CAM AREA A QUALIFYING EXAM

May 31, 2002, 9:00 a.m.–12:00 noon

Work any 6 of the following 7 problems.

1. Solve the following initial-value problem for $x(t)$, $t \geq 0$, using the Laplace transform:

$$\ddot{x} + 2\dot{x} = H(t-1) \ ,$$
$$x(0) = 0 \ ,$$
$$\dot{x}(0) = 1 \ ,$$

wherein H is the Heaviside function. You will also need to use the Residue Theorem to compute the inverse Laplace transform.

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and $\Omega \subset \mathbb{R}^n$ a bounded set. Prove Jensen's inequality: If $f \in L^1(\Omega)$ is real, then

$$\varphi\left(\frac{1}{m(\Omega)} \int_\Omega f(x)\, dx\right) \leq \frac{1}{m(\Omega)} \int_\Omega \varphi(f(x))\, dx \ .$$

[Hint: It may be easier to first prove the result assuming that $\varphi \in C^2(\mathbb{R})$, so then $\varphi'' \geq 0$.]

3. Suppose that $\Omega = (-1, 1)$ and $\mathcal{P}^1(\Omega)$ is the set of linear polynomials on Ω. Let $F : \mathcal{P}^1(\Omega) \to \mathbb{R}$ be the continuous linear functional defined by

$$F(p) = p(0) \ .$$

(a) State the Hahn-Banach Extension Theorem.
(b) Explain why F is not defined on $L^1(\Omega)$.
(c) Define a continuous linear functional $\mathcal{F} : L^1(\Omega) \to \mathbb{R}$ such that $\mathcal{F}$ restricted to $\mathcal{P}^1(\Omega)$ is F.
(d) Show that infinitely many such extensions exist.

4. Consider the problem

$$\frac{du}{dt} = 1 + tu^2 \ ,$$
$$u(0) = 1 \ .$$

(a) Define what is meant by a *contraction* map, and state the Banach Contraction Mapping Theorem.
(b) Restate the nonlinear initial-value problem above as a fixed point problem on some Banach space.
(c) Use the Banach Contraction Mapping Theorem to prove the local existence and uniqueness of a solution.
(d) Give a concrete number t_0 such that the solution exists in the interval $(0, t_0)$.

5. Let $V(x) = \log|x|$ in $\mathbb{R}^2$.

(a) Show that $\Delta V = 2\pi\delta_0$ in the sense of distributions, where δ_0 is the Dirac distribution supported at 0. You should use polar coordinates. Then

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

(b) Find a formal solution to

$$\Delta u = e^{-|x|^2} .$$

(c) Justify your solution.

6. Let V be the Hilbert space $H^1(\alpha, \beta)$, and let $b \geq 0$, $\gamma \in [\alpha, \beta]$, and $F \in L^2(\alpha, \beta)$ be given. Define a bilinear form and a linear functional on V by

$$\begin{aligned} a(u, v) &= (u, v)_{H^1} + b\, u(\gamma)\, v(\gamma) , \quad && \forall u, v \in V , \\ f(v) &= (F, v)_{L^2} , && \forall v \in V . \end{aligned}$$

(a) Show there exists exactly one solution $u \in V$ of

$$a(u, v) = f(v) , \quad \forall v \in V .$$

(b) Find the boundary-value-problem that characterizes the solution of the minimization problem, and *prove* the equivalence.

7. Find the C^2 curve $y(t)$ that minimizes the functional

$$\int_0^1 \left[(y(t))^2 + (y'(t))^2 \right] dt$$

subject to the endpoint constraints

$$y(0) = 0 \quad \text{and} \quad y(1) = 1$$

and the constraint

$$\int_0^1 y(t)\, dt = 0 .$$