# Rational Parametrizations of Non-singular Real Cubic Surfaces

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**Abstract**

Real cubic algebraic surfaces may be described by either implicit or parametric equations. One particularly useful representation is the rational parametrization, where the three spatial coordinates are given by rational functions of two parameters. These parametrizations take on different forms for different classes of cubic surfaces. Classification of real cubic algebraic surfaces into five families for the nonsingular case is based on the configuration of twenty-seven lines on them. We provide a method of extracting all these lines by constructing and solving a polynomial of degree twenty-seven. Simple roots of this polynomial correspond to real lines on the surface, and real skew lines are used to form rational parametrizations for three of these families. Complex conjugate skew lines are used to parametrize surfaces from the fourth family. The parametrizations for these four families involve quotients of polynomials of degree no higher than four. Each of these parametrizations covers the whole surface except for a few points, lines, or conic sections. The parametrization for the fifth family, as noted previously in the literature, requires a square root. We also analyze the image of the derived rational parametrization for both real and complex parameter values, together with "base" points where the parametrizations are ill-defined.

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# 1 Introduction

Low degree real algebraic surfaces (quadrics, cubics and quartics) play a significant role in constructing accurate computer models of physical objects and environments for purposes of simulation and prototyping [7]. While quadrics such as spheres, cones, hyperboloids and paraboloids prove sufficient for constructing restricted classes of models, cubic algebraic surface patches are sufficient to model the boundary of objects with arbitrary topology in a $C^1$ piecewise smooth manner [8].

Real cubic algebraic surfaces are the real zeros of a *polynomial* equation $f(x, y, z) = 0$ of degree three. In this representation the cubic surface is said to be in *implicit* form. The irreducible cubic surface which is not a cylinder or cone of a nonsingular cubic curve, can alternatively be described explicitly by rational functions of parameters $u$ and $v$:

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)}, \tag{1}$$

where $f_i$, $i = 1 \ldots 4$ are polynomials. In this case the cubic surface is said to be in *rational parametric* form.

Real cubic algebraic surfaces thus possess dual implicit-parametric representations and this property proves important for the efficiency of a number of geometric modeling and computer graphics display operations [7]. For example, with dual available representations the intersection of two surfaces or surface patches reduces simply to the sampling of an algebraic curve in the planar parameter domain [5]. Similarly, point-surface patch incidence classification, a prerequisite for boolean set operations and ray casting for graphics display, is greatly simplified in the case when both the implicit and parametric representations are available [5]. Additional examples in the computer graphics domain which benefit from dual implicit-parametric representations are the rapid triangulation for curved surface display [9] and image texture mapping on curved surface patches [16].

Deriving the rational parametric form from the implicit representation of algebraic surfaces, is a process known as rational parametrization. Algorithms for the rational parametrization of cubic algebraic surfaces have been given in [3, 26], based on the classical theory of skew straight lines and rational curves on the cubic surface [11, 17, 27]. Finite rational parametrizations, possibly in Bernstein-

Bézier or B-spline bases, also provide dual representations useful in computer aided geometric design applications [10, 15, 20]. One of the main results of our current paper is to constructively address the parametrization of cubic surfaces based on the reality of the straight lines on the real cubic surface. In doing so we provide an algorithm to construct all twenty-seven straight lines (real and complex) on the real nonsingular cubic surface. Given a pair of real skew lines on the cubic surface, one can easily generate a rational bi-quadratic Bézier representation for cubic surface patches [20]. We demonstrate this in the subsequent section.

A singular cubic surface $f(x, y, z) = 0$ is one on which there exists a point $\mathbf{p}$ such that $\nabla f(\mathbf{p}) = \mathbf{0}$; a nonsingular cubic surface is one with no such points. We prove that the parametrizations of the real cubic surface components are constructed using a pair of real skew lines for those three families which have them, and remarkably using a complex conjugate pair of skew lines in a fourth family. In each of these four families, the components $(x, y, z)$ are given as the quotient of a quartic and a cubic polynomial in two parameters. There does not appear to be a similar rational parametrization for the fifth family that covers all or almost all of the surface, so instead we use two disjoint parametrizations which involve one square root each. A rational parametrization is described in [26], but that covering is in general two-to-one instead of one-to-one. All of the parametrizations described in this paper are one-to-one, meaning that for any point on the cubic surface there can be just one set of values $(u, v)$ which give rise to that point.

We also analyze the image of the derived rational parametrization for both real and complex parameter values, together with "base" points where the parametrizations are ill-defined. These base points cause a finite number (at most five) of lines and points, and possibly two conic sections lying on the surface, to be missed by the parametrizations. One of these conics can be attained by letting $u \to \pm\infty$ and the other with $v \to \pm\infty$ separately.

## 2 Preliminaries

One of the gems of classical algebraic geometry has been the theorem that twenty-seven distinct straight lines lie completely on a nonsingular cubic surface [24]. See figure 1. Schläfli's double-six notation elegantly captures the complicated and many-fold symmetry of the configurations of the twenty-seven lines [25]. He also partitions all nonsingular cubic surfaces $f(x, y, z) = 0$ into five families $F_1, \ldots, F_5$

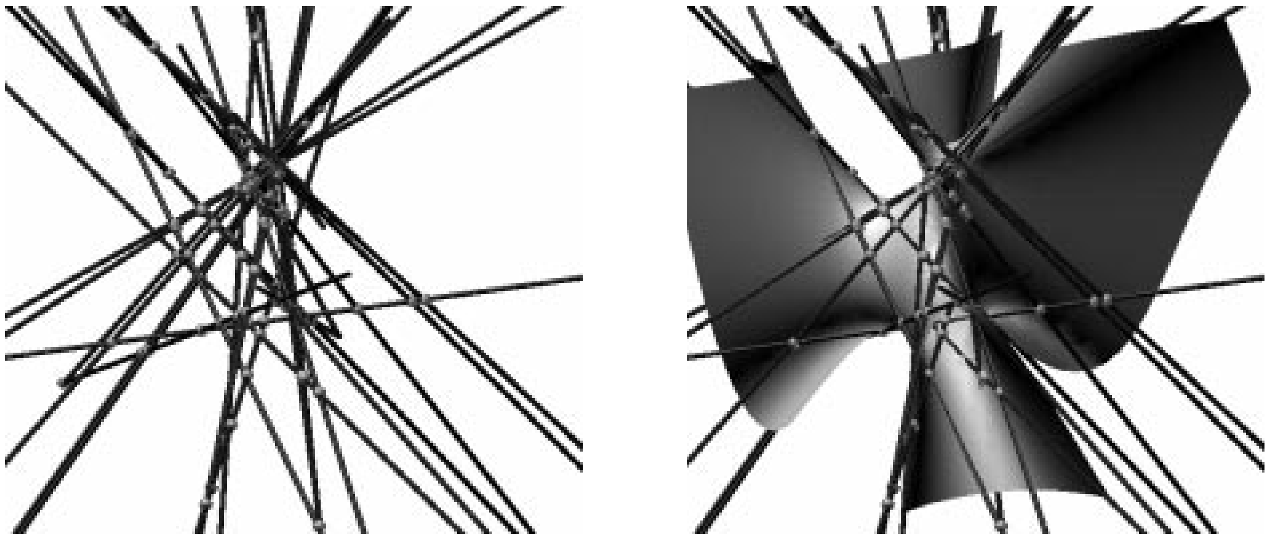

Figure 1: A configuration of twenty-seven real lines of a cubic surface shown with and without the surface. Intersections of the coplanar straight lines are also shown.

based on the reality of the twenty-seven lines. Family $F_1$ contains 27 real straight lines, family $F_2$ contains 15 real lines, and family $F_3$ contains 7 real lines while families $F_4$ and $F_5$ contain 3 real lines each. What distinguishes $F_4$ from $F_5$ is that while 6 of the 12 conjugate complex line pairs of $F_4$ are skew (and 6 pairs are coplanar), each of the 12 conjugate pairs of complex line pairs of $F_5$ is coplanar. When a nonsingular cubic surface $F$ tends to a singular cubic surface $G$ with an isolated double point, 12 of $F$'s straight lines (constituting a double six) tend to 6 lines through the double point of $G$ [27]. Hence singular cubic surfaces have only twenty-one distinct straight lines.

Alternatively a classification of cubic surfaces can be obtained from computing all 'base' points of its parametric representation,

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)},$$

Base points of a surface parametrization are those isolated parameter values which simultaneously satisfy $f_1 = f_2 = f_3 = f_4 = 0$. It is known that any nonsingular cubic surface can be expressed as a rational parametric cubic with six base points. The classification of nonsingular real cubic surfaces then follows from:

1. If all six base points are real, then all 27 lines are real, i.e. the $F_1$ case.

2. If two of the base points are a complex conjugate pair then 15 of the straight lines are real, i.e. the $F_2$ case.

3. If four of the base points are two complex conjugate pairs then 7 of the straight lines are real, i.e. the $F_3$ case.

4. If all base points are complex then three of the straight lines are real. In this case the three real lines are all coplanar, i.e. the $F_4$ and $F_5$ cases.

# 3 Real and Rational Points on Cubic Surfaces

We first begin by computing a simple real point (with a predefined bit precision) on a given real cubic surface $f(x, y, z) = 0$. For obvious reasons of exact calculations with bounded precision it is very desirable to choose the simple point to have rational coordinates. Mordell in his 1969 book [22] mentions that no method is known for determining whether rational points exist on a general cubic surface $f(x, y, z) = 0$, or finding all of them if any exist. We are unaware if a general criterion or method now exists or whether the conjecture of Mordell given below has been resolved.

The following theorems and conjecture exhibit the difficulty of this problem, and are repeated here for information.

**Theorem**[[22],chap 11]: All rational points on a cubic surface can be found if it contains two lines whose equations are defined by conjugate numbers of a quadratic field and in particular by rational numbers.

**Theorem**[[22],chap 11]: The general cubic equation (irreducible cubic and not a function of two independent variables nor a homogeneous polynomial in linear functions of its variables) has either none or an infinity of rational solutions.

**Conjecture**[[22],chap 11]: The cubic equation $F(X, Y, Z, W) = 0$ is solvable if and only if the congruence $F(X, Y, Z, W) \equiv 0 \pmod{p^r}$ is solvable for all primes $p$ and integers $r > 0$ with $(X, Y, Z, W, p) = 1$.

We present a straightforward search procedure to determine a real point on $f(x, y, z) = 0$, and if lucky one with rational coordinates. First, homogenize the cubic polynomial with a new variable $w$, so that we have the homogeneous cubic $F(w, x, y, z) = 0$. Set each of $\{w, x, y, z\}$ in turn to zero to obtain a homogeneous cubic. For $z = 0$, for example, we obtain $F_3(w, x, y)$, representing the component at infinity in the $z$ direction. Recursively determine if $F_3(w, x, y) = 0$ has a real/rational point (other than $(0,0,0)$). Being homogeneous, one only needs to check for $F_3(w,x,1)=0$ and $F_3(w,x,0)=0$, which are both polynomials in one less variable, and hence the recursion is in dimension. Now for a univariate polynomial equation $g(x)=0$ we use the technique of [21] to determine the existence and coordinates of a rational root. If not, one computes a real root having the desired bit precision [13, 19].

Additionally, if no rational points are found for $F(w,x,y,z)=0$ when any one of $\{w,x,y,z\}$ is zero, we search for an extreme real/rational point on a closed component of the surface. We can compute the resultant and linear subresultants of $f$ and $f_x$, (extreme points in the $x$ direction) eliminating $x$ to yield new polynomials $f_1(\tilde{y},\tilde{z})$ and $\tilde{x}f_2(\tilde{y},\tilde{z})+f_3(\tilde{y},\tilde{z})$, where $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are linearly related to $x,y$, and $z$ (see [6] for details of this computation). One then computes the rational points of $f_1(\tilde{y},\tilde{z})=0$, and uses the equation $\tilde{x}f_2(\tilde{y},\tilde{z})+f_3(\tilde{y},\tilde{z})=0$ to determine the rational $\tilde{x}$ coordinate given rational $\tilde{y}$ and $\tilde{z}$ coordinates of the point, if rational $\tilde{y},\tilde{z}$ satisfying $f_1(\tilde{y},\tilde{z})=0$ are found and $\tilde{x},\tilde{y}$, and $\tilde{z}$ are rationally linearly related to $x$, $y$ and $z$. Otherwise, one computes a real point having the desired bit precision. The variables $x$, $y$, and $z$ may of course be permuted throughout these operations, and are easily recovered from $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$.

In the general case, we are forced to take a real simple point on the cubic surface. We can bound the required precision of this real simple point so that comparisons between algebraic numbers (or the sign of algebraic numbers) in the cubic surface parametrization algorithm of the next section, are performed correctly. The lower bound of this value can be estimated using bit approximations and the gap theorem in [13].

## 4 Algebraic Reduction

Given two skew lines $\mathbf{l}_1(u)=\begin{bmatrix} x_1(u) \\ y_1(u) \\ z_1(u) \end{bmatrix}$ and $\mathbf{l}_2(v)=\begin{bmatrix} x_2(v) \\ y_2(v) \\ z_2(v) \end{bmatrix}$ on the cubic surface $f(x,y,z)=0$, the cubic parametrization formula for a point $\mathbf{p}(u,v)$ on the surface is :

$$\mathbf{p}(u,v)=\begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}=\frac{a\mathbf{l}_1+b\mathbf{l}_2}{a+b}=\frac{a(u,v)\mathbf{l}_1(u)+b(u,v)\mathbf{l}_2(v)}{a(u,v)+b(u,v)} \tag{2}$$

where

$$a = a(u,v) = \nabla f(\mathbf{l}_2(v)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]$$
$$b = b(u,v) = \nabla f(\mathbf{l}_1(u)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]$$

The total degree of the numerator of the parametrization formula in $\{u, v\}$ is 4 while the denominator total degree is 3. Note that if the lines are coplanar, formula (2) can only produce points on the plane of the lines, hence the search for skew lines on the cubic surface. Similar parameter representations from skew lines on cubic surfaces for Bernstein-Bézier polynomial representations are given in [20].

Following the notation of [3], a real cubic surface has an implicit representation of the form

$$f(x,y,z) = Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hxz^2 + Iyz^2 + Jxyz + Kx^2 + Ly^2 + Mz^2 + Nxy + Oxz + Pyz + Qx + Ry + Sz + T = 0 \ .$$

Compute a simple (nonsingular) point $(x_0, y_0, z_0)$ on the surface. We can move the simple point to the origin by a translation $x = x' + x_0,\ y = y' + y_0,\ z = z' + z_0$, producing

$$f'(x', y', z') = Q'x' + R'y' + S'z' + \ldots \text{ terms of higher degree.}$$

Next, we wish to rotate the tangent plane to $f(x', y', z')$ at the origin to the plane $z'' = 0$. This can be done by the transformation

$$\begin{array}{ll} x' = x'',\ y' = y'',\ z' = (z'' - Q'x'' - R'y'')/S' & \text{if } S' \neq 0 \\ x' = x'',\ y' = (z'' - Q'x'')/R',\ z' = y'' & \text{if } S' = 0 \text{ and } R \neq 0 \\ x' = z''/Q',\ y' = x'',\ z' = y'' & \text{if } S' = 0,\ R' = 0, \text{ and } Q' \neq 0 \ . \end{array}$$

Fortunately $Q', R'$, and $S'$ cannot all be zero, because then the selected point $(x_0, y_0, z_0)$ would be a singular point on the cubic surface.

The transformed surface can be put in the form

$$f''(x'', y'', z'') = z'' + [f_2(x'', y'') + f_1(x'', y'')z'' + f_0 z''^2] + [g_3(x'', y'') + g_2(x'', y'')z'' + g_1(x'', y'')z''^2 + g_0 z''^3] \ ,$$

where $f_j(x'', y'')$ and $g_j(x'', y'')$ are terms of degree $j$ in $x''$ and $y''$. In general, this surface intersects the tangent plane $z'' = 0$ in a cubic curve with a double point at the origin (as its lowest degree terms are quadratic). This curve can be rationally parametrized as

$$x'' \ = \ K(t) \ = \ -\frac{L''t^2 + N''t + K''}{B''t^3 + F''t^2 + D''t + A''}$$

$$y'' = L(t) = tK(t) = -\frac{L''t^3 + N''t^2 + K''t}{B''t^3 + F''t^2 + D''t + A''} \qquad (3)$$
$$z'' = 0 \ ,$$

where $A'', B'', \ldots$ are the coefficients in $f''(x'', y'', z'')$ that are analogous to $A, B, \ldots$ in $f(x, y, z)$. In the special case that the singular cubic curve is reducible (a conic and a line or three lines), a parametrization of the conic is taken instead.

We transform the surface again to bring a general point on the parametric curve specified by $t$ to the origin by the translation

$$x'' = \overline{x} + K(t) \ , \qquad y'' = \overline{y} + L(t) \ , \qquad z'' = \overline{z} \ .$$

The cubic surface can now be expressed by

$$\bar{f}(\overline{x}, \overline{y}, \overline{z}) = \overline{Q}(t)\overline{x} + \overline{R}(t)\overline{y} + \overline{S}(t)\overline{z} + \cdots \text{ terms of higher degree} \ .$$

We make the tangent plane of the surface at the origin coincide with the plane $\hat{z} = 0$ by applying the transformation

$$\overline{x} = \hat{x} \ , \qquad \overline{y} = \hat{y} \ , \qquad \overline{z} = -\frac{\overline{Q}(t)}{\overline{S}(t)}\hat{x} - \frac{\overline{R}(t)}{\overline{S}(t)}\hat{y} + \frac{1}{\overline{S}(t)}\hat{z} \ .$$

The equation of the surface now has the form

$$f(\hat{x}, \hat{y}, \hat{z}) = \hat{z} + [\hat{f}_2(\hat{x}, \hat{y}) + \hat{f}_1(\hat{x}, \hat{y})\hat{z} + \hat{f}_0\hat{z}^2] + [\hat{g}_3(\hat{x}, \hat{y}) + \hat{g}_2(\hat{x}, \hat{y})\hat{z} + \hat{g}_1(\hat{x}, \hat{y})\hat{z}^2 + \hat{g}_0\hat{z}^3] \ .$$

The intersection of this surface with $\hat{z} = 0$ gives

$$\hat{f}_2(\hat{x}, \hat{y}) + \hat{g}_3(\hat{x}, \hat{y}) = 0 \ . \qquad (4)$$

Recall that $\hat{x}$ and $\hat{y}$, and hence $\hat{f}_2$ and $\hat{g}_3$, are functions of $t$. As shown in [3], equation (4) is reducible, and hence contains a linear factor, for those values of $t$ for which $\hat{f}_2(\hat{x}, \hat{y})$ and $\hat{g}_3(\hat{x}, \hat{y})$ have a linear or quadratic factor in common. These factors correspond to lines on the cubic surface, and our goal is to find the values of $t$ which produce these lines.

The way in which a common factor of $\hat{f}_2(\hat{x}, \hat{y})$ and $\hat{g}_3(\hat{x}, \hat{y})$ corresponds to a line on the cubic surface is as follows. A linear factor is of the form $c_1\hat{x} + c_2\hat{y}$, and a quadratic factor is of the form $c_1\hat{x}^2 + c_2\hat{x}\hat{y} + c_3\hat{y}^2$ and can be split into two such linear factors, possibly with complex coefficients. Since $c_1\hat{x} + c_2\hat{y}$ was obtained by intersecting the plane $\hat{z} = 0$ with the surface, this implies that the line $c_1\hat{x} + c_2\hat{y} = 0, \hat{z} = 0$ lies on the surface. The substitutions described earlier in this section may be traced backwards in order to obtain the line in the original $(x, y, z)$ coordinates. Thus each value of $t$ for which $\hat{f}_2(\hat{x}(t), \hat{y}(t))$ and $\hat{g}_3(\hat{x}(t), \hat{y}(t))$ have a common factor gives rise to a line on the surface.

The values of $t$ may be obtained by taking the resultant of $\hat{f}_2(\hat{x}, \hat{y}, t)$ and $\hat{g}_3(\hat{x}, \hat{y}, t)$ by eliminating either $\hat{x}$ or $\hat{y}$. Since $\hat{f}_2$ and $\hat{g}_3$ are homogeneous in $\{\hat{x}, \hat{y}\}$ it does not matter with respect to which variable the resultant is taken[28]; the result will have the other variable raised to the sixth power as a factor. Apart from the factor of $\hat{x}^6$ or $\hat{y}^6$, the resultant consists of an 81st degree polynomial $P_{81}(t)$ in $t$. At first glance it would appear that there could be 81 values of $t$ for which a line on the cubic surface is produced, but this is not the case:

**Theorem 1:** The polynomial $P_{81}(t)$ obtained by taking the resultant of $\hat{f}_2$ and $\hat{g}_3$ factors as $P_{81}(t) = P_{27}(t)[P_3(t)]^6[P_6(t)]^6$, where $P_3(t) = B''t^3 + F''t^2 + D''t + A''$, the denominator of $K(t)$ and $L(t)$, and $P_6(t)$ is the numerator of $\overline{S}(t)$ $\quad (P_6(t) = \overline{S}(t)[P_3(t)^2])$.

**Sketch of proof:** This proof was performed through the use of the symbolic manipulation program Maple [14]. When expanded out in full, $P_{81}(t)$ contains hundreds of thousands of terms, so a direct approach was not possible. Instead, $P_{81}(t)$ was shown to be divisible by both $[P_3(t)]^6$ and $[P_6(t)]^6$.

When $\hat{f}_2$ and $\hat{g}_3$ were expressed in terms of the numerators of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$, it was possible to take the resultant without overflowing the memory capabilities of the machine. The resultant could be factored, and $[P_6(t)]^6$ was found to be one of the factors.

The factor $[P_3(t)]^6$ proved to be more difficult to obtain. After the factor $[P_6(t)]^6$ was removed, the remaining factor was split into several pieces, according to which powers of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$ they contained. These pieces were each divided by $[P_3(t)]^6$, and the remainders taken. The remainders were expressed as certain polynomials times various powers of $P_3(t)$, as in $a_0(t) + a_1(t)P_3(t) + a_2(t)[P_3(t)]^2 + a_3(t)[P_3(t)]^3 + a_4(t)[P_3(t)]^4 + a_5(t)[P_3(t)]^5$. We were able to show that $a_0(t)$ is in fact divisible by $P_3(t)$. Then we could show that $a_0(t)/P_3(t) + a_1(t)$ is also divisible by $P_3(t)$, and so on up the line until we could show the whole remaining factor is divisible by $[P_3(t)]^6$. Details are given in Appendix B.

The solutions of $P_{27}(t) = 0$ correspond to the 27 lines on the cubic surface. A method of partial classification is suggested by considering the number of real roots of $P_{27}(t)$: if it has 27, 15, or 7 real roots the cubic surface is $F_1$, $F_2$, or $F_3$, respectively, and if $P_{27}(t) = 0$ has three real roots the surface can be either $F_4$ or $F_5$. However, this is not quite accurate. In exceptional cases, $P_{27}(t)$ may have a double root at $t = t_0$, which corresponds to $\hat{f}_2$ and $\hat{g}_3$ sharing a quadratic factor. If this quadratic factor is reducible over the reals, the double root corresponds to two (coplanar) real lines; if the quadratic factor has no real roots it corresponds to two coplanar complex conjugate lines.

**Theorem 2:** *Simple real roots of $P_{27}(t) = 0$ correspond to real lines on the surface.*

**Proof:** Let $t_0$ be a simple real root of $P_{27}(t) = 0$. Since $P_{27}(t)$ is a factor of the resultant of $\hat{f}_2$ and $\hat{g}_3$ obtained by eliminating $\hat{x}$ or $\hat{y}$, $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ must have a linear or quadratic factor in common. If $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ have just a linear factor in common, then that factor is of the form $c_1\hat{x} + c_2\hat{y}$ where $c_1$ and $c_2$ are real constants since all the coefficients of $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ are real and $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ are homogeneous in $\hat{x}$ and $\hat{y}$. In this case the real line $c_1\hat{x} + c_2\hat{y} = 0, \hat{z} = 0$ lies on the surface.

If $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ have a quadratic factor in common, then that factor is of the form $c_1\hat{x}^2 + c_2\hat{x}\hat{y} + c_3\hat{y}^2$. We will show that if this is the case, then $P_{27}(t)$ has at least a double root at $t = t_0$. This will be sufficient to prove that simple roots of $P_{27}(t)$ can only correspond to common linear factors of $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$, and hence real lines on the cubic surface.

If we write $\hat{f}_2(\hat{x}, \hat{y}, t) = Q_1(t)\hat{x}^2 + Q_2(t)\hat{x}\hat{y} + Q_3(t)\hat{y}^2$ and $\hat{g}_3(\hat{x}, \hat{y}, t) = Q_4(t)\hat{x}^3 + Q_5(t)\hat{x}^2\hat{y} + Q_6(t)\hat{x}\hat{y}^2 + Q_7(t)\hat{y}^3$, then the resultant of $\hat{f}_2(\hat{x}, \hat{y}, t)$ and $\hat{g}_3(\hat{x}, \hat{y}, t)$ obtained by eliminating $\hat{x}$ is

$$R(\hat{f}_2, \hat{g}_3) = \begin{vmatrix} Q_1(t) & Q_2(t) & Q_3(t) & 0 & 0 \\ 0 & Q_1(t) & Q_2(t) & Q_3(t) & 0 \\ 0 & 0 & Q_1(t) & Q_2(t) & Q_3(t) \\ Q_4(t) & Q_5(t) & Q_6(t) & Q_7(t) & 0 \\ 0 & Q_4(t) & Q_5(t) & Q_6(t) & Q_7(t) \end{vmatrix} \hat{y}^6 \ . \tag{5}$$

We need to show that if $\hat{f}_2(\hat{x}, \hat{y}, t)$ and $\hat{g}_3(\hat{x}, \hat{y}, t)$ have a quadratic factor in common when $t = t_0$, then $R(\hat{f}_2, \hat{g}_3)/\hat{y}^6$ has a double root at $t = t_0$. This is equivalent to showing that $R(\hat{f}_2(t_0), \hat{g}_3(t_0)) = 0$ and $(d/dt)[R(\hat{f}_2(t_0), \hat{g}_3(t_0))] = 0$. If $\hat{f}_2(\hat{x}, \hat{y}, t_0)$ and $\hat{g}_3(\hat{x}, \hat{y}, t_0)$ have a quadratic factor in common, then $\hat{g}_3(t_0) = k(c_1\hat{x} - c_2\hat{y})\hat{f}_2(t_0)$ for some real constants $k$, $c_1$, and $c_2$. Thus $Q_4(t_0) = kc_1Q_1(t_0)$, $Q_5(t_0) = k[c_1Q_2(t_0) - c_2Q_1(t_0)]$, $Q_6(t_0) = k[c_1Q_3(t_0) - c_2Q_2(t_0)]$, and $Q_7(t_0) = -kc_2Q_3(t)$. Making these substitutions in (5), we find that indeed both $R(\hat{f}_2(t_0), \hat{g}_3(t_0)) = 0$ and $(d/dt)[R(\hat{f}_2(t_0), \hat{g}_3(t_0))] = 0$. ■

To summarize, the simple real roots of $P_{27}(t) = 0$ correspond to real lines on the cubic surface. Double real roots may correspond to either real or complex lines, depending on whether the quadratic factor $\hat{f}_2(\hat{x}, \hat{y}, t)$ and $\hat{g}_3(\hat{x}, \hat{y}, t)$ have in common is reducible or not over the reals. Higher order roots

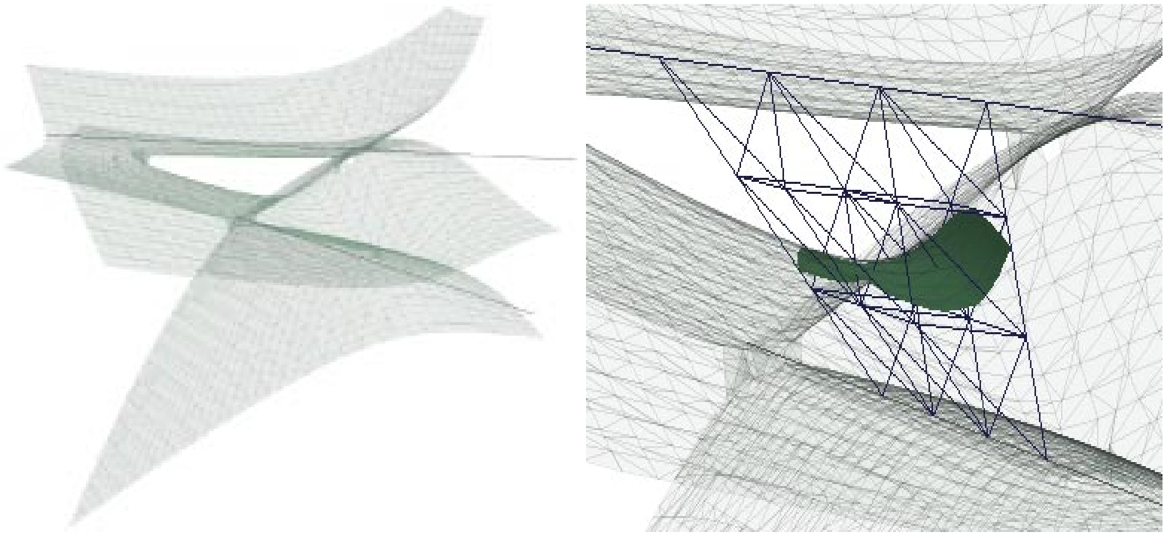

Figure 2: An $F_1$ cubic surface with two skew lines out of its 27 real straight lines (left), and a zoom-in on that surface showing the Bézier patch with its bounding tetrahedron, determined by the two skew lines as opposite edges (right).

indicate some type of singularity. Complex roots can only correspond to complex lines in nonsingular cases. If $t_0$, a complex root of $P_{27}(t) = 0$, corresponded to a real line $c_1\hat{x} - c_2\hat{y}$ on the surface, then $\overline{t_0}$ would correspond to the same line, as a real line is its own complex conjugate. Thus one real line would be leading to two distinct values for $t_0$.

## 5 Parametrizations with Real Skew Lines

When the cubic surface is of class $F_1$, $F_2$, or $F_3$, it contains at least two real skew lines, and the parametrization in [3] is used. Figures 2, 3, and 4 show $F_1$, $F_2$ and $F_3$ surfaces, respectively.

The picture on the right in figure 2 shows a patch entirely within a tetrahedron with two of its edges along the skew lines, and each point of the displayed patch is the third point of intersection of the cubic surface with a line passing through a point on each of the skew edges. Having obtained skew lines $\mathbf{l}_1(u) = [x_1(u) \;\; y_1(u) \;\; z_1(u)]$ and $\mathbf{l}_2(v) = [x_1(v) \;\; y_1(v) \;\; z_1(v)]$, we consider the net of lines passing through a point on each. This is given by

$$\frac{z - z_1}{x - x_1} = \frac{z_2 - z_1}{x_2 - x_1} \qquad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \, .$$

Solving these for $y$ and $z$ in terms of $x$, and substituting into the cubic surface $f(x, y, z) = 0$ gives a

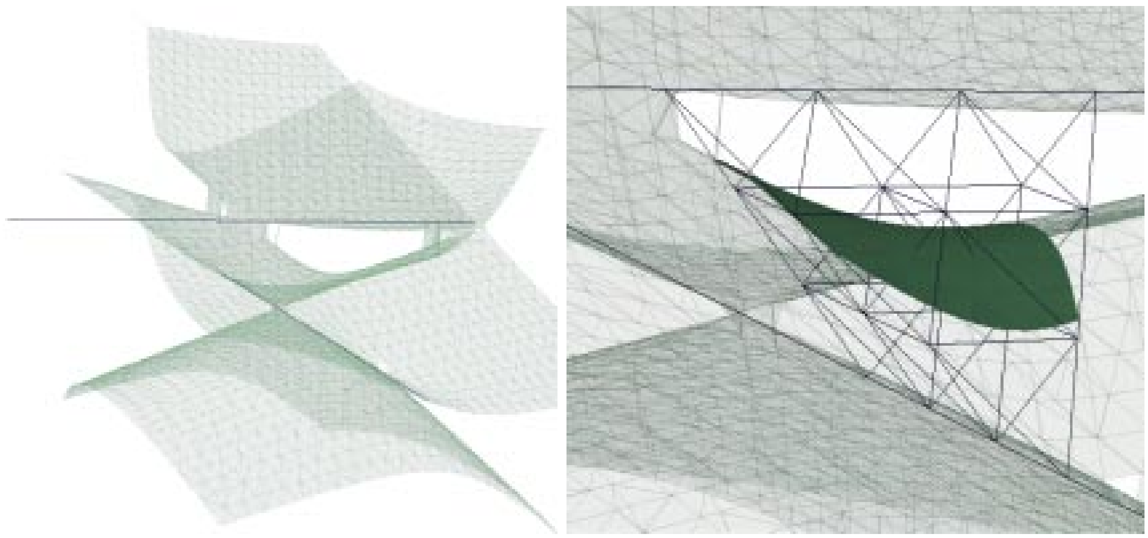

Figure 3: An $F_2$ cubic surface with two skew lines out of its 15 real straight lines (left), and a zoom-in on that surface showing the Bézier patch with its bounding tetrahedron, determined by the two skew lines as opposite edges (right).

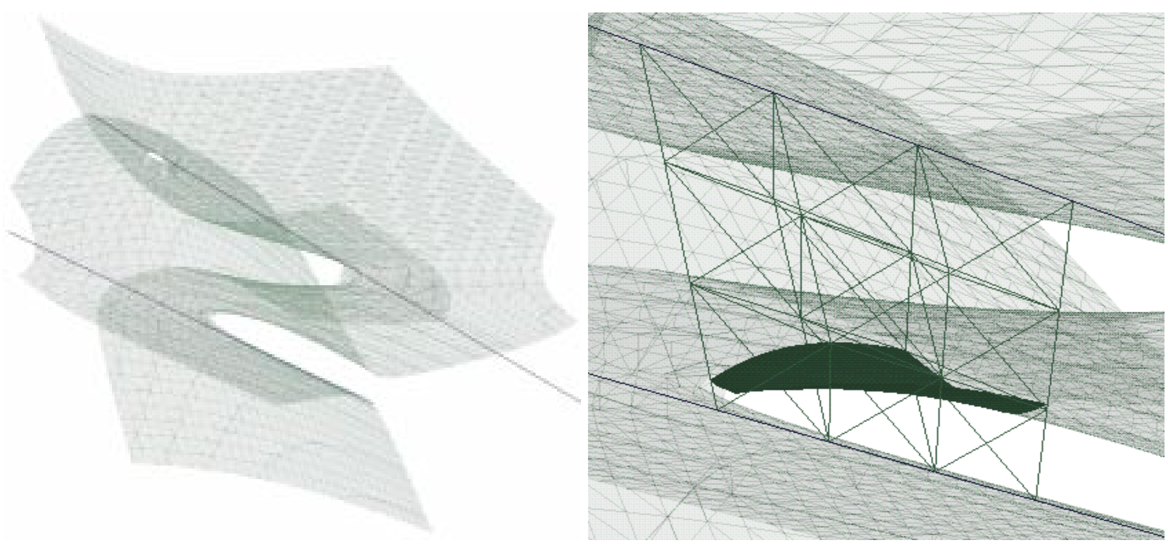

Figure 4: An $F_3$ cubic surface with two skew lines out of its 7 real straight lines (left), and a zoom-in on that surface showing the Bézier patch with its bounding tetrahedron, determined by the two skew lines as opposite edges (right).

cubic equation in $x$ with coefficients in $u$ and $v$, say $G(x,u,v) = 0$. Since $x = x_1$ and $x = x_2$ satisfy this equation, $G(x,u,v)$ is divisible by $x - x_1$ and $x - x_2$, and we have that

$$H(u,v,x) = \frac{G(x,u,v)}{[x - x_1(u)][x - x_2(v)]} \tag{6}$$

is a linear polynomial in $x$. This is solved for $x$ as a rational function of $u$ and $v$. Rational functions for $y$ and $z$ are obtained analogously.

The parametrization (1) is then computed as in (2):

$$(x,\ y,\ z) = (x(u,v),\ y(u,v),\ z(u,v)) = (f_1(u,v)/f_4(u,v),\ \ f_2(u,v)/f_4(u,v),\ f_3(u,v)/f_4(u,v))$$

where

$$\begin{aligned} f_1(u,v) &= a(u,v)x_1(u) + b(u,v)x_2(v) \\ f_2(u,v) &= a(u,v)y_1(u) + b(u,v)y_2(v) \\ f_3(u,v) &= a(u,v)z_1(u) + b(u,v)z_2(v) \\ f_4(u,v) &= a(u,v) + b(u,v)\ \ , \end{aligned} \tag{7}$$

with

$$a(u,v) = \nabla f(\mathbf{l}_2(v)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]\ , \qquad b(u,v) = \nabla f(\mathbf{l}_1(u)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]$$

In this notation $-f_1(u,v)$ and $f_4(u,v)$ are the coefficients of $x^0$ and $x^1$, respectively, in $H(u,v,x)$. The symbolic manipulation program Maple was used to verify that the expressions $f_1(u,v)/f_4(u,v)$, $f_2(u,v)/f_4(u,v)$, and $f_3(u,v)/f_4(u,v)$ do simplify to $x$, $y$, and $z$ respectively.

Using floating-point arithmetic, it may be the case that some terms with very small coefficients appear in $f_1(u,v)$, $f_2(u,v)$, $f_3(u,v)$, and $f_4(u,v)$ when the coefficients should in fact be zero. Specifically, these are the terms containing $u^3$, $v^3$, $u^4$, $v^4$, $u^3v$ and $uv^3$ in $f_1$, $f_2$, and $f_3$, and terms containing $u^3$ and $v^3$ in $f_4$. These coefficients were shown to be zero using Maple, so in the algorithm they are subtracted off in case they appear in the construction of $f_1$, $f_2$, $f_3$, and $f_4$.

## 6 Parametrizations without Real Skew Lines

When the cubic surface is of class $F_4$ or $F_5$ it does not contain any pair of real skew lines. In the $F_4$ case we derive a parametrization using complex conjugate skew lines, and in the $F_5$ case we obtain a

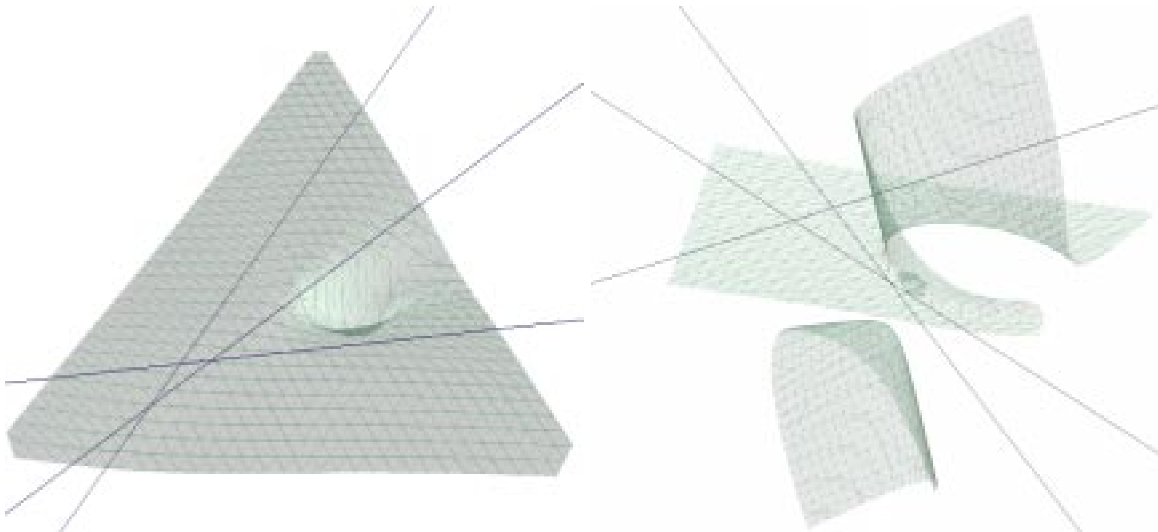

Figure 5: (Left) An $F_4$ cubic surface with all its three real lines, which are coplanar. The two skew complex conjugate lines used in the parametrization are not displayed. (Right) An $F_5$ cubic surface, together with all its three real lines, which are coplanar. An $F_5$ cubic surface has no skew lines, real or complex. This particular example has multiple real sheets.

parametrization by parametrizing conic sections which are the further intersections of the cubic surface with planes through a real line on the surface.

### 6.1 The $F_4$ Case

In this case there are 12 pairs of complex conjugate lines. For 6 of these pairs, the two lines intersect (at a real point). In the other 6 pairs, the two lines are skew. Let one pair of complex conjugate skew lines be given by $(x_1(w),\ y_1(w),\ z_1(w))$ and $(x_1(\overline{w}),\ y_1(\overline{w}),\ z_1(\overline{w}))$, where $w = w_R + w_I i$ is a complex-valued parameter with $w_R$ and $w_I$ as its real and imaginary parts. Here $x_1$, $y_1$, and $z_1$ are (linear) complex functions of a complex variable, and $x_2$, $y_2$, $z_2$ may be considered to be the complex conjugates of $x_1$, $y_1$, $z_1$. Also the real parameters $w_R$ and $w_I$ are unrestricted. Then the parametrization is again given by (7), with $u$ and $v$ being replaced by $w = w_R + w_I i$ and $\overline{w} = w_R - w_I i$, respectively. Even though the quantities $x_i$, $y_i$, and $z_i$ are complex, the expressions for $x(u, v) = x(w, \overline{w})$, $y(u, v) = y(w, \overline{w})$, and $z(u, v) = z(w, \overline{w})$ turn out to be real when $x_2$, $y_2$, and $z_2$ are the complex conjugates of $x_1$, $y_1$, and $z_1$. The symbolic manipulation program Maple was used to verify that the quantities $f_1(w, \overline{w})/i$, $f_2(w, \overline{w})/i$, $f_3(w, \overline{w})/i$ and $f_4(w, \overline{w})/i$ are all real when $(x_1, y_1, z_1)$ and

$(x_2, y_2, z_2)$ are complex conjugates. Note that the functions $f_i(w, \overline{w})$ may be regarded as functions of the two real variables $w_R$ and $w_I$.

Using floating-point arithmetic, it may be the case that some terms with very small coefficients appear in $f_1(w, \overline{w})$, $f_2(w, \overline{w})$, and $f_3(w, \overline{w})$ when the coefficients should in fact be zero. Specifically, these are the terms containing $w_R^3 w_I$ and $w_R w_I^3$. These coefficients were shown to be zero using Maple, so in the algorithm they are subtracted off in case they appear in the construction of $f_1$, $f_2$, and $f_3$.

**Theorem 3:** The parametrization described in Section 5 provides a valid parametrization of an $F_4$ cubic surface when $u$ and $v$ are replaced by $w = w_R + w_I i$ and $\overline{w} = w_R - w_I i$, respectively. The parameters $w_R$ and $w_I$ range over all real values. Each real point on the $F_4$ surface, except for those corresponding to base points of the parametrization, is obtained for exactly one complex value of $w$.

**Proof of Theorem 3:** A classical result from line geometry asserts that two skew complex conjugate lines possess a two-parameter family of real lines intersecting them, and that every real point in space lies on exactly one of these lines of the two-parameter family. Thus given an arbitrary real point $(x_0, y_0, z_0)$ and two skew complex conjugate lines $\mathbf{l}_1(u)$ and $\overline{\mathbf{l}_1}(v)$ on the cubic surface, there is a unique pair of real numbers $(a_0, b_0)$ such that the three points $(x_0, y_0, z_0)$, $\mathbf{l}_1(a_0 + b_0 i)$, and $\overline{\mathbf{l}_1}(a_0 - b_0 i)$ are collinear. This value of $(a_0, b_0)$, when inserted into the parametrization (7), gives back $(x_0, y_0, z_0)$, unless $(a_0, b_0)$ happens to make the fractions in (7) 0/0, which means that $(a_0, b_0)$ is a base point of the parameter map. ∎

### 6.2 The $F_5$ Case

When the cubic surface is of class $F_5$ (example shown in figure 5 (right)) it does not have any complex conjugate skew lines. One could attempt to use one real line and one complex line, or two non-conjugate complex skew lines, and proceed as before. However, there is no simple way to describe the values the parameters $u$ and $v$ may take on. In the $F_1$, $F_2$, and $F_3$ cases, $u$ and $v$ were unrestricted real parameters. In the $F_4$ case, we let $u = w_R + w_I i$ and $v = w_R - w_I i$, and obtained a parametrization in which $w_R$ and $w_I$ are unrestricted. If we try the same idea with one real and one complex line, or two complex lines which are not conjugates, and let $w_R$ and $w_I$ be unrestricted, then $w_R$ and $w_I$ are complicated functions of $u$ and $v$, typically seventh degree polynomials.

In [26], a rational parametrization based on tangent planes at points lying on a real line is given. However, this parametrization is two-to-one, meaning that there are typically two values of $(u, v)$

corresponding to points on the cubic surface, instead of the one-to-one map resulting when both curves used in the parametrization are line, as in the $F_1$ through $F_4$ cases. Another approach used in [26] involves a square root of a fourth degree polynomial in two variables. The surface is rotated so that the $z^3$ term vanishes, and then the quadratic formula may be applied by regarding the surface as a quadratic in $z$. While this method is quite workable, it does not lend itself readily to geometric interpretation as the skew line parametrizations do.

With this in mind, we propose to parametrize the surface by parametrizing planes through one of the real lines on the surface, and then by parametrizing the conic sections which are the further intersections of these planes with the cubic surface. Thus the curves traced out when one of the parameters is held constant will be these conics. The parametrization of the conics will be that of [2]. With this procedure we have to use two distinct parametrizations; one which works when the conics are ellipses and the other for hyperbolas. Each of these requires one square root of a univariate polynomial.

The procedure for finding the parametrization starts out like the ones for the $F_1$ through $F_4$ cases. In this case three coplanar real lines and 24 complex lines are determined, and the complex lines are found to come in 12 coplanar conjugate pairs. Since the methods of the other cases involving skew lines do not work here, one of the real lines is chosen to be mapped into the $x$-axis and the plane of the three real lines is mapped into the $xy$-plane. Specifically, suppose a real line $\mathbf{l}$ is given by $\mathbf{l}(u) = (A + Bu, C + Du, E + Fu)$ and that the normal to the plane is given by $\mathbf{N} = (N_1, N_2, N_3)$. $\mathbf{N}$ is obtained by taking the cross product of the (unit) direction vectors of two of the real lines, or by taking any unit vector perpendicular to the real lines if they are all parallel. Next, let $\mathbf{B} = (B_1, B_2, B_3)$ be the cross product of the direction vector of $\mathbf{l}$ with $\mathbf{N}$. We move a point on $\mathbf{l}$ to the origin by the translation $x = x' + A$, $y = y' + C$, $z = z' + E$, and then apply the transformation

$$\begin{aligned} x' &= (B_2N_3 - B_3N_2)x'' + (FN_2 - DN_3)y'' + (DB_3 - FB_2)z'' \\ y' &= (B_3N_1 - B_1N_3)x'' + (BN_3 - FN_1)y'' + (FB_1 - BB_3)z'' \\ z' &= (B_1N_2 - B_2N_1)x'' + (DN_1 - BN_2)y'' + (BB_2 - DB_1)z'' \ . \end{aligned} \tag{8}$$

This brings $\mathbf{l}$ to the $x''$ axis and the plane of the real lines to $z'' = 0$.

Planes through the $x''$-axis can be parametrized by $y'' = uz''$ for real values of $u$. All planes through the $x''$-axis are obtained except for $z'' = 0$, the plane containing the three real lines already found. The cubic surface now has an equation of the form $f''(x'', y'', z'') = 0$, and satisfies $f''(x'', 0, 0) = 0$.

If we now make the substitution $y'' = uz''$ into $f''(x'', y'', z'')$, we obtain an equation that factors as $z''g''(x'', z'') = 0$, where $g''(x'', z'')$ is a quadratic in $x''$ and $z''$. The factor of $z''$ indicates that the line $z'' = 0$ is in the intersection of the cubic surface and the plane $y'' = uz''$ for any real $u$. The conic section $g(x'', z'') = 0$ is parametrized as in [2]: Let $g(x'', z'') = ax''^2 + bz''^2 + cx''z'' + dx'' + ez'' + f$, and the discriminant $k = c^2 - 4ab$. The quantities $a$ through $f$ are polynomials in $u$.

If $k < 0$, the conic is an ellipse, and is parametrized by

$$
\begin{aligned}
x'' &= \frac{[af(ce-2bd) - d(t_2+t_3)]v^2 + [df(ce-2bd) - 2ft_3]v + f^2(ce-2bd)}{a(t_1+t_3)v^2 - df(c^2-4ab)v + f(t_1-t_3)} \\
z'' &= \frac{f(c^2-4ab)(av^2+dv+f)}{a(t_1+t_3)v^2 - df(c^2-4ab)v + f(t_1-t_3)} \ ,
\end{aligned}
$$

where

$$
t_1 = ae^2 + bd^2 - cde \ , \quad t_2 = t_1 + f(c^2 - 4ab) \ , \quad t_3 = \sqrt{t_1 t_2} \ .
$$

This gives real points only when the terms $t_1$ and $t_2$ have the same sign or are zero. If $t_1$ and $t_2$ have opposite sign, $g(x'', z'') = 0$ has no real points, and geometrically this means that the plane $y'' = uz''$ intersects the cubic surface only in the $x''$-axis. Thus values of $u$ should be restricted to those that give non-negative values for $t_1 t_2$. Upon back substitution using $y'' = uz''$ and (8), in the final parametrization $x$, $y$, and $z$ are given by quotients of functions of the form $Q_1(u,v) + Q_2(u,v)\sqrt{Q_3(u)}$, where $Q_1(u,v)$ is of degree six in $u$ and two in $v$, $Q_2(u,v)$ is of degree one in $u$ and two in $v$, and $Q_3(u)$ is of degree nine in $u$ alone. Due to the use of floating-point arithmetic, a nonzero coefficient for $u^{10}$ may appear in $Q_3(u)$, and this is subtracted off in case it does show up.

If $k \geq 0$, the conic is a hyperbola or parabola, and is parametrized by

$$
\begin{aligned}
x'' &= \frac{a(c + \sqrt{c^2-4ab})v^2 + 2aev + f(c - \sqrt{c^2-4ab})}{2a\sqrt{c^2-4ab}\,v + 2ae - cd + d\sqrt{c^2-4ab}} \\
z'' &= \frac{-2a(av^2+dv+f)}{2a\sqrt{c^2-4ab}\,v + 2ae - cd + d\sqrt{c^2-4ab}} \ .
\end{aligned}
$$

Here real values are given for all $u$ and $v$ for which the denominators are nonzero. In the final parametrization $x$, $y$, and $z$ are given by quotients of functions of the form $[Q_1(u,v) + Q_2(u,v)\sqrt{Q_3(u)}]$ $/[Q_4(u) + Q_5(u,v)\sqrt{Q_3(u)}]$, where $Q_1(u,v)$ is of degree three in $u$ and two in $v$, $Q_2(u,v)$ is of degree one in $u$ and two in $v$, $Q_3(u)$ is of degree four in $u$ alone, $Q_4(u)$ is of degree three in $u$ alone, and $Q_5(v)$ is of degree one in each of $u$ and $v$.

This parametrization, partly by hyperbolas/parabolas and partly by ellipses, sweeps out the entire surface except possibly for the three real lines on the plane $z'' = 0$. These lines cannot normally be reached as $u$ would have to approach $\pm\infty$ in view of the relation $y'' = uz''$. In some cases one of the lines, specifically $y'' = 0, z'' = 0$, may be obtained for a specific value of $u$ when the intersection of the plane $y'' = uz''$ with the cubic surface consists of two lines, with the line $y'' = 0, z'' = 0$ counting as having been hit twice. This would be the case if the intersection was of the form $z''^2(z'' - kx'')$ for some constant $k$ for the particular value of $u$. This transition of iso-parameter curves from hyperbolas to ellipses is analogous to the transition of planar cross sections of a right circular cone.

# 7 Classification and Straight Lines from Parametric Equations

We also consider the question of deriving a classification and generating the straight lines of the cubic surface given its rational parametric equations (equation (1) above):

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)},$$

Note that given an arbitrary parametrization, the fact that it belongs to a cubic surface can be computed by determining the parametrization base points and multiplicities.

The computation of real base points which are the simultaneous zeros of $f_1 = f_2 = f_3 = f_4 = 0$, are obtained by first computing the real zeros of $f_1 = f_2 = 0$ using resultants and subresultants, via the method of birational maps [6] and then keeping those zeros which also satisfy $f_3 = f_4 = 0$. The classification follows from the reality of the base points, as detailed in the preliminaries section.

Having determined the base points, the straight lines on the cubic surface are then determined by the image of these points and combinations of them. In general there can be six real base points for cubic surfaces. The image of each of the six base points under the parametrization map yields a straight line on the surface. Next the fifteen pairs of base points define lines in the $u, v$ parameter space, whose images under the parametrization map also yield straight lines. Finally the six different conics in the $u, v$ parameter space which pass through distinct sets of five base points, also yield straight line images under the parametrization map. See Bajaj and Royappa [9] for techniques to find parametric representations of the straight lines which are images of these base points. The question of determining parametric representations of the straight lines which are the images of parameter lines or parameter conics is for now, open.

Normally a cubic surface parametrization has six base points, but in the case of our parametrizations for the $F_1$, $F_2$, $F_3$, and $F_4$ surfaces, the number of base points is reduced to five. This happens because the degree of the parametrization is sufficiently small. In the $F_1$, $F_2$, and $F_3$ cases, neither $u$ nor $v$ appears to a power higher than the second. Consider the intersection of the parametrized surface with a line in 3-space. Let the line be given as the intersection of two planes $a_ix + b_iy + c_iz + d_i = 0$ for $i = 1, 2$. Then when the substitutions $x = f_1(u,v)/f_4(u,v)$, $y = f_2(u,v)/f_4(u,v)$, $z = f_3(u,v)/f_4(u,v)$ are made into the equations of the lines, we obtain polynomials of degree two in each of $u$ and $v$. When resultants of these polynomials are taken to eliminate either $u$ or $v$, univariate polynomials of degree eight are obtained. This indicates that there could be as many as eight intersection points of the line with the surface. However, a cubic surface will intersect the line in only three (possibly complex) points, counting multiplicity and solutions at infinity. The difference between these two results (eight and three) is the number of base points. In the $F_4$ case, the degrees of the numerators and denominator of the components of the parametrization are 4 and 3, respectively. However, making the linear change of variables $w_R = (u+v)/2, w_I = [(v-u)/2]i$ yields a parametrization in which $u$ and $v$ each appear to powers at most 2, just as in the $F_1$, $F_2$, and $F_3$ cases. The computation of the location of the base points does not depend on the coordinate system used, and since these computations are done over the field of complex numbers, such a complex-valued linear transformation is permissible. Thus the same argument applies, and there are five values of $(u, v)$, and consequently five values of $(w_R, w_I)$, which make $f_1, f_2, f_3$, and $f_4$ all equal to zero. A general cubic parametrization would have led to nine possible intersection points when considering the algebraic equations, and hence six, the difference of nine and three, is the number of base points for such a parametrization. Our parametrization for $F_5$ surfaces may have six base points.

Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be the two skew lines used in the parametrization, whether they be real or complex. The base points $(u, v)$ correspond to lines on the cubic surface which intersect both $\mathbf{l}_1$ and $\mathbf{l}_2$. Real base points correspond to real lines and complex base points correspond to complex lines. One of the many useful results on nonsingular cubic surfaces is that given any two (real or complex) skew lines on the surface, there are exactly five lines that intersect both [27]. For an $F_1$ surface, the five transversal lines, and the base points, are all real. Thus those five real lines are missed by the parametrization (1). For an $F_2$ surface, three of the base points are real and the other two form a complex conjugate pair. The parametrization (1) consequently misses the three real lines incident to both $\mathbf{l}_1$ and $\mathbf{l}_2$. In addition, if the two transversal complex conjugate lines are coplanar and have a real intersection point, that point is also missed. For both $F_3$ and $F_4$ surfaces, one base point is real and the other four form two conjugate pairs. In each of these cases there is one real line through both $\mathbf{l}_1$ and $\mathbf{l}_2$, and that line is missed. Again, if a pair of transversal complex conjugate lines is coplanar, their real intersection point is missed, so there may be two such isolated points for $F_3$ and $F_4$. the missing points on the surface can be approached as $(u, v)$ approaches the corresponding base point in an appropriate manner. Skew complex conjugate lines corresponding to complex base points result in no missed real surface points.

In addition to the transversal lines, two conic sections are also missed in the parametrization of the $F_1$, $F_2$, and $F_3$ surfaces. One conic is obtained as follows: take the intersection of the plane containing $\mathbf{l}_1(u)$ and perpendicular to $\mathbf{l}_2(v)$ with the cubic surface. This intersection consists of $\mathbf{l}_1$ plus a conic. It turns out that the value of $v$ at which $\mathbf{l}_2$ intersects this plane tends to $\pm\infty$. Thus points on the conic are not obtained for finite values of $v$, even though the line $\mathbf{l}_1$ does turn out to be reachable. The other missing conic is found by interchanging the roles of $\mathbf{l}_1$ and $\mathbf{l}_2$. These two conics lie on parallel planes, and are obtained if $u$ or $v$, respectively, is allowed to approach $\pm\infty$. In the $F_4$ case, no real conics are missed. The plane containing $\mathbf{l}_1$ and perpendicular to $\mathbf{l}_2$ intersects the cubic surface in the complex line $\mathbf{l}_1$ and a complex, not a real, conic section.

## 8 Conclusion and Future Research

We have presented a method of extracting real straight lines and from there a rational parametrization of four out of five families of nonsingular cubic surfaces. The parametrizations of the real cubic surface components are constructed using a pair of real skew lines for those three families which have them, and remarkably using a complex conjugate pair of skew lines, in a fourth family. In each of these, the entire real surface is covered except for one, three, or five lines which intersect both skew lines, one or two isolated points, and two conic sections. The missing conics can be recovered through the use of projective instead of real coordinates. For the last family, in which two real skew lines do not exist, in order to cover the whole surface we had to use two separate parametrizations, each involving a square root. Fortunately many graphics applications, such as the triangulation of a real surface, will involve only the classes of cubics which do contain real skew lines. These real skew lines will correspond to non-intersecting edges of the tetrahedra. This procedure may also be used when the cubic surface is given in Bernstein-Bézier form as shown in figures 2, 3, and 4. Open problems remain in computing the images of curves on the cubic surface corresponding to real base points of high multiplicity, as well as in efficiently generating Bernstein-Bézier forms for the $F_5$ case. All figures of the cubic surfaces shown in this paper were made using the GANITH and SPLINEX toolkits of the SHASTRA system [4].

An additional associated line of future research is in computing invariants for cubic surfaces based on their straight lines. In Computer Vision, as pointed out in [12, 18, 23], it is essential to derive properties of curves and surfaces which are invariant to perspective projection and to be able to compute these invariants reliably from perspective image intensity data. In connection with the First Fundamental Theorem of Invariant Theory, referring to [1, 23] for details, we attempt to calculate complete systems of symbolic invariants of cubic surfaces. In doing these calculations, it is important to know all the relations among a set of invariants which is the content of the Second Fundamental Theorem of Invariant Theory.

## Appendix A: Examples

In this appendix we provide examples of the parametrization of $F_1$, $F_4$, and $F_5$ cubic surfaces.

### An $F_1$ Surface

The $F_1$ surface is given by the implicit equation

$$f(x,y,z) = 16x^3 - 10y^3 - 156z^3 + 3x^2y + 101x^2z - 38xy^2 + 72y^2z + 39xz^2 - 74yz^2 - 81xyz \\ - 389x^2 - 98y^2 + 1988z^2 + 470xy - 291xz + 318yz + 332x - 718y - 8114z + 11082 = 0 \ .$$

The point $(1,2,3)$ lies on this surface. Using this point, the polynomial $P_{27}(t)$ as computed by the algorithm in Section 4 factors as

$$t(t-2)(t+4)(t+8)(2t-1)(2t+1)(3t-1)(4t-3)(4t+7)(4t+17)(4t-43) \\ (5t-1)(5t+11)(5t-16)(5t+24)(5t+32)(5t+74)(8t-11)(20t+71) \\ (25t-8)(37t-32)(205t-116)(215t+32)(295t+1216)(755t-4576) \ .$$

When the solution $t=0$ is substituted into the expressions $\hat{f}_2(\hat{x},\hat{y})$ and $\hat{g}_3(\hat{x},\hat{y})$ (from equation (4)), it is found that their common factor is $\hat{x}+\hat{y}$. Thus the line $\hat{x}+\hat{y}=0, \hat{z}=0$ lies on the cubic surface, and transforming back to the original coordinates, this turns out to be the line $\mathbf{l}_1(u) = (x,y,z) = (u+3, -u+2, -u+3)$. When the solutions $t = 1/2$ is chosen, the common factor of $\hat{f}_2(\hat{x},\hat{y})$ and $\hat{g}_3(\hat{x},\hat{y})$ is $\hat{y}$. Therefore the line $\hat{y}=0, \hat{z}=0$ is on the surface, and in the original coordinates this is $\mathbf{l}_2(v) = (2, v-2, v/3+3)$. These two lines are skew, and many other choices are possible. See figure 2.

With these lines, we obtain the parametrization $(x(u,v), y(u,v), z(u,v)) = (f_1(u,v)/f_4(u,v), f_2(u,v)$ $/f_4(u,v), f_3(u,v)/f_4(u,v))$ where

$$\begin{aligned}
f_1 &= 185u^2v^2 - 2151u^2v + 1602u^2 + 652uv^2 - 9972uv + 21708u + 291v^2 - 6981v + 19890 \\
f_2 &= 55u^2v^2 - 369u^2v - 1602u^2 + 603uv^2 - 6747uv + 11502u + 812v^2 - 10134v + 24660 \\
f_3 &= -105u^2v^2 + 2511u^2v - 14202u^2 + 568uv^2 - 5352uv + 324u + 497v^2 - 7503v + 16470 \quad (9) \\
f_4 &= 240u^2v - 2520u^2 + 185uv^2 - 2301uv + 3078u + 97v^2 - 2121v + 5490 \ .
\end{aligned}$$

Similarly, we can easily obtain the rational parametric biquadratic Bézier form [15, 20]

$$p(s,t) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij}c_{ij} \binom{2}{i} s^i(1-s)^{2-i} \binom{2}{j} t^j(1-t)^{2-j}}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} \binom{2}{i} s^i(1-s)^{2-i} \binom{2}{j} t^j(1-t)^{2-j}} ,$$

where

$$\begin{array}{ccc}
c_{00} = (1154, 1365, 6254)/1427, & c_{01} = (4020, 3827, 17672)/4093, & c_{02} = (428, 337, 1560)/367, \\
c_{10} = (4662, 3395, 16262)/3981, & c_{11} = (14370, 9105, 44874)/11081, & c_{12} = (2752, 1523, 7728)/1925, \\
c_{20} = (1814, 665, 5014)/1307, & c_{21} = (5257, 1684, 13523)/3535, & c_{22} = (317, 88, 759)/199, \\
w_{00} = -11416/3, & w_{01} = -8186/3, & w_{02} = -5872/3, \\
w_{10} = -2654, & w_{11} = -11081/6, & w_{12} = -3850/3, \\
w_{20} = -5228/3, & w_{21} = -3535/3, & w_{22} = -796.
\end{array}$$

At the end of Section 4 it was mentioned that some small coefficients arising from imperfect floating point computations are removed from the $f_i(u,v)$. In this example, when 15 digits precision is used, the terms truncated from $f_1, f_2, f_3$, and $f_4$ are

$$\begin{aligned}
&-2.7 \cdot 10^{-11}u^3 - 3.6 \cdot 10^{-13}uv^3 - 1.08 \cdot 10^{-12}v^3 \ , \\
&-1.35 \cdot 10^{-11}u^3v + 2.7 \cdot 10^{-11}u^3 + 3.6 \cdot 10^{-13}uv^3 - 7.2 \cdot 10^{-13}v^3 \ , \\
&-4.5 \cdot 10^{-12}u^3v - 4.05 \cdot 10^{-11}u^3 + 3.6 \cdot 10^{-13}uv^3 - 1.08 \cdot 10^{-12}v^3 \ , \\
&\text{and } 1.35 \cdot 10^{-11}u^3 - 3.6 \cdot 10^{-13}v^3 \ ,
\end{aligned}$$

respectively.

The five base points, where $f_1 = f_2 = f_3 = f_4 = 0$, are $(u,v) = (-1,\ 9/2)$, $(-5/4,\ 5)$, $(-12,\ 114/11)$, $(-37/29,\ 81/16)$, and $(-29/15,\ 156/23)$. These correspond to the lines $(2,\ w+4,\ -w+3)$, $(w +$

$1,\ -w+4,\ 5/3\,w+3)$, $(w+303/47,\ -62/121\,w+286/47,\ -94/121\,w+3)$, $(w+166/191,\ -99/128\,w+752/191,\ 191/128\,w+3)$, and $(w-502/113,\ 293/322\,w-122/113,\ 113/322\,w+3)$, respectively. As an example of what is meant by this correspondence, consider an arbitrary point $(x,y,z)$ in 3-space. The values of $u_0$ and $v_0$ for which the points $(x,y,z)$, $(u_0+3,-u_0+2,-u_0+3)$ and $(2,v_0-2,v_0/3+3)$ are collinear are given by

$$(u_0,\,v_0)=\left(\frac{-4x+y-3z+19}{2x-y+3z-15}\,,\,\frac{3(4x-y+5z-25)}{2x-y+3z-13}\right)\ . \tag{10}$$

When $(x,y,z)=(2,w+4,-w+3)$ is plugged into this expression, we obtain $(u_0,\,v_0)=(-1,\,9/2)$. Since this is a base point, however, plugging this into (9) yields $0/0$ for $x$, $y$, and $z$.

It is evident from (10) that a point $(x,y,z)$ on the cubic surface will be missed when a denominator is zero while the corresponding numerator is not. In this example these points lie on the planes $E_1$, given by $2x-y+3z=15$, and $E_2$, given by $2x-y+3z=13$. $E_1$ contains $\mathbf{l}_2$ while $E_2$ contains $\mathbf{l}_1$, and $E_1$ and $E_2$ are parallel.

The intersection of $E_1$ with the cubic surface consists of the line $\mathbf{l}_2$ and a conic section. It turns out that $\mathbf{l}_2$ may be obtained by (9), but not the conic cannot. In this example, substituting $(x,y,z)=(2,w-2,w/3+3)$ into (10) gives $185uv^2-2631uv+6642u+97v^2-2739v+8910=0$, and each point on this curve in the parameter space, except for the base point $(-1,9/2)$, gives rise to a point on $\mathbf{l}_2$. The conic may be parametrized by letting $u\to\pm\infty$ in (9). In this example we have

$$(x,y,z)=\left(\frac{185v^2-2151v+1602}{120(2v-21)},\frac{55v^2-369v-1602}{120(2v-21)},\frac{-35v^2+837v-4734}{40(2v-21)}\right)\ .$$

Symmetric arguments apply showing that $\mathbf{l}_1$ is obtained by the parametrization, and the other missing conic is found by letting $v\to\pm\infty$ in (9).

**An $F_4$ Surface**

The $F_4$ surface (shown in figure 5 (left)) is given by the implicit equation

$$\begin{aligned}f(x,y,z)=1696x^3-1196y^3+881z^3-2984x^2y-62x^2z+2424xy^2+1174y^2z-913xz^2-781yz^2\\+450xyz-1802x^2+443y^2-1217z^2+1786xz+266xy-1596yz+1696z=0\ .\end{aligned}$$

The polynomial $P_{27}(t)$ as computed by the algorithm in Section 4 is $(11t+1)(t^2-2t+2)P_{24}(t)$, where $P_{24}(t)$ is a polynomial of degree 24 with two real and 22 complex roots. The two complex roots of the factor $t^2-2t+2$, namely $t=1\pm i$, yield two skew complex conjugate lines. When $t=1+i$ is substituted into the expressions for $\hat{f}_2(\hat{x}, \hat{y})$ and $\hat{g}_3(\hat{x}, \hat{y})$, it is found that their common factor is $(3-i)\hat{x}+2\hat{y}$. Thus the line $(3-i)\hat{x}+2\hat{y}=0, \hat{z}=0$ lies on the cubic surface, and transforming back to the original coordinates, this is the line $\mathbf{l}_1(w)=(x,y,z)=((1-i)w+1+i,\ (-1+2i)w+2-i,\ (-2-3i)w+3+2i)$, where $w=w_r+w_I i$ is a complex-valued parameter. When $t=1-i$, the complex conjugate line $\mathbf{l}_2(w)=((1+i)w+1-i,\ (-1-2i)w+2+i,\ (-2+3i)w+3-2i)$ is obtained. With these lines, we obtain the parametrization $(x(w_R, w_I), y(w_R, w_I), z(w_R, w_I)) = (f_1(w_R, w_I)/f_4(w_R, w_I), f_2(w_R, w_I)/f_4(w_R, w_I), f_3(w_R, w_I)/f_4(w_R, w_I))$ where

$$\begin{aligned}
f_1 &= 68358w_R^4 - 69411w_R^3 + 136716w_R^2w_I^2 + 42607w_R^2w_I - 22381w_R^2 - 69411w_Rw_I^2 \\
&\quad - 39230w_Rw_I + 43253w_R + 68358w_I^4 + 42607w_I^3 - 5775w_I^2 + 8221w_I - 11755 \\
f_2 &= -68958w_R^4 + 284194w_R^3 - 137916w_R^2w_I^2 + 4441w_R^2w_I - 366491w_R^2 + 284194w_Rw_I^2 \\
&\quad + 11300w_Rw_I + 193570w_R - 68958w_I^4 + 4441w_I^3 - 124361w_I^2 - 8901w_I - 36677 \qquad (11) \\
f_3 &= -133716w_R^4 + 417667w_R^3 - 267432w_R^2w_I^2 - 37422w_R^2w_I - 466042w_R^2 + 417667w_Rw_I^2 \\
&\quad + 58622w_Rw_I + 224171w_R - 133716w_I^4 - 37422w_I^3 - 164742w_I^2 - 22866w_I - 39654 \\
f_4 &= 2(33879w_R^3 + 300w_R^2w_I - 62530w_R^2 + 33879w_Rw_I^2 + 3994w_Rw_I \\
&\quad + 38739w_R + 300w_I^3 - 22624w_I^2 - 2804w_I - 8072)\ .
\end{aligned}$$

The real base point is $(w_R, w_I) = (2/3, -1/6)$, which corresponds to the line $(w+1,\ 3w+1/6,\ 2w+1/6)$. The four complex base points, $(0.67336 \pm 0.02735i,\ -0.07294 \pm 0.11195i)$ and $(0.69678 \pm 0.02251i,\ -0.05028 \mp 0.13900i)$ correspond to the pairs of skew complex conjugate lines $(w+0.16675 \mp 0.18781i,\ (0.93864 - 0.59824i)w + 0.72700 \mp 0.06977i,\ (0.55461 \mp 0.58502i)w)$ and $(w + 1.45840 + 0.89959i,\ (1.26868 + 1.30057i)w + 0.09568 + 0.80755i,\ (0.31897 + 1.11820i)w)$, respectively. Since these complex conjugate lines are skew, no isolated real points are missed by the $F_4$ parametrization here. Also, since the lines $\mathbf{l}_1$ and $\mathbf{l}_2$ are complex, there are no real conics missed that lie in the planes containing one of these lines and perpendicular to the other, as was the case in the $F_1$ example. Indeed, if we let $w_R$ and/or $w_I$ approach $\pm\infty$ in (11), all three of $(x, y, z)$ become infinite. Because of this property it may be desirable to use the skew complex-line parametrization in the other cases in which it may be used, namely the $F_2$ and $F_3$ surfaces.

**An $F_5$ Surface**

The $F_5$ surface (shown in figure 5 (right) ) is given by the implicit equation

$$f(x,y,z) = 1816584x^3 + 5756616y^3 + 1816584z^3 - 7289736x^2y - 9033502x^2z - 14543124xy^2 + 4366603y^2z + 3281094xz^2 + 10858818yz^2 - 18019466xyz + 7087008x^2 + 5512596y^2 + 4779161z^2 + 1406184xy - 4714206xz + 5102202yz + 1816584z = 0 \ .$$

The polynomial $P_{27}(t)$ as computed by the algorithm in Section 4 is $(t+1)(69t+55)(18t+25)P_{24}(t)$, where $P_{24}(t)$ is a polynomial of degree 24 with 24 complex roots. When $t = 1$ is substituted into the expressions for $\hat{f}_2(\hat{x},\hat{y})$ and $\hat{g}_3(\hat{x},\hat{y})$, it is found that their common factor is $9\hat{x} + 11\hat{y}$. Thus the line $9\hat{x} + 11\hat{y} = 0, \hat{z} = 0$ lies on the cubic surface, and transforming back to the original coordinates, this is the line $\mathbf{l}_1(u) = (x,y,z) = (u+1,\ -9/11u - 1,\ 12/11u)$. When $t = -55/69$, the corresponding line is $2\hat{x} + 7\hat{y} = 0, \hat{z} = 0$, or $\mathbf{l}_2(u) = (u + 23/11, -2/7u - 5/3, -12/7u)$. When $t = -25/18$, the corresponding line is $107\hat{x} + 108\hat{y} = 0, \hat{z} = 0$, or $\mathbf{l}_3(u) = (u + 1/2, -107/108u - 25/36, 2u)$. These three lines lie in the plane $132x + 216y + 41z + 84 = 0$. The discriminant $k$ of Section 6.2 is

$$u^4 + 62.3281u^3 - 4080.61u^2 + 1509.67u + 6291.89 \ ;$$

This is positive when $u < -102.340$, $-1.06325 < u < 1.45963$, or $u > 39.6151$. Thus when $u$ is in one of these ranges, we obtain the parametrization

$$x = \frac{Q_1(u,v) + Q_2(u,v)\sqrt{Q_7(u)}}{Q_8(u,v) + Q_9(u,v)\sqrt{Q_7(u)}} \ , \ y = \frac{Q_3(u,v) + Q_4(u,v)\sqrt{Q_7(u)}}{Q_8(u,v) + Q_9(u,v)\sqrt{Q_7(u)}} \ , \ z = \frac{Q_5(u,v) + Q_6(u,v)\sqrt{Q_7(u)}}{Q_8(u,v) + Q_9(u,v)\sqrt{Q_7(u)}} \ ,$$

where

$$\begin{aligned}
Q_1(u,v) &= (8.99587 \cdot 10^{-4}v^2 + 3.51462 \cdot 10^{-5}v - 5.36254 \cdot 10^{-6})u^6 \\
&\quad + (0.105133v^2 - 2.39248 \cdot 10^{-3}v - 1.42488 \cdot 10^{-3})u^5 \\
&\quad + (-20.0872v^2 - 2.81225v - 0.120394)u^4 \\
&\quad + (-2185.20v^2 - 260.513v - 7.67265)u^3 + (-38310.8v^2 - 5524.64v - 252.070)u^2 \\
&\quad + (-105756v^2 - 14774.2v - 638.200)u - 61685.8v^2 - 7016.59v - 177.247 \\
Q_2(u,v) &= (-5.80221v^2 - 1.35397v - 6.10370 \cdot 10^{-2})u - 976.728v^2 - 237.966v - 10.7275 \\
Q_3(u,v) &= (-1.03635 \cdot 10^{-3}v^2 - 4.04895 \cdot 10^{-5}v + 6.17781 \cdot 10^{-6})u^6 \\
&\quad + (-9.01630 \cdot 10^{-2}v^2 + 4.01661 \cdot 10^{-3}v + 1.61011 \cdot 10^{-3})u^5 \\
&\quad + (21.3779v^2 + 3.09571v + 9.94279 \cdot 10^{-2})u^4
\end{aligned}$$

$$+ (1847.63v^2 + 206.918v + 0.796964)u^3 + (16437.1v^2 + 1161.86v - 88.7687)u^2$$
$$+ (16986.2v^2 - 3377.13v - 738.127)u + 7559.01v^2 - 3574.96v - 621.892$$
$$Q_4(u,v) = (6.60893v^2 + 1.10780v + 6.10370 \cdot 10^{-2})u + 690.007v^2 + 194.699v + 10.7275$$
$$Q_5(u,v) = (1.07033 \cdot 10^{-3}v^2 + 4.18170 \cdot 10^{-5}v - 6.38036 \cdot 10^{-6})u^6$$
$$+ (0.225780v^2 - 6.86940 \cdot 10^{-3}v - 2.34092 \cdot 10^{-3})u^5$$
$$+ (-30.6381v^2 - 2.66067v - 0.241823)u^4$$
$$+ (-2228.79v^2 - 52.2212v - 6.63043)u^3 + (-44926.3v^2 - 4190.55v - 316.786)u^2$$
$$+ (-81241.7v^2 - 968.908v - 67.8675)u - 22013.4v^2 + 8162.85v + 615.892$$
$$Q_6(u,v) = (4.84035v^2 - 1.47706v)u - 1720.32v^2 - 259.599v$$
$$Q_7(u) = -1.23442 \cdot 10^{-6}u^9 - 8.98903 \cdot 10^{-5}u^8 + 1.44803 \cdot 10^{-2}u^7 + 2.34577u^6$$
$$+ 105.322u^5 + 1142.72u^4 + 3359.87u^3 + 2594.14u^2 - 1217.84u - 1506.36$$
$$Q_8(u,v) = (7.28844 \cdot 10^{-4}v^2 + 2.84754 \cdot 10^{-5}v - 4.34472 \cdot 10^{-6})u^6$$
$$+ (-4.67672 \cdot 10^{-2}v^2 - 3.34107 \cdot 10^{-3}v - 7.39510 \cdot 10^{-4})u^5$$
$$+ (-8.66808v^2 - 2.21378v + 5.73177 \cdot 10^{-2})u^4$$
$$+ (-163.316v^2 - 89.9454v + 13.4150)u^3 + (39916.3v^2 + 7250.72v + 719.566)u^2$$
$$+ (15667.4v^2 - 2717.48v + 268.812)u - 27542.8v^2 - 11191.5v - 492.248$$
$$Q_9(u,v) = -10.2392uv^2 - 6.10370 \cdot 10^{-2}u + 600.236v^2 - 10.7275 .$$

When $-102.340 \leq u \leq -1.06325$ or $1.45963 \leq u \leq 39.6151$, the parametrization is

$$x = \frac{Q_1(u,v) + Q_2(u,v)\sqrt{Q_7(u)}}{Q_8(u) + Q_9(u,v)\sqrt{Q_7(u)}} \ , \ y = \frac{Q_3(u,v) + Q_4(u,v)\sqrt{Q_7(u)}}{Q_8(u) + Q_9(u,v)\sqrt{Q_7(u)}} \ , \ z = \frac{Q_5(u,v) + Q_6(u,v)\sqrt{Q_7(u)}}{Q_8(u) + Q_9(u,v)\sqrt{Q_7(u)}} \ ,$$

where

$$Q_1(u,v) = (-37.9245v^2 + 10.3515v + 1.29295)u^3 + (494.075v^2 - 128.591v - 2.94159)u^2$$
$$+ (93044.0v^2 - 29296.9v - 2915.68)u + 487668v^2 + 74012.5v + 4973.82$$
$$Q_2(u,v) = (61.5568v^2 + 11.0999v + 0.583780)u - 3608.55v^2 - 650.692v - 12.5729$$
$$Q_3(u,v) = (50.3928v^2 - 7.71286v - 1.36727)u^3 + (-2521.47v^2 - 202.038v - 2.39279)u^2$$
$$+ (-27524.8v^2 + 37257.0v + 4336.48)u + 126848v^2 + 64003.1v + 6008.88$$
$$Q_4(u,v) = (-50.3646v^2 - 11.0999v - 0.517062)u + 2952.45v^2 + 650.692v + 24.2988$$

$$
\begin{aligned}
Q_5(u,v) &= (-143.386v^2 + 7.30692v + 0.854738)u^3 + (12450.7v^2 + 1507.99v + 126.109)u^2 \\
&\quad + (-243361v^2 - 114213v - 4201.03)u + 364834v^2 + 41143.5v + 6520.34 \\
Q_6(u,v) &= (67.1529v^2 + 0.400306)u - 3936.60v^2 + 70.3552 \\
Q_7(u) &= 1.45651u^4 + 90.7814u^3 - 5943.45u^2 + 2198.85u + 9164.20 \\
Q_8(u) &= 1.06687u^3 - 52.9820u^2 - 4776.83u - 3741.83 \\
Q_9(u,v) &= (11.0999v + 0.216832)u - 650.692v - 77.0652\ .
\end{aligned}
$$

## Appendix B: Proof of Theorem 1

**Theorem 1:** The polynomial $P_{81}(t)$ obtained by taking the resultant of $\hat{f}_2$ and $\hat{g}_3$ factors as $P_{81}(t) = P_{27}(t)[P_3(t)]^6[P_6(t)]^6$, where $P_3(t) = B''t^3 + F''t^2 + D''t + A''$, the denominator of $K(t)$ and $L(t)$, and $P_6(t)$ is the numerator of $\overline{S}(t)$ $\quad (P_6(t) = \overline{S}(t)[P_3(t)^2])$.

**Proof:** This proof was performed through the use of Maple. When expanded out in full, $P_{81}(t)$ contains hundreds of thousands of terms, so a direct approach was not possible. Instead, $P_{81}(t)$ was shown to be divisible by both $[P_3(t)]^6$ and $[P_6(t)]^6$.

The quantities $\hat{f}_2$ and $\hat{g}_3$ were expressed in terms of the numerators of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$, and the numerator and denominator of $K(t)$. Let $K(t) = P_2(t)/P_3(t)$, where

$$
\begin{aligned}
P_2(t) &= -(Lt^2 + Nt + K) \\
P_3(t) &= Bt^3 + Ft^2 + Dt + A
\end{aligned}
\tag{12}
$$

(For brevity in this appendix we drop the double primes on the coefficients $A''$ through $P''$ of $f(x'', y'', z'')$.) Then we have

$$
\begin{aligned}
\overline{Q}(t) &= \frac{[(Ft^2 + 2Dt + 3A)P_2(t) + (Nt + 2K)P_3(t)]P_2(t)}{[P_3(t)]^2} = \frac{Q^*}{P_3^2} \\
\overline{R}(t) &= \frac{[(3Bt^2 + 2Ft + D)P_2(t) + (2Lt + N)P_3(t)]P_2(t)}{[P_3(t)]^2} = \frac{R^*}{P_3^2} \\
\overline{S}(t) &= \frac{(Gt^2 + Jt + E)[P_2(t)]^2 + (Pt + O)P_2(t)P_3(t) + S[P_3(t)]^2}{[P_3(t)]^2} = \frac{S^*}{P_3^2}\ .
\end{aligned}
\tag{13}
$$

Then we obtain

$$
\begin{aligned}
\hat{f}_2 &= \{[(It + H)P_2 + MP_3]Q^{*2} - [(Jt + 2E)P_2 + oP_3]Q^*S^* + [(Dt + 3A)P_2 + KP_3]S^{*2}\}\hat{x}^2 \\
&\quad + \{[(2It + 2H)P_2 + 2MP_3]Q^*R^* - [(2Gt + J)P_2 + PP_3]Q^*S^*
\end{aligned}
$$

$$- [(Jt+2E)P_2 + OP_3]R^*S^* + [(2Ft+2D)P_2 + NP_3]S^{*2}\}\hat{x}\hat{y}$$
$$+ \{[(It+H)P_2 + MP_3]R^{*2} - [(2Gt+J)P_2 + PP_3]R^*S^* + [(3Bt+F)P_2 + LP_3]S^{*2}\}\hat{y}^2 \ ,$$

$$\begin{aligned}\hat{g}_3 \ &= \ (-CQ^{*3} + HQ^{*2}S^* - EQ^*S^{*2} + AS^{*3})\hat{x}^3 \\ &+ (-3CQ^{*2}R^* + IQ^{*2}S^* + 2HQ^*R^*S^* - JQ^*S^{*2} - ER^*S^{*2} + DS^{*3})\hat{x}^2\hat{y} \\ &+ (-3CQ^*R^{*2} + 2IQ^*R^*S^* - GQ^*S^{*2} + HR^{*2}S^* - JR^*S^{*2} + FS^{*3})\hat{x}\hat{y}^2 \\ &+ (-CR^{*3} + IR^{*2}S^* - GR^*S^{*2} + BS^{*3})\hat{y}^3 \ .\end{aligned}$$

With this representation it was possible to take the resultant of $\hat{f}_2$ and $\hat{g}_3$ with respect to $\hat{x}$ without overflowing the memory capabilities of the machine. The resultant could be factored, and $[P_6(t)]^6$ was found to be one of the factors.

The factor $[P_3(t)]^6$ proved to be more difficult to obtain. After the factor $[P_6(t)]^6$ was removed from the resultant, the substitution $Q^* = P_2^2 P_3 - tR^*$ was used to eliminate $Q^*$ from the remaining factor. This remaining factor was split into 28 terms as follows:

$$\begin{aligned}&A_1R^{*6} + A_2R^{*5}S^* + A_3R^{*5} + A_4R^{*4}S^{*2} + A_5R^{*4}S^* + A_6R^{*4} + A_7R^{*3}S^{*3} \\ &+ A_8R^{*3}S^{*2} + A_9R^{*3}S^* + A_{10}R^{*3} + A_{11}R^{*2}S^{*4} + A_{12}R^{*2}S^{*3} + A_{13}R^{*2}S^{*2} + A_{14}R^{*2}S^* \\ &+ A_{15}R^{*2} + A_{16}R^*S^{*5} + A_{17}R^*S^{*4} + A_{18}R^*S^{*3} + A_{19}R^*S^{*2} + A_{20}R^*S^* + A_{21}R^* \\ &+ A_{22}S^{*6} + A_{23}S^{*5} + A_{24}S^{*4} + A_{25}S^{*3} + A_{26}S^{*2} + A_{27}S^* + A_{28}\end{aligned} \tag{14}$$

The coefficients $A_i$ are functions of $A$ through $P$, $P_2$, and $P_3$, and range from 76 terms in the case of $A_{22}$ to 1674 terms for $A_5$. Thus these coefficients must be omitted here for space considerations. Next these substitutions were made:

$$\begin{aligned}R^* &= M_2P_2^2 + N_2P_2P_3 \\ S^* &= M_3P_2^2 + N_3P_2P_3 + SP_3^2 \ .\end{aligned} \tag{15}$$

Later on these substitutions will be made:

$$\begin{aligned}M_2 &= 3Bt^2 + 2Ft + D \quad & M_3 &= Gt^2 + Jt + E \\ N_2 &= 2Lt + N & N_3 &= Pt + O\end{aligned} \tag{16}$$

so that the system (15, 16) agrees with the definitions of (12, 13). The reason behind these substitutions is to express the resultant in terms of $P_3$ as much as possible so as to be able to more readily determine what powers of $P_3$ divide into the coefficients $A_i$.

Upon making the substitutions in (15), each of the terms $A_i R^{*j} S^{*k}$ becomes a term $B_i$, where the $B_i$ are functions of $A$ through $P$, $P_2$, and $P_3$. The number of terms in the $B_i$ ranges from 140 for $B_{28}$ to 48960 for $B_7$. Each $B_i$ can be regarded as a polynomial in $P_3$. The highest power of $P_3$ appearing in any term is $P_3^{15}$, in $B_{28}$. Since we are trying to show that $\sum_{i=1}^{28} B_i$ is divisible by $P_3^6$, we need only consider the terms of the $B_i$ which do not contain any power of $P_3$ greater than or equal to six. That is,

$$\text{if } B_i = \sum_{i=0}^{15} b_i P_3^i \ ,$$
$$\text{let } C_i = \sum_{i=0}^{5} b_i P_3^i \ .$$

It turns out that each of the $C_i$ is divisible by $P_2^{10}$, so let

$$D_i = C_i / P_2^{10} \ .$$

We now make the substitutions

$$A = P_3 - Bt^3 - Ft^2 - Dt$$
$$K = -P_2 - Lt^2 - Nt$$

into the terms $D_i$ to produce more terms $E_i$. The latter are now functions of $B, C, \ldots, J, L, M, \ldots, P, P_2$, and $P_3$. Each of the $E_i$ turns out to be divisible by $P_3^2$. As was the case with the $B_i$, we remove powers of $P_3$ greater than or equal to six from the $E_i$. When we do that, all of the resulting terms are divisible by $P_2$. Thus,

$$\text{if } E_i = \sum_{i=2}^{8} e_i P_3^i \ ,$$
$$\text{let } F_i = (\sum_{i=2}^{5} b_i P_3^i)/P_2 \ .$$

(the highest power of $P_3$ appearing in the $E_i$ is 8, in seven of the $E_i$.)

The sum of all the terms of the $F_i$ is 61170. Since this is less than $2^{16}$, all of the $F_i$ can be added together in Maple to obtain one large expression. This can be expressed as a polynomial in $P_2$ and $P_3$ as follows:

$$\begin{aligned}(G_1 P_2^4 + G_2 P_2^3 + G_3 P_2^2 + G_4 P_2 + G_5) P_3^5 + (G_6 P_2^4 + G_7 P_2^3 + G_8 P_2^2 + G_9 P_2) P_3^4 \\ + (G_{10} P_2^4 + G_{11} P_2^3 + G_{12} P_2^2) P_3^3 + (G_{13} P_2^4 + G_{14} P_2^3) P_3^2 \ .\end{aligned} \tag{17}$$

Through the use of Maple we were able to show that each of the four terms enclosed in parentheses in (17) vanish. The fourth term, $(G_{13}P_2^4 + G_{14}P_2^3)$, was shown to be zero by making the three substitutions of (16), namely $N_2 = 2Lt + N$, $N_3 = Pt + O$, and (after simplifying) $M_3 = Gt^2 + Jt + E$, then determining that the result was divisible by $M_2 - 3Bt^2 - 2Ft - D$. The same procedure worked for the third term in parentheses in (17), $(G_{10}P_2^4 + G_{11}P_2^3 + G_{12}P_2^2)$, and for these combinations: $(G_6P_2^4 + G_7P_2^3)$, $G_8P_2^2$, $G_9P_2$, $(G_1P_2^4 + G_2P_2^3 + G_3P_2^2)$, $G_4P_2$, and $G_5$. Thus the expression in (17) vanishes, and since this is the remainder of the resultant (14) upon division by $P_3^6$, we conclude that the entire expression (14) is divisible by $P_3^6$. ■

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