\mathcal{D}} \psi$ as $\epsilon \rightarrow 0$.

(b) If $u \in \mathcal{D}'$, show that $u * \varphi_\epsilon \xrightarrow{\mathcal{D}'} u$ as $\epsilon \rightarrow 0$.

7. State Banach's Contraction Mapping Theorem. Use it to prove convergence of the Simplified Newton Method: Let X and Y be Banach spaces and $f : X \rightarrow Y$ be differentiable in a neighborhood of $B = B_r(x_0)$, the ball of radius $r > 0$ about $x_0 \in X$. If $y \in Y$, $A = Df(x_0)$ has a bounded inverse, and $\|I - A^{-1}Df(x)\| \leq K < 1$ for all $x \in B$, then $F(x) = y$ has a unique solution in B provided that $\|f(x_0) - y\| \leq (1 - K)r/\|A^{-1}\|$.

8. Let $\Omega \subset \mathbb{R}^d$ be a domain and $f \in L^2(\Omega)$. Show that the boundary value problem

$$\begin{aligned} -\Delta u + 2u &= f, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

has a unique solution in $H_0^1(\Omega)$. Moreover, if

$$J(v) = \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 + v^2 - f v \right) dx,$$

show that $u \in H_0^1(\Omega)$ is the solution if, and only if, $J(u) = \min_{v \in H_0^1(\Omega)} J(v)$